MAT6684W: Sieve Methods, Fall 2012

Homework assignment #4

The exercise and theorem numbers refer to the course notes

Exercise 5.1.3 (5 marks). Show that

$$\rho(u) = \left(\frac{e + o(1)}{u \log u}\right)^u \quad (u \to \infty).$$

Hint: For each fixed $\epsilon > 0$, show that

$$\left(\frac{e - \epsilon}{u \log u}\right)^u \ll \rho(u) \ll \left(\frac{e + \epsilon}{u \log u}\right)^u \quad (u \geq 2).$$

Exercise (5 marks). Consider the Laplace transform of Dickman's function

$$\hat{\rho}(s) = \int_0^\infty \rho(t) e^{-st} dt \quad (s \in \mathbb{C}).$$

Show that $\hat{\rho}$ satisfies the differential equation

$$\frac{d}{ds}(s\hat{\rho}(s)) = e^{-s}\hat{\rho}(s).$$

Derive a formula for $\hat{\rho}$ (this formula might not be closed).

Exercise 6.1.3 (7 marks). Fix $\epsilon > 0$. Show that, for every prime p, we have that

$$n(p) := \min\left\{m \in \mathbb{N} : \left(\frac{m}{p}\right) = -1\right\} \ll_\epsilon p^{1/(2\sqrt{e})+\epsilon}.$$

Hint: Use Theorem B.3.1 and the fact that $(\frac{m}{p}) = 1$ for all m with $P^+(m) \leq n(p)$.

Exercise 7.4.2 (8 marks). Show that, for $1 \leq Q \leq x^{2/3}$,

$$\begin{aligned} S(x;Q) &:= \sum_{Q<q\leq 2Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \pmod q} \max_{y \leq x} \left|\sum_{n \leq y} \chi(n)\Lambda(n)\right| \\ &\ll (\log x)^6 (x + x^{1/2}Q^2 + x^{5/6}Q), \end{aligned}$$

which is an improvement over Theorem 7.1.1 when $Q \geq x^{1/9}$.
Hint: Write

$$\sum_{\substack{ab=n \\ a\leq UV}} \left(\sum_{\substack{cd=a \\ c\leq V,\, d\leq U}} \mu(c)\Lambda(d)\right) = \sum_{\substack{ab=n \\ a\leq U}} \left(\sum_{\substack{cd=a \\ c\leq V,\, d\leq U}} \mu(c)\Lambda(d)\right) + \sum_{\substack{ab=n \\ U<a\leq UV}} \left(\sum_{\substack{cd=a \\ c\leq V,\, d\leq U}} \mu(c)\Lambda(d)\right).$$