# THE NUMBER OF PRIME FACTORS IN $h$-FREE AND $h$-FULL POLYNOMIALS OVER FUNCTION FIELDS 

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#### Abstract

We study the distribution of the number of prime divisors function $\Omega$ over polynomials of the function field $\mathbb{F}_{q}(T)$ when restricted to $h$-free polynomials and to $h$-full polynomials. We use an adaptation of the Selberg-Delange method to arithmetical semigroups due to Warlimont to compute the first and second moments in each case and show that a generalization of the Erdős-Kac Theorem is true.


## 1. Introduction

Let $q$ be a prime power, and let $\mathbb{F}_{q}$ be the finite field with $q$ elements. Given $f \in \mathbb{F}_{q}[T]$, let

$$
\begin{equation*}
f=\alpha P_{1}^{e_{1}} \cdots P_{r}^{e_{r}} \tag{1}
\end{equation*}
$$

be its prime factorization, where each $P_{j} \in \mathbb{F}_{q}[T]$ is monic irreducible, $e_{j} \geq 1$, and $\alpha \in \mathbb{F}_{q}$.
Let $h \geq 2$ be an integer. We say that $f$ is $h$-free if $e_{j}<h$ for $j=1, \ldots, r$ in (1). In particular, the case $h=2$ yields the square-free polynomials. We denote by $\mathcal{S}_{h}$ the set of $h$-free monic polynomials. Analogously, we say that $f$ is $h$-full if $e_{j} \geq h$ for $j=1, \ldots, r$ in (11). In particular, the case $h=2$ yields the square-full polynomials. We denote by $\mathcal{N}_{h}$ the set of $h$-full monic polynomials.

The number of prime divisors function is defined by $\Omega(f):=e_{1}+\cdots+e_{r}$. The distribution of values of this function, along with that of the number of distinct prime divisors $\omega(f):=r$, have been widely studied over the natural numbers. The average number of $\omega(n)$ is known to satisfy

$$
\frac{1}{x} \sum_{n \leq x} \omega(n)=\log \log x+O(1)
$$

More precise expressions for this average were given by Sathe [Sat53], Selberg Sel54, Delange [Del53, Del71], and Saidak [Sai02]. Hardy and Ramanujan [HR00] famously proved that $\omega(n)$ has normal order $\log \log n$, which means that $\omega(n) \sim \log \log n$ for almost all $n$. Turán Tur34 proved that the variance of $\omega(n)$ for $n \leq x$ is $\ll x \log \log x$. Erdős and Kac EK40] used the central limit theorem and Brun's sieve to famously prove that $(\omega(n)-\log \log n) /(\log \log n)^{1 / 2}$ for $n \geq 3$ has a limiting normal distribution. This result was recovered by Delange [Del53, Del71] by the method of moments, and it was later extended by Halberstam Hal55. The method of moments was further simplified by Billingsley [Bil69] and by Granville and Soundararajan GS07]. The function field version of Turán's and Erdős-Kac's theorems was proven by Liu Liu04b, Liu04a] as consequences of much more general results. Rhoades Rho09 considered an application of Granville-Soundararajan's method to the function field case. The function $\Omega(n)$ is closely related to $\omega(n)$ and satisfies similar properties including the Erdős-Kac Theorem.

[^0]In this note we are interested in studying the distribution of $\omega(f)$ and $\Omega(f)$ as $f$ goes over the $h$ free and the $h$-full polynomials of $\mathbb{F}_{q}[T]$. We focus on $\Omega$ because there is a very natural argument for obtaining the moments in the case of $h$-full polynomials, but we also state the analogous results for $\omega$. Our results imply that these functions satisfy limiting normal distributions even when restricted to $h$-free or $h$-full polynomials, and therefore versions of Erdős-Kac type of statements are true in these restricted contexts. This is expected in both cases, but the $h$-full case is somehow more surprising, since the family is sparse, as will be discussed later. The current work generalizes the results obtained in [JL22] for the first moments of $\Omega(n)$.

Let

$$
B_{1}=\gamma+\sum_{P}\left(\log \left(1-\frac{1}{|P|}\right)+\frac{1}{|P|}\right) \text { and } B_{2}=\gamma+\sum_{P}\left(\log \left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}\right)
$$

denote the function field analogues of the Mertens constants, where

$$
\gamma=\lim _{n \rightarrow \infty}\left(\sum_{k=1}^{n} \frac{1}{k}-\log n\right) \approx 0.57721566 \ldots
$$

is the Euler-Mascheroni constant. Mertens constants $B_{1}$ and $B_{2}$ appear in estimates for $\sum_{n \leq x} \omega(n)$ and $\sum_{n \leq x} \Omega(n)$ respectively (see HW08, Theorem 430]).

Above and for the rest of the article, $\sum_{P}$ and $\prod_{P}$ denote the sum and product over all monic irreducible polynomials in $\mathbb{F}_{q}[T]$. We let $\mathcal{M}_{n}$ denote the set of monic polynomials of $\mathbb{F}_{q}[T]$ of degree $n$. For a polynomial $F(T) \in \mathbb{F}_{q}[T]$, its norm or absolute value is defined by $|F(T)|=q^{\operatorname{deg}(F)}$. We denote by $\zeta_{q}(s)$ the zeta function associated to $\mathbb{F}_{q}[T]$, defined precisely in (4). Throughout this article, the error terms have implied functions depending on $q$ and $h$. This does not pose a problem, as we assume $q$ and $h$ to be fixed.

We compute the first and second moments of $\Omega$ for $h$-free polynomials.
Theorem 1.1. For any $\varepsilon>0$ and as $n \rightarrow \infty$, the first moment of $\Omega$ over the $h$-free polynomials of degree $n$ is given by

$$
\begin{equation*}
\sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} \Omega(f)=\frac{q^{n} \log (n)}{\zeta_{q}(h)}+\frac{q^{n}}{\zeta_{q}(h)}\left(B_{2}-\sum_{P} \frac{h}{|P|^{h}-1}\right)+O\left(\frac{q^{n}}{n^{1-\varepsilon}}\right) \tag{2}
\end{equation*}
$$

while the second moment is given by

$$
\begin{align*}
\sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} \Omega(f)^{2}= & \frac{q^{n}(\log (n))^{2}}{\zeta_{q}(h)}+\frac{q^{n}(\log (n))}{\zeta_{q}(h)}\left[2\left(B_{2}-\sum_{P} \frac{h}{|P|^{h}-1}\right)+1\right] \\
& +\frac{q^{n}}{\zeta_{q}(h)}\left[B_{2}^{2}+B_{2}-\zeta(2)+\sum_{P}\left(\frac{1}{(|P|-1)^{2}}-\frac{h^{2}}{|P|^{h}-1}-\frac{h^{2}}{\left(|P|^{h}-1\right)^{2}}\right)\right. \\
& \left.+\left(\sum_{P} \frac{h}{|P|^{h}-1}\right)^{2}-2 B_{2} \sum_{P} \frac{h}{|P|^{h}-1}\right]+O\left(\frac{q^{n}}{n^{1-\varepsilon}}\right) . \tag{3}
\end{align*}
$$

In particular, for the square-free polynomials, we have

$$
\frac{1}{\left|\mathcal{S}_{2} \cap \mathcal{M}_{n}\right|} \sum_{f \in \mathcal{S}_{2} \cap \mathcal{M}_{n}} \Omega(f)=\log (n)+\left(B_{2}-\sum_{P} \frac{2}{|P|^{2}-1}\right)+O\left(\frac{1}{n^{1-\varepsilon}}\right)
$$

and

$$
\begin{aligned}
\frac{1}{\left|\mathcal{S}_{2} \cap \mathcal{M}_{n}\right|} \sum_{f \in \mathcal{S}_{2} \cap \mathcal{M}_{n}} \Omega(f)^{2}= & (\log (n))^{2}+\log (n)\left[2\left(B_{2}-\sum_{P} \frac{2}{|P|^{2}-1}\right)+1\right] \\
& +\left[B_{2}^{2}+B_{2}-\zeta(2)-\sum_{P} \frac{3|P|+1}{(|P|-1)(|P|+1)^{2}}\right. \\
& \left.+\left(\sum_{P} \frac{2}{|P|^{2}-1}\right)^{2}-2 B_{2} \sum_{P} \frac{2}{|P|^{2}-1}\right]+O\left(\frac{1}{n^{1-\varepsilon}}\right) .
\end{aligned}
$$

We remark that the asymptotics of equation (2) over the integers was given in JL22, Theorem 1]. Equation 2 shows that the asymptotics of the mean value of $\Omega$ over the $h$-free polynomials of degree $n$ is $\log (n)$, which is analogous to the result over the integers if we interpret the degree of a polynomial as analogous to the logarithm of a positive integer. The monic $h$-free polynomials of degree $n$ are a comparatively large subset of the monic polynomials of degree $n$ : their sizes are $\frac{q^{n}}{\zeta_{q}(h)}$ and $q^{n}$ respectively (see Lemma 3.3). Therefore it is not surprising to obtain results for $h$-free polynomials that are analogous to what is obtained when considering the whole set of polynomials.

The proof for Theorem (1.1) rests in a version of the Selberg-Delange method for function fields developed by Warlimont War93].

The above result allows us to compute the variance.
Theorem 1.2. For any $\varepsilon>0$ and as $n \rightarrow \infty$, the variance of $\Omega$ over the $h$-free polynomials of degree $n$ is given by

$$
\begin{aligned}
\operatorname{Var}_{h-\text { free }, n}(\Omega): & =\frac{1}{\left|\mathcal{S}_{h} \cap \mathcal{M}_{n}\right|} \sum_{f \in \mathcal{\mathcal { S } _ { h } \cap \mathcal { M } _ { n }}} \Omega(f)^{2}-\left(\frac{1}{\left|\mathcal{S}_{h} \cap \mathcal{M}_{n}\right|} \sum_{f \in \mathcal{\mathcal { S } _ { h } \cap \mathcal { M } _ { n }}} \Omega(f)\right)^{2} \\
& =\log (n)+B_{2}-\zeta(2)+\sum_{P}\left(\frac{1}{(|P|-1)^{2}}-\frac{h^{2}}{|P|^{h}-1}-\frac{h^{2}}{\left(|P|^{h}-1\right)^{2}}\right)+O\left(\frac{1}{n^{1-\varepsilon}}\right) .
\end{aligned}
$$

Finally, we obtain an analogue of Erdős-Kac theorem in this case.
Theorem 1.3. As $n \rightarrow \infty, \Omega(f)$ with $f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}$ approaches a normal distribution, namely, for $\alpha \leq \beta$,

$$
\frac{1}{\left|\mathcal{S}_{h} \cap \mathcal{M}_{n}\right|}\left|\left\{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}: \alpha \leq \frac{\Omega(f)-\log (n)}{\sqrt{\log (n)}} \leq \beta\right\}\right| \rightarrow \frac{1}{\sqrt{2 \pi}} \int_{\alpha}^{\beta} e^{-t^{2} / 2} d t
$$

This result is proven by computing the moments of $\frac{\Omega(f)-\log (n)}{\sqrt{\log (n)}}$ and using Stirling numbers of the second kind to understand their growth.

We also compute the first and second moments of $\Omega$ for $h$-full polynomials.
Let $\xi_{h}$ denote a primitive $h$-root of unity in $\mathbb{C}$.

Theorem 1.4. For any $\varepsilon>0$ and as $n \rightarrow \infty$, the first moment of $\Omega$ over the $h$-full polynomials of degree $n$ is given by

$$
\begin{aligned}
\sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}} \Omega(f)= & \frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& \times\left[h\left(\log \left(\frac{n}{h}\right)+B_{2}\right)\right. \\
& \left.+\sum_{j=1}^{h-1}(h+j) \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}}\right] \\
& +O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right)
\end{aligned}
$$

while the second moment is given by

$$
\begin{aligned}
\sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}} \Omega(f)^{2}= & \frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& \times\left[h^{2}\left(\log \left(\frac{n}{h}\right)^{2}+\left(2 B_{2}+1\right) \log \left(\frac{n}{h}\right)+B_{2}^{2}+B_{2}-\zeta(2)+\sum_{P} \frac{1}{(|P|-1)^{2}}\right)\right. \\
& +\sum_{j=1}^{h-1}(h+j)\left(2 h\left(\log \left(\frac{n}{h}\right)+B_{2}\right)+(h+j)\right. \\
& \left.+\sum_{\ell=1}^{h-1}(h+\ell) \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+\ell)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}}\right) \\
& \times \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}} \\
& \left.-\sum_{1 \leq j, \ell \leq h-1}(h+j)(h+\ell) \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+\ell)}}{\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right)^{2}}\right] \\
& +O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right) .
\end{aligned}
$$

In particular, for the square-full polynomials, we have the following,

$$
\begin{aligned}
\sum_{f \in \mathcal{N}_{2} \cap \mathcal{M}_{n}} \Omega(f)= & q^{\frac{n}{2}}\left(\frac{1-q^{-2}}{1-q^{-\frac{1}{2}}}\left[\log \left(\frac{n}{2}\right)+B_{2}+\frac{3}{2} \sum_{P} \frac{q^{-\frac{3}{2} \operatorname{deg}(P)}}{1+q^{-\frac{3}{2} \operatorname{deg}(P)}}\right]\right. \\
& \left.+(-1)^{n} \frac{1+q^{-2}}{1+q^{-\frac{1}{2}}}\left[\log \left(\frac{n}{2}\right)+B_{2}+\frac{3}{2} \sum_{P} \frac{(-1)^{\operatorname{deg}(P)} q^{-\frac{3}{2} \operatorname{deg}(P)}}{\left.1+(-1)^{\operatorname{deg}(P)} q^{-\frac{3}{2} \operatorname{deg}(P)}\right)}\right]\right) \\
& +O\left(\frac{q^{\frac{n}{n}}}{n^{1-\varepsilon}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{f \in \mathcal{N}_{2} \cap \mathcal{M}_{n}} \Omega(f)^{2}= & \frac{q^{\frac{n}{2}}}{2}\left(\frac { 1 - q ^ { - 2 } } { 1 - q ^ { - \frac { 1 } { 2 } } } \left[4\left(\log \left(\frac{n}{2}\right)^{2}+\left(2 B_{2}+1\right) \log \left(\frac{n}{2}\right)+B_{2}^{2}+B_{2}-\zeta(2)+\sum_{P} \frac{1}{(|P|-1)^{2}}\right)\right.\right. \\
& +3\left(4\left(\log \left(\frac{n}{2}\right)+B_{2}\right)+3+3 \sum_{P} \frac{q^{-\frac{3}{2} \operatorname{deg}(P)}}{1+q^{-\frac{3}{2} \operatorname{deg}(P)}}\right) \sum_{P} \frac{q^{-\frac{3}{2} \operatorname{deg}(P)}}{1+q^{-\frac{3}{2} \operatorname{deg}(P)}} \\
& \left.-9 \sum_{P} \frac{q^{-3 \operatorname{deg}(P)}}{\left(1+q^{-\frac{3}{2} \operatorname{deg}(P)}\right)^{2}}\right]+(-1)^{n} \frac{1+q^{-2}}{1+q^{-\frac{1}{2}}} \\
& \times\left[4\left(\log \left(\frac{n}{2}\right)^{2}+\left(2 B_{2}+1\right) \log \left(\frac{n}{2}\right)+B_{2}^{2}+B_{2}-\zeta(2)+\sum_{P} \frac{1}{(|P|-1)^{2}}\right)\right. \\
& +3\left(4\left(\log \left(\frac{n}{2}\right)+B_{2}\right)+3+3 \sum_{P} \frac{(-1)^{\operatorname{deg}(P)} q^{-\frac{3}{2} \operatorname{deg}(P)}}{1+(-1)^{\operatorname{deg}(P)} q^{-\frac{3}{2} \operatorname{deg}(P)}}\right) \sum_{P} \frac{(-1)^{\operatorname{deg}(P)} q^{-\frac{3}{2} \operatorname{deg}(P)}}{1+(-1)^{\operatorname{deg}(P)} q^{-\frac{3}{2} \operatorname{deg}(P)}} \\
& \left.\left.-9 \sum_{P} \frac{q^{-3 \operatorname{deg}(P)}}{\left(1+(-1)^{\operatorname{deg}(P)} q^{-\frac{3}{2} \operatorname{deg}(P)}\right)^{2}}\right]\right)+O\left(\frac{q^{\frac{n}{n}}}{n^{1-\varepsilon}}\right) .
\end{aligned}
$$

Unlike the $h$-free polynomials, the $h$-full polynomials are a sparse set: the number of monic $h$-full polynomials of degree $n$ is asymptotically

$$
\frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right)
$$

(See Lemma 5.3.) For the case $h=2$, this reduces to

$$
\frac{q^{\frac{n}{2}}}{2}\left[\prod_{P}\left(1+q^{-\frac{3}{2} \operatorname{deg}(P)}\right)+(-1)^{n} \prod_{P}\left(1+(-1)^{\operatorname{deg}(P)} q^{-\frac{3}{2} \operatorname{deg}(P)}\right)\right]=q^{\left\lfloor\frac{n-1}{2}\right\rfloor}\left(1+q^{-1}\right) .
$$

The fact that the set is sparse explains why the results of Theorem 1.4 are very different from what is obtained in the cases of the $h$-free polynomials and the whole family.

Theorem 1.4 allows us to compute the variance.

Theorem 1.5. For any $\varepsilon>0$ and as $n \rightarrow \infty$, the variance of $\Omega$ over the $h$-full polynomials is given by

$$
\begin{aligned}
& \operatorname{Var}_{h-\mathrm{full}, n}(\Omega):=\frac{1}{\left|\mathcal{N}_{h} \cap \mathcal{M}_{n}\right|} \sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}} \Omega(f)^{2}-\left(\frac{1}{\left|\mathcal{N}_{h} \cap \mathcal{M}_{n}\right|} \sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}} \Omega(f)\right)^{2} \\
& =\left[h^{2}\left(\log \left(\frac{n}{h}\right)+B_{2}-\zeta(2)+\sum_{P} \frac{1}{(|P|-1)^{2}}\right)\right. \\
& +\left(\sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right)\right)^{-1} \\
& \times \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& \times\left[\sum_{j=1}^{h-1}(h+j)\left((h+j)+\sum_{\ell=1}^{h-1}(h+\ell) \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+\ell)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}}\right)\right. \\
& \times \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}} \\
& \left.-\sum_{1 \leq j, \ell \leq h-1}(h+j)(h+\ell) \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+\ell)}}{\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right)^{2}}\right] \\
& -\left[\left(\sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right)\right)^{-1}\right. \\
& \times \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& \left.\times \sum_{j=1}^{h-1}(h+j) \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)-\operatorname{deg}(P)(h+1)+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)-\operatorname{deg}(P)(2 h-1)}\right]^{2}+O\left(\frac{1}{n^{1-\varepsilon}}\right) .
\end{aligned}
$$

The above statements are proven by considering the factorization of each $f$ to codify the $h$-full condition and by exploiting the fact that $\Omega$ is completely additive. As we will remark later, it is possible to give a proof that avoids using the completely additive property, that therefore can be applied to extend the results to $\omega$. We have chosen to keep the method of proof that employs the completely additive property here because it yields clearer formulas for the first and second moments and the variant of $\Omega$.

Finally, we also obtain an analogue of Erdős-Kac theorem in this case.
Theorem 1.6. As $n \rightarrow \infty, \Omega(f)$ with $f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}$ approaches a normal distribution, namely, for $\alpha \leq \beta$,

$$
\frac{1}{\left|\mathcal{N}_{h} \cap \mathcal{M}_{n}\right|}\left|\left\{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}: \alpha \leq \frac{\Omega(f)-h \log \left(\frac{n}{h}\right)}{h \sqrt{\log \left(\frac{n}{h}\right)}} \leq \beta\right\}\right| \rightarrow \frac{1}{\sqrt{2 \pi}} \int_{\alpha}^{\beta} e^{-t^{2} / 2} d t
$$

This result is again proven by computing the moments of $\frac{\Omega(f)-h \log \left(\frac{n}{h}\right)}{h \sqrt{\log \left(\frac{n}{h}\right)}}$, using similar techniques as the ones for the $h$-free polynomial case.

Let us remark that we can recover the results over all monic polynomials with no restrictions if we specialize at $h=1$ in the formulas for $h$-full polynomials. Our error terms depend on $h$. If they did not, letting $h \rightarrow \infty$ in the formulas for $h$-free polynomials would also recover the results for all monic polynomials.

This article is organized as follows. Section 2 provides some notation and preliminary statements. The distribution of $\Omega$ over the $h$-free polynomials is considered in detail in Section 3, while Section 4 describes the corresponding results for $\omega$. Similarly, the distribution of $\Omega$ over the $h$-full polynomials is treated in Section 5, including a discussion on the proof strategy for Theorem 1.4 in Section 5.3 . Section 6 states the corresponding results for $\omega$. Finally we discuss the results in Section 7 .

## 2. Notation and preliminary statements

We denote by $\mathcal{M}$ the set of monic polynomials over $\mathbb{F}_{q}[T]$ and by $\mathcal{M}_{n}$ (respectively $\mathcal{M}_{\leq n}$ ) the subset of $\mathcal{M}$ containing the polynomials of degree $n$ (respectively degree $\leq n$ ). Let $\mathcal{P}$ be the set of monic irreducible polynomials, and let $\mathcal{P}_{n}, \mathcal{P}_{\leq n}$ be defined analogously to the corresponding subsets of $\mathcal{M}$. The zeta function of $\mathbb{F}_{q}[T]$ is given by

$$
\begin{equation*}
\zeta_{q}(s)=\sum_{f \in \mathcal{M}} \frac{1}{|f|^{s}}=\prod_{P}\left(1-\frac{1}{|P|^{s}}\right)^{-1} \tag{4}
\end{equation*}
$$

The above sum and product converge for $\operatorname{Re}(s)>1$. However, one can also see that

$$
\zeta_{q}(s)=\frac{1}{1-q^{1-s}},
$$

and this provides a meromorphic continuation for $\zeta_{q}(s)$ to the whole complex plane, with a single pole at $s=1$. By making the change of variables $u=q^{-s}$ we can write

$$
\mathcal{Z}_{q}(u)=\sum_{f \in \mathcal{M}} u^{\operatorname{deg}(f)}=\prod_{P}\left(1-u^{\operatorname{deg}(P)}\right)^{-1}
$$

which converges absolutely for $|u|<\frac{1}{q}$, and has a meromorphic continuation to the complex plane with a pole at $u=\frac{1}{q}$.

We recall Perron's formula over $\mathbb{F}_{q}[T]$ which we will use throughout this article. (See for example [Mur08, 4.4.15] for the classical statement, and [DFL22, Lemma 2.2] for the function field version.)

Theorem 2.1 (Perron's Formula). If the generating series $\mathcal{A}(u)=\sum_{f \in \mathcal{M}} a(f) u^{\operatorname{deg}(f)}$ is absolutely convergent in $|u| \leq r<1$, then

$$
\sum_{f \in \mathcal{M}_{n}} a(f)=\frac{1}{2 \pi i} \oint_{|u|=r} \frac{\mathcal{A}(u)}{u^{n}} \frac{d u}{u}
$$

where, in the usual notation, we take $\oint$ to signify the integral over the circle oriented counterclockwise.

The following result was first proven by Warlimont in the context of arithmetical semigroups and was recently rediscovered by Afshar and Porritt in the function field setting. It provides an extension of the Selberg-Delange method for function fields and will be crucial for obtaining the first and second moments in all cases.

Theorem 2.2. War93, Lemma 2], AP19, Proposition 2.3] Let $C(u, z)=\sum_{n>0} C_{z}(n) u^{n}$ and $B(u, z)=\sum_{n \geq 0} B_{z}(n) u^{n}$ be power series with coefficients depending on $z$ satisfying $C(u, z)=$ $B(u, z) \mathcal{Z}_{q}(u)^{z}$. Suppose also that, uniformly for $|z| \leq A$,

$$
\sum_{n \geq 0} \frac{\left|B_{z}(n)\right|}{q^{n}} n^{2 A+2} \ll A A 1
$$

Then, uniformly for $|z| \leq A$ and $n \geq 1$, we have

$$
C_{z}(n)=q^{n} \frac{n^{z-1}}{\Gamma(z)} B(1 / q, z)+O_{A}\left(q^{n} n^{\operatorname{Re}(z)-2}\right)
$$

where $\Gamma(z)$ is the gamma function defined by

$$
\Gamma(z)=\int_{0}^{\infty} t^{z-1} e^{-t} d t
$$

We will need the following extension of the above theorem.
Theorem 2.3 (Generalized Theorem 2.2). Let $C(u, z)=\sum_{n \geq 0} C_{z}(n) u^{n}$ and $B(u, z)=\sum_{n \geq 0} B_{z}(n) u^{n}$ be power series with coefficients depending on $z$ satisfying $C(u, z)=B(u, z) \mathcal{Z}_{q}\left(u^{h}\right)^{z}$, where $h$ is a positive integer. Suppose also that, uniformly for $|z| \leq A$,

$$
\sum_{n \geq 0} \frac{\left|B_{z}(n)\right|}{q^{\frac{n}{h}}} n^{2 A+2}<_{A} 1
$$

Then, uniformly for $|z| \leq A$ and $n \geq 1$, we have

$$
C_{z}(n)=\frac{q^{\frac{n}{h}} n^{z-1}}{h^{z} \Gamma(z)} \sum_{k=0}^{h-1} \xi_{h}^{k n} B\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)+O_{A}\left(q^{\frac{n}{n}} n^{\operatorname{Re}(z)-2}\right),
$$

where $\xi_{h}$ denotes a primitive $h$-root of unity in $\mathbb{C}$.
Proof. Following Lemma 2.1 and Corollary 2.2 in [AP19], we define $D_{z}(n)$ for $n \geq 0$ via $\mathcal{Z}_{q}\left(u^{h}\right)^{z}=$ $\sum_{n=0}^{\infty} D_{z}(n) u^{n}$. The binomial theorem gives

$$
\mathcal{Z}_{q}\left(u^{h}\right)^{z}=\left(1-q u^{h}\right)^{-z}=\sum_{n \geq 0}\binom{n+z-1}{n} q^{n} u^{h n}
$$

and therefore,

$$
\begin{equation*}
D_{z}(n)=\frac{1}{h} \sum_{k=0}^{h-1}\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{n}\binom{n / h+z-1}{n / h} \tag{5}
\end{equation*}
$$

By AP19, Corollary 2.2], $\frac{\Gamma(m+z)}{\Gamma(m+1)}=m^{z-1}\left(1+O_{A}\left(\frac{1}{m}\right)\right)$. Applying this to equation (5), we have

$$
D_{z}(n)=\frac{1}{h} \sum_{k=0}^{h-1}\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{n} \frac{\Gamma\left(\frac{n}{h}+z\right)}{\Gamma\left(\frac{n}{h}+1\right) \Gamma(z)}=\frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \frac{\left(\frac{n}{h}\right)^{z-1}}{\Gamma(z)}\left(1+O_{A}\left(\frac{1}{n}\right)\right) .
$$

We now follow the proof of [AP19, Proposition 2.3]. First express the coefficients $C_{z}$ as convolutions of $B_{z}$ and $D_{z}$ to obtain

$$
\begin{align*}
C_{z}(n) & =\sum_{0 \leq a \leq n} B_{z}(a) D_{z}(n-a) \\
(6) & =\frac{q^{\frac{n}{h}}}{h^{z}} \sum_{0 \leq a<n} \frac{B_{z}(a) \sum_{k=0}^{h-1} \xi_{h}^{(n-a) k}(n-a)^{z-1}}{q^{\frac{a}{h}} \Gamma(z)}+B_{z}(n)+O_{A}\left(q^{\frac{n}{h}} \sum_{0 \leq a<n} \frac{\left|B_{z}(a)\right|}{q^{\frac{a}{h}}}(n-a)^{\operatorname{Re}(z)-2}\right) . \tag{6}
\end{align*}
$$

We split the first sum at $\frac{n}{2}$ and use that

$$
(n-a)^{z-1}= \begin{cases}n^{z-1}\left(1+O_{A}(a / n)\right), & \text { if } 0 \leq a \leq \frac{n}{2} \\ O_{A}\left(n^{A-1}\right), & \text { if } \frac{n}{2}<a<n\end{cases}
$$

Then the first sum in (6) can be written as

$$
\begin{aligned}
& \sum_{0 \leq a<n} \frac{B_{z}(a) \sum_{k=0}^{h-1} \xi_{h}^{(n-a) k}(n-a)^{z-1}}{q^{\frac{a}{h}}} \\
& =\sum_{0 \leq a \leq \frac{n}{2}} \frac{B_{z}(a) \sum_{k=0}^{h-1} \xi_{h}^{(n-a) k} n^{z-1}}{q^{\frac{a}{h}}}\left(1+O_{A}(a / n)\right)+O_{A}\left(\sum_{\frac{n}{2}<a<n} \frac{B_{z}(a) n^{A-1}}{q^{\frac{a}{h}}}\right) \\
& =n^{z-1} \sum_{0 \leq a \leq \frac{n}{2}} \sum_{k=0}^{h-1} \frac{B_{z}(a) \xi_{h}^{n k}}{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{a}}+O_{A}\left(n^{\operatorname{Re}(z)-2}\left(\sum_{0 \leq a \leq \frac{n}{2}} \frac{\left|B_{z}(a)\right| a}{q^{\frac{a}{h}}}+\sum_{\frac{n}{2}<a<n} \frac{\left|B_{z}(a)\right| a^{2 A+1}}{q^{\frac{a}{h}}}\right)\right),
\end{aligned}
$$

where we have used that $n^{\operatorname{Re}(z)-2} a^{2 A+1} \gg n^{-A-2} n^{2 A+1}=n^{A-1}$ for $\frac{n}{2}<a<n$. Completing the sum over $a$, the above then equals

$$
\begin{align*}
& n^{z-1} \sum_{k=0}^{h-1} \xi_{h}^{k n} B\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)+O_{A}\left(n^{\operatorname{Re}(z)-1} \sum_{\frac{n}{2}<a} \frac{B_{z}(a)}{q^{\frac{a}{h}}}+n^{\operatorname{Re}(z)-2}\right) \\
& =n^{z-1} \sum_{k=0}^{h-1} \xi_{h}^{k n} B\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)+O_{A}\left(n^{\operatorname{Re}(z)-2} \sum_{\frac{n}{2}<a} \frac{B_{z}(a) n}{q^{\frac{a}{h}}}+n^{\operatorname{Re}(z)-2}\right) \\
& =n^{z-1} \sum_{k=0}^{h-1} \xi_{h}^{k n} B\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)+O_{A}\left(n^{\operatorname{Re}(z)-2}\right) . \tag{7}
\end{align*}
$$

For the other terms in (6), we get

$$
\begin{equation*}
B_{z}(n) \ll n^{\operatorname{Re}(z)-2} n^{A+2}\left|B_{z}(n)\right| \ll q^{\frac{n}{h}} n^{\operatorname{Re}(z)-2} \sum_{a=0}^{\infty} \frac{\left|B_{z}(a)\right|}{q^{\frac{a}{h}}} a^{A+2} \ll q^{\frac{n}{h}} n^{\operatorname{Re}(z)-2}, \tag{8}
\end{equation*}
$$

and

$$
\begin{align*}
& \sum_{0 \leq a<n} \frac{\left|B_{z}(a)\right|}{q^{\frac{a}{h}}}(n-a)^{\operatorname{Re}(z)-2} \\
& =\sum_{0 \leq a \leq \frac{n}{2}} \frac{\left|B_{z}(a)\right|}{q^{\frac{a}{h}}} n^{\operatorname{Re}(z)-2}\left(1+O_{A}(a / n)\right)+O_{A}\left(\sum_{\frac{n}{2} a<n} \frac{\left|B_{z}(a)\right|}{q^{\frac{a}{h}}} n^{A-2}\right) \\
& <_{A} n^{\operatorname{Re}(z)-2} \sum_{0 \leq a \leq \frac{n}{2}} \frac{\left|B_{z}(a)\right|}{q^{\frac{a}{h}}}+n^{\operatorname{Re}(z)-3} \sum_{0 \leq a \leq \frac{n}{2}} \frac{\left|B_{z}(a)\right| a}{q^{\frac{a}{h}}}+n^{\operatorname{Re}(z)-3} \sum_{\frac{n}{2}<a<n} \frac{\left|B_{z}(a)\right| a^{2 A+1}}{q^{\frac{a}{h}}} \\
& <_{A} n^{\operatorname{Re}(z)-2 .} \tag{9}
\end{align*}
$$

Combining (7) with (8) and (9) in (6), we get the result.

## 3. The distribution of $\Omega$ over $h$-free polynomials

Recall that $h \geq 2$ is a fixed integer and that we denote by $\mathcal{S}_{h}$ the set of $h$-free monic polynomials. In order to prove Theorem 1.1, we start by considering the following Euler product (converging for $\operatorname{Re}(s)>1$ ), which gives the generating Dirichlet series for the $h$-free polynomials:

$$
\begin{equation*}
\frac{\mathcal{Z}_{q}(u)}{\mathcal{Z}_{q}\left(u^{h}\right)}=\prod_{P}\left(\frac{1-u^{h \operatorname{deg}(P)}}{1-u^{\operatorname{deg}(P)}}\right)=\prod_{P}\left(1+u^{\operatorname{deg}(P)}+\cdots+u^{(h-1) \operatorname{deg}(P)}\right)=\sum_{f \in \mathcal{S}_{h}} u^{\operatorname{deg}(f)} \tag{10}
\end{equation*}
$$

We incorporate the coefficient $\Omega(f)$ by introducing it as the exponent of an extra variable, which we will later differentiate. This preserves the additive structure of $\Omega(f)$.

$$
\begin{equation*}
\mathcal{C}(u, z):=\prod_{P}\left(1+z u^{\operatorname{deg}(P)}+\cdots+z^{h-1} u^{(h-1) \operatorname{deg}(P)}\right)=\sum_{f \in \mathcal{S}_{h}} z^{\Omega(f)} u^{\operatorname{deg}(f)} \tag{11}
\end{equation*}
$$

Extracting the singularity at $u=\frac{1}{q}$ leads us to consider

$$
\begin{align*}
\mathcal{B}(u, z) & :=\mathcal{Z}_{q}(u)^{-z} \mathcal{C}(u, z) \\
& =\prod_{P}\left(1-u^{\operatorname{deg}(P)}\right)^{z}\left(\frac{1-z^{h} u^{h \operatorname{deg}(P)}}{1-z u^{\operatorname{deg}(P)}}\right) \tag{12}
\end{align*}
$$

In order to estimate the derivative of $\mathcal{C}(u, z)$ we need Theorem 2.2. We claim that for $\mathcal{B}(u, z)$ given by (12), if $\mathcal{B}(u, z)=\sum_{n \geq 0} \mathcal{B}_{z}(n) u^{n}$, then uniformly for $|z| \leq A$, we have $\sum_{n \geq 0} \frac{\mid \mathcal{\mathcal { B } _ { z } ( n ) |}}{q^{n}} n^{2 A+2}<_{A} 1$. Indeed, we have the following result.
Lemma 3.1. For $|z| \leq A, n \geq 2$ and $\sigma>\frac{1}{2}$,

$$
\sum_{0 \leq a \leq n} \frac{\left|\mathcal{B}_{z}(a)\right|}{q^{\sigma a}} \leq c_{A, \sigma}
$$

where $c_{A, \sigma}$ is a constant depending on $A$ and $\sigma$.
Proof. Our proof follows very similar steps to those in the proof of AP19, Proposition 2.5]. Let $b_{z}(f)$ be the function defined on the powers of monic irreducible polynomials $P$ by

$$
\begin{equation*}
1+\sum_{k \geq 1} b_{z}\left(P^{k}\right) u^{k}=\left(1+z u+\cdots+z^{h-1} u^{h-1}\right)(1-u)^{z}=\left(\frac{1-z^{h} u^{h}}{1-z u}\right)(1-u)^{z} \tag{13}
\end{equation*}
$$

and extended multiplicatively to all $f \in \mathcal{M}$.

Then $\mathcal{B}(u, z)=\sum_{f \in \mathcal{M}} b_{z}(f) u^{\operatorname{deg}(f)}$, and therefore, $\mathcal{B}_{z}(n)=\sum_{f \in \mathcal{M}_{n}} b_{z}(f)$. Expanding the righthand side of (13), we see that $b_{z}(P)=0$. By Cauchy's formula integrating (13) over $|u|=\sqrt{\frac{2}{3}}$, we obtain

$$
b_{z}\left(P^{k}\right)=\frac{1}{2 \pi i} \oint_{|u|=\sqrt{\frac{2}{3}}}\left(\frac{1-z^{h} u^{h}}{1-z u}\right)(1-u)^{z} \frac{d u}{u^{k+1}}
$$

Thus,

$$
\left|b_{z}\left(P^{k}\right)\right| \leq\left(\frac{3}{2}\right)^{\frac{k}{2}} M_{A}
$$

for $k \geq 2$, where

$$
M_{A}:=\sup _{|z| \leq A,|u| \leq \sqrt{\frac{2}{3}}}\left|\left(1+z u+\cdots+z^{h-1} u^{h-1}\right)(1-u)^{z}\right|
$$

is a constant depending on $A$. Applying this bound, we have

$$
\begin{aligned}
\sum_{0 \leq a \leq n} \frac{\left|\mathcal{B}_{z}(a)\right|}{q^{\sigma a}} & \leq \sum_{f \in \mathcal{M}_{\leq n}} \frac{\left|b_{z}(f)\right|}{q^{\sigma \operatorname{deg}(f)}} \\
& \leq \prod_{P \in \mathcal{P}_{\leq n}}\left(1+\sum_{k \geq 2} \frac{\left|b_{z}\left(P^{k}\right)\right|}{q^{k \sigma \operatorname{deg}(P)}}\right) \\
& \leq \prod_{P \in \mathcal{P}_{\leq n}}\left(1+M_{A} \sum_{k \geq 2}\left(\frac{\sqrt{3 / 2}}{q^{\sigma \operatorname{deg}(P)}}\right)^{k}\right) \\
& =\prod_{P \in \mathcal{P}_{\leq n}}\left(1+\frac{M_{A} 3 / 2}{q^{\sigma \operatorname{deg}(P)}\left(q^{\sigma \operatorname{deg}(P)}-\sqrt{3 / 2}\right)}\right)
\end{aligned}
$$

Taking the logarithm, noticing that $q^{\sigma \operatorname{deg}(P)}-\sqrt{3 / 2} \geq q^{\sigma \operatorname{deg}(P)} \frac{\sqrt{2}-1}{\sqrt{2}}$ for $q \geq 3$ and $\operatorname{deg}(P) \geq 1$, and applying the Prime Polynomial Theorem [Ros02, Theorem 2.2] that states that there are $\frac{q^{n}}{n}+O\left(\frac{q^{\frac{n}{2}}}{n}\right)$ monic irreducible polynomials of degree $n$, we have

$$
\sum_{P \in \mathcal{P}_{\leq n}} \log \left(1+\frac{M_{A} 3 / 2}{q^{\sigma \operatorname{deg}(P)}\left(q^{\sigma \operatorname{deg}(P)}-\sqrt{3 / 2}\right)}\right) \ll M_{A} \sum_{1 \leq d \leq n} \frac{q^{d(1-2 \sigma)}}{d} \leq \frac{M_{A}}{q^{2 \sigma-1}-1} .
$$

The statement follows by applying the exponential function.
Since $a^{2 A+2}<q^{a / 3}$ as $a$ approaches infinity, it follows from Lemma 3.1 that

$$
\sum_{a \geq 0} \frac{\left|\mathcal{B}_{z}(a)\right|}{q^{a}} a^{2 A+2}<\sum_{a \geq 0} \frac{\left|\mathcal{B}_{z}(a)\right|}{q^{\frac{2 a}{3}}} \ll{ }_{A} 1
$$

uniformly for $|z| \leq A$. Thus we can apply Theorem 2.2 to $\mathcal{B}(u, z)$.
Before proceeding to the proof of Theorem 1.1, we prove an auxiliary result that will allow us to differentiate inside an error term after applying Theorem 2.2 .

Lemma 3.2. Let $G_{n}(z), f(z)$ be complex functions, analytic at $z=1$ such that $G_{n}(\mathbb{R}), f(\mathbb{R}) \subset \mathbb{R}$ and

$$
G_{n}(z)=f(z) n^{\operatorname{Re}(z)-1}+O\left(n^{\operatorname{Re}(z)-2}\right)
$$

in a neighborhood of $z=1$. Then, for any small $\varepsilon>0$, we have

$$
\begin{align*}
& G_{n}^{\prime}(1)=\left.\frac{\partial}{\partial z}\left(f(z) n^{\mathrm{Re}(z)-1}\right)\right|_{z=1}+O\left(n^{-1+\varepsilon}\right)  \tag{14}\\
& G_{n}^{\prime \prime}(1)=\left.\frac{\partial^{2}}{\partial z^{2}}\left(f(z) n^{\mathrm{Re}(z)-1}\right)\right|_{z=1}+O\left(n^{-1+\varepsilon}\right) \tag{15}
\end{align*}
$$

Proof. For ease of notation, we will assume that $z$ is real and $z>1$ to compute the derivative. (This does not restrict the result, since the derivatives at $z=1$ exists by hypothesis.) Write $z=1+\ell$. Then for any $K>0$ there is an $M=M_{K}$ such that if $n \geq M$,

$$
f(z) n^{\ell}-K n^{\ell-1} \leq G_{n}(z) \leq f(z) n^{\ell}+K n^{\ell-1}
$$

in a neighborhood of $z=1$. Thus,

$$
\left(f(z) n^{\ell}-K n^{\ell-1}\right)-\left(f(1)+K n^{-1}\right) \leq G_{n}(z)-G_{n}(1) \leq\left(f(z) n^{\ell}+K n^{\ell-1}\right)-\left(f(1)-K n^{-1}\right)
$$

and

$$
\left|\frac{f(z) n^{\ell}-f(1)}{\ell}-\frac{G_{n}(z)-G_{n}(1)}{\ell}\right| \leq \frac{K\left(n^{\ell}+1\right)}{\ell n} .
$$

By enlarging $M$ if necessary, we can assume that, for $n \geq M$,

$$
\frac{K}{n f(z)}, \frac{K}{n f(1)} \leq \ell n^{-1+\varepsilon}
$$

We have

$$
K n^{\ell-1}=f(z) n^{\ell}\left(\frac{K}{n f(z)}\right) \leq f(z) n^{\ell}\left(\ell n^{-1+\epsilon}\right)=\ell f(z) n^{\ell-1+\varepsilon}
$$

and similarly,

$$
K n^{-1} \leq \ell f(1) n^{-1+\varepsilon}
$$

Thus

$$
\left|\frac{f(z) n^{\ell}-f(1)}{\ell}-\frac{G_{n}(z)-G_{n}(1)}{\ell}\right| \leq \frac{f(z) n^{\ell}+f(1)}{n^{1-\varepsilon}} \ll \frac{1}{n^{1-\ell-\varepsilon}} .
$$

Taking $\ell$ to be arbitrarily small in the above equation, we obtain

$$
G_{n}^{\prime}(1)=\left.\left(f(z) n^{z-1}\right)^{\prime}\right|_{z=1}+O\left(\frac{1}{n^{1-\varepsilon}}\right),
$$

as desired. This completes the proof of equation (14).
To prove (15) we proceed similarly. Writing $z_{1}=1+\ell$ and $z_{2}=z_{1}+m$ with $z_{2}>z_{1}>1$ for ease of notation, we have that for $K>0$ and $M=M_{K}$ and $n \geq M$,

$$
\left|\frac{f\left(z_{1}\right) n^{\ell}-f(1)}{\ell}-\frac{G_{n}\left(z_{1}\right)-G(1)}{\ell}\right| \leq \frac{K\left(n^{\ell}+1\right)}{\ell n}
$$

and

$$
\left|\frac{f\left(z_{2}\right) n^{m+\ell}-f\left(z_{1}\right) n^{\ell}}{m}-\frac{G_{n}\left(z_{2}\right)-G_{n}\left(z_{1}\right)}{m}\right| \leq \frac{K n^{\ell}\left(n^{m}+1\right)}{m n} .
$$

By combining the above inequalities, we get

$$
\begin{aligned}
& \left|\frac{\frac{f\left(z_{2}\right) n^{m+\ell}-f\left(z_{1}\right) n^{\ell}}{m}-\frac{f\left(z_{1}\right) n^{\ell}-f(1)}{\ell}}{\ell+m}-\frac{\frac{G\left(z_{2}\right)-G\left(z_{1}\right)}{m}-\frac{G\left(z_{1}\right)-G(1)}{\ell}}{\ell+m}\right| \\
& \leq \frac{K n^{\ell}\left(n^{m}+1\right)}{m(\ell+m) n}+\frac{K\left(n^{\ell}+1\right)}{\ell(\ell+m) n} .
\end{aligned}
$$

As before, by enlarging $M$ if necessary, we obtain

$$
\frac{K}{n f\left(z_{1}\right)}, \frac{K}{n f\left(z_{2}\right)}, \frac{K}{n f(1)} \leq \frac{\ell m(\ell+m)}{n^{1-\varepsilon}} .
$$

The above leads to

$$
\begin{aligned}
& \left|\frac{\frac{f\left(z_{2}\right) n^{m+\ell}-f\left(z_{1}\right) n^{\ell}}{m}-\frac{f\left(z_{1}\right) n^{\ell}-f(1)}{\ell}}{\ell+m}-\frac{\frac{G\left(z_{2}\right)-G\left(z_{1}\right)}{m}-\frac{G\left(z_{1}\right)-G(1)}{\ell}}{\ell+m}\right| \\
& \leq \frac{f\left(z_{2}\right) n^{\ell+m} \ell+f\left(z_{1}\right) n^{\ell} \ell}{n^{1-\varepsilon}}+\frac{f\left(z_{1}\right) n^{\ell} m+f(1) m}{n^{1-\varepsilon}} \\
& \ll \frac{1}{n^{1-\epsilon-\ell-m}} .
\end{aligned}
$$

Taking $\ell, m$ arbitrarily small, we obtain

$$
G^{\prime \prime}(1)=\left.\left(f(z) n^{z-1}\right)^{\prime \prime}\right|_{z=1}+O\left(\frac{1}{n^{1-\varepsilon}}\right) .
$$

This concludes the proof of (15).
3.1. The first and second moments of $\Omega$ over $h$-free polynomials. We now have all the elements to proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. Starting from equation (11), we have

$$
\sum_{f \in \mathcal{S}_{h}} z^{\Omega(f)} u^{\operatorname{deg}(f)}=\mathcal{C}(u, z)=\mathcal{B}(u, z) \mathcal{Z}_{q}(u)^{z} .
$$

Applying Theorem 2.2, we obtain

$$
\begin{equation*}
\sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} z^{\Omega(f)}=q^{n} \frac{n^{z-1}}{\Gamma(z)} \mathcal{B}(1 / q, z)+O_{A}\left(q^{n} n^{\operatorname{Re}(z)-2}\right) \tag{16}
\end{equation*}
$$

Differentiating both sides of (16) with respect to $z$ for $z$ close to 1 , and applying (14), we obtain

$$
\begin{equation*}
\sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} \Omega(f) z^{\Omega(f)-1}=\left(\frac{\mathcal{B}(1 / q, z)}{\Gamma(z)}\right)^{\prime} q^{n} n^{z-1}+\frac{\mathcal{B}(1 / q, z)}{\Gamma(z)} q^{n} \log (n) n^{z-1}+O\left(\frac{q^{n}}{n^{1-\varepsilon}}\right) \tag{17}
\end{equation*}
$$

Evaluating (17) at $z=1$ gives

$$
\begin{equation*}
\sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} \Omega(f)=\frac{\frac{\partial}{\partial z} \mathcal{B}(1 / q, 1) \Gamma(1)-\mathcal{B}(1 / q, 1) \Gamma^{\prime}(1)}{\Gamma(1)^{2}} q^{n}+\frac{\mathcal{B}(1 / q, 1)}{\Gamma(1)} q^{n} \log (n)+O\left(\frac{q^{n}}{n^{1-\varepsilon}}\right) \tag{18}
\end{equation*}
$$

Notice that $\Gamma(1)=1$ and $\Gamma^{\prime}(1)=-\gamma$. We also have that

$$
\mathcal{B}(1 / q, 1)=\prod_{P}\left(1-\frac{1}{|P|^{h}}\right)=\frac{1}{\zeta_{q}(h)} .
$$

In addition, the logarithmic derivative gives

$$
\frac{\frac{\partial}{\partial z} \mathcal{B}(1 / q, z)}{\mathcal{B}(1 / q, z)}=\sum_{P}\left(\log \left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-z}-\frac{h z^{h-1}}{|P|^{h}-z^{h}}\right) .
$$

Applying the above identities to (18), we obtain the proof of (22).
We now proceed to prove (3). Multiplying (17) by $z$, differentiating both sides with respect to $z$ for $z$ close to 1 , and applying (15), we obtain

$$
\begin{align*}
\sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} \Omega(f)^{2} z^{\Omega(f)-1}= & \left(\frac{\mathcal{B}(1 / q, z)}{\Gamma(z)}\right)^{\prime \prime} q^{n} z n^{z-1}+\left(\frac{\mathcal{B}(1 / q, z)}{\Gamma(z)}\right)^{\prime} q^{n}\left(n^{z-1}+2 z n^{z-1} \log (n)\right) \\
& +\frac{\mathcal{B}(1 / q, z)}{\Gamma(z)} q^{n} \log (n)\left(n^{z-1}+z n^{z-1} \log (n)\right) \\
& +O\left(\frac{q^{n}}{n^{1-\varepsilon}}\right) \tag{19}
\end{align*}
$$

We remark that

$$
\begin{aligned}
\left(\frac{\mathcal{B}(1 / q, z)}{\Gamma(z)}\right)^{\prime \prime} & =\left(\frac{\frac{\partial}{\partial z} \mathcal{B}(1 / q, z) \Gamma(z)-\mathcal{B}(1 / q, z) \Gamma^{\prime}(z)}{\Gamma(z)^{2}}\right)^{\prime} \\
& =\frac{\frac{\partial^{2}}{\partial z^{2}} \mathcal{B}(1 / q, z) \Gamma(z)^{2}-2 \frac{\partial}{\partial z} \mathcal{B}(1 / q, z) \Gamma(z) \Gamma^{\prime}(z)+\mathcal{B}(1 / q, z)\left(2 \Gamma^{\prime}(z)^{2}-\Gamma(z) \Gamma^{\prime \prime}(z)\right)}{\Gamma(z)^{3}} .
\end{aligned}
$$

In addition, the second derivative of $\mathcal{B}(1 / q, z)$ gives

$$
\begin{aligned}
\frac{\partial^{2}}{\partial z^{2}} \mathcal{B}(1 / q, z)= & \frac{\partial}{\partial z} \mathcal{B}(1 / q, z) \sum_{P}\left(\log \left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-z}-\frac{h z^{h-1}}{|P|^{h}-z^{h}}\right) \\
& +\mathcal{B}(1 / q, z) \sum_{P}\left(\frac{1}{(|P|-z)^{2}}-\frac{h(h-1) z^{h-2}}{|P|^{h}-z^{h}}-\frac{h^{2} z^{2 h-2}}{\left(|P|^{h}-z^{h}\right)^{2}}\right)
\end{aligned}
$$

Evaluating the above at $z=1$, we obtain

$$
\begin{align*}
\frac{\partial^{2}}{\partial z^{2}} \mathcal{B}(1 / q, 1)= & \frac{1}{\zeta_{q}(h)}\left(\sum_{P}\left(\log \left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}-\frac{h}{|P|^{h}-1}\right)\right)^{2} \\
& +\frac{1}{\zeta_{q}(h)} \sum_{P}\left(\frac{1}{(|P|-1)^{2}}-\frac{h(h-1)}{|P|^{h}-1}-\frac{h^{2}}{\left(|P|^{h}-1\right)^{2}}\right) \tag{20}
\end{align*}
$$

Evaluating (19) at $z=1$ gives

$$
\begin{aligned}
\sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} \Omega(f)^{2}= & \frac{\frac{\partial^{2}}{\partial z^{2}} \mathcal{B}(1 / q, 1) \Gamma(1)^{2}-2 \frac{\partial}{\partial z} \mathcal{B}(1 / q, 1) \Gamma(1) \Gamma^{\prime}(1)+\mathcal{B}(1 / q, 1)\left(2 \Gamma^{\prime}(1)^{2}-\Gamma(1) \Gamma^{\prime \prime}(1)\right)}{\Gamma(1)^{3}} q^{n} \\
& +\frac{\frac{\partial}{\partial z} \mathcal{B}(1 / q, 1) \Gamma(1)-\mathcal{B}(1 / q, 1) \Gamma^{\prime}(1)}{\Gamma(1)^{2}} q^{n}(1+2 \log (n)) \\
& +\frac{\mathcal{B}(1 / q, 1)}{\Gamma(1)} q^{n} \log (n)(1+\log (n))+O\left(\frac{q^{n}}{n^{1-\varepsilon}}\right)
\end{aligned}
$$

Replacing (20) in the above equation, and using the fact that $\Gamma^{\prime \prime}(1)=\gamma^{2}+\zeta(2)$, we obtain

$$
\begin{aligned}
\sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} \Omega(f)^{2}= & \frac{q^{n}(\log (n))^{2}}{\zeta_{q}(h)} \\
& +\frac{q^{n}(\log (n))}{\zeta_{q}(h)}\left(2 \sum_{P}\left(\log \left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}-\frac{h}{|P|^{h}-1}\right)+2 \gamma+1\right) \\
& +\frac{q^{n}}{\zeta_{q}(h)}\left(\sum_{P}\left(\log \left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}-\frac{h}{|P|^{h}-1}\right)\right)^{2} \\
& +\frac{q^{n}}{\zeta_{q}(h)} \sum_{P}\left(\frac{1}{(|P|-1)^{2}}-\frac{h(h-1)}{|P|^{h}-1}-\frac{h^{2}}{\left(|P|^{h}-1\right)^{2}}\right) \\
& +\frac{2 \gamma+1}{\zeta_{q}(h)} q^{n} \sum_{P}\left(\log \left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}-\frac{h}{|P|^{h}-1}\right)+\frac{\gamma^{2}+\gamma-\zeta(2)}{\zeta_{q}(h)} q^{n} \\
& +O\left(\frac{q^{n}}{n^{1-\varepsilon}}\right)
\end{aligned}
$$

We can compute the variance from Theorem 1.1. Before doing this, we need the following auxiliary result.

Lemma 3.3. Let $n \geq h$. We have

$$
\sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} 1=\frac{q^{n}}{\zeta_{q}(h)}
$$

Proof. Recall from (10) that

$$
\sum_{f \in \mathcal{S}_{h}} u^{\operatorname{deg}(f)}=\frac{\mathcal{Z}_{q}(u)}{\mathcal{Z}_{q}\left(u^{h}\right)}
$$

By Perron's formula (Theorem 2.1), for $0<\delta<1 / q$, we have

$$
\sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} 1=\frac{1}{2 \pi i} \oint_{|u|=\delta} \frac{1-q u^{h}}{1-q u} \frac{d u}{u^{n+1}}=-\operatorname{Res}_{u=1 / q} \frac{1-q u^{h}}{u^{n+1}(1-q u)}=\frac{q^{n}}{\zeta_{q}(h)} .
$$

Proof of Theorem 1.2. By combining the results of Theorem 1.1 with Lemma 3.3, we have

$$
\begin{aligned}
\mathbb{E}_{h, n}\left(\Omega^{2}\right)-\left(\mathbb{E}_{h, n}(\Omega)\right)^{2}= & \frac{1}{\left|\mathcal{S}_{h} \cap \mathcal{M}_{n}\right|} \sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} \Omega(f)^{2}-\left(\frac{1}{\left|\mathcal{S}_{h} \cap \mathcal{M}_{n}\right|} \sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} \Omega(f)\right)^{2} \\
= & \log (n)+\sum_{P}\left(\frac{1}{(|P|-1)^{2}}-\frac{h(h-1)}{|P|^{h}-1}-\frac{h^{2}}{\left(|P|^{h}-1\right)^{2}}\right)+(\gamma-\zeta(2)) \\
& +\sum_{P}\left(\log \left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}-\frac{h}{|P|^{h}-1}\right)+O\left(\frac{1}{n^{1-\varepsilon}}\right) .
\end{aligned}
$$

3.2. Higher moments of $\Omega$ and an Erdős-Kac result for $h$-free polynomials. The goal of this section is to prove Theorem [1.3, namely the Erdős-Kac theorem for $\Omega$ over the $h$-free polynomials. To this end we will prove that as $n \rightarrow \infty$,

$$
\frac{1}{\left|\mathcal{S}_{h} \cap \mathcal{M}_{n}\right|} \sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}}\left(\frac{\Omega(f)-\log (n)}{\sqrt{\log (n)}}\right)^{v} \rightarrow C_{v}
$$

where

$$
C_{v}= \begin{cases}\frac{v!}{2^{\frac{v}{2}}\left(\frac{v}{2}\right)!} & v \text { even }  \tag{21}\\ 0 & v \text { odd }\end{cases}
$$

Notice that

$$
\frac{1}{\left|\mathcal{S}_{h} \cap \mathcal{M}_{n}\right|} \sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}}\left(\frac{\Omega(f)-\log (n)}{\sqrt{\log (n)}}\right)^{v}=\frac{1}{\log (n)^{\frac{v}{2}}} \sum_{\ell=0}^{v}\binom{v}{\ell} \mathbb{E}\left(\Omega^{\ell}\right)(-1)^{v-\ell} \log (n)^{v-\ell} .
$$

Consider again the moment generating function (11)

$$
\mathcal{C}\left(u, e^{t}\right):=\prod_{P}\left(1+e^{t} u^{\operatorname{deg}(P)}+\cdots+e^{(h-1) t} u^{(h-1) \operatorname{deg}(P)}\right)=\sum_{f \in \mathcal{\mathcal { S } _ { h } \cap \mathcal { M }}} e^{\Omega(f) t} u^{\operatorname{deg}(f)}
$$

After extracting the singularity at $u=\frac{1}{q}$, we obtain

$$
\begin{aligned}
\mathcal{B}\left(u, e^{t}\right): & :=\mathcal{Z}_{q}(u)^{-e^{t}} \mathcal{C}\left(u, e^{t}\right) \\
& =\prod_{P}\left(1-u^{\operatorname{deg}(P)}\right)^{e^{t}}\left(\frac{1-e^{h t} u^{h \operatorname{deg}(P)}}{1-e^{t} u^{\operatorname{deg}(P)}}\right) .
\end{aligned}
$$

Applying Theorem 2.2, we have

$$
\sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} e^{\Omega(f) t}=q^{n} n^{e^{t}-1} \frac{\mathcal{B}\left(1 / q, e^{t}\right)}{\Gamma\left(e^{t}\right)}+O_{A}\left(q^{n} n^{\operatorname{Re}\left(e^{t}\right)-2}\right)
$$

By the property of the moment generating function, we have

$$
\begin{equation*}
\mathbb{E}\left(\Omega^{\ell}\right)=\left.\mathbb{E}\left(e^{\Omega t}\right)^{(\ell)}\right|_{t=0} \tag{22}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\mathbb{E}\left(\Omega^{\ell}\right)=\left.\zeta_{q}(h) \sum_{j=0}^{\ell}\binom{\ell}{j}\left(n^{e^{t}-1}\right)^{(j)}\left(\frac{\mathcal{B}\left(1 / q, e^{t}\right)}{\Gamma\left(e^{t}\right)}\right)^{(\ell-j)}\right|_{t=0}+O\left(\frac{1}{n^{1-\varepsilon}}\right) \tag{23}
\end{equation*}
$$

We claim that for $j \geq 1$,

$$
\left(n^{e^{t}-1}\right)^{(j)}=n^{e^{t}-1} \sum_{m=1}^{j}\left\{\begin{array}{c}
j  \tag{24}\\
m
\end{array}\right\}\left(e^{t} \log (n)\right)^{m}=n^{e^{t}-1} T_{j}\left(e^{t} \log (n)\right)
$$

where the $\left\{\begin{array}{l}j \\ m\end{array}\right\}$ are the Stirling numbers of the second kind, and the $T_{j}$ are the Touchard polynomials Tou39. Indeed, proceeding by induction, when $j=1$, both sides of (24) become $n^{e^{t}-1} e^{t} \log (n)$. Now assume that (24) is true for $j$. Differentiating, we have

$$
\left(n^{e^{t}-1}\right)^{(j+1)}=n^{e^{t}-1} \sum_{m=1}^{j}\left\{\begin{array}{c}
j \\
m
\end{array}\right\} m\left(e^{t} \log (n)\right)^{m}+n^{e^{t}-1} \sum_{m=1}^{j}\left\{\begin{array}{c}
j \\
m
\end{array}\right\}\left(e^{t} \log (n)\right)^{m+1}
$$

and the induction step follows from the fact that $m\left\{\begin{array}{c}j \\ m\end{array}\right\}+\left\{\begin{array}{c}j \\ m-1\end{array}\right\}=\left\{\begin{array}{c}j+1 \\ m\end{array}\right\}$.
Combining (24) with (23), we have

$$
\mathbb{E}\left(\Omega^{\ell}\right)=\left.\zeta_{q}(h) \sum_{j=0}^{\ell}\binom{\ell}{j} T_{j}(\log (n))\left(\frac{\mathcal{B}\left(1 / q, e^{t}\right)}{\Gamma\left(e^{t}\right)}\right)^{(\ell-j)}\right|_{t=0}+O\left(\frac{1}{n^{1-\varepsilon}}\right)
$$

and

$$
\frac{1}{\log (n)^{\frac{v}{2}}} \sum_{\ell=0}^{v}\binom{v}{\ell} \mathbb{E}\left(\Omega^{\ell}\right)(-1)^{v-\ell} \log (n)^{v-\ell}
$$

$$
\begin{equation*}
=\left.\frac{\zeta_{q}(h)}{\log (n)^{\frac{v}{2}}} \sum_{\ell=0}^{v}\binom{v}{\ell} \sum_{j=0}^{\ell}\binom{\ell}{j} T_{j}(\log (n))\left(\frac{\mathcal{B}\left(1 / q, e^{t}\right)}{\Gamma\left(e^{t}\right)}\right)^{(\ell-j)}\right|_{t=0}(-1)^{v-\ell} \log (n)^{v-\ell}+O\left(\frac{1}{n^{1-\varepsilon}}\right) . \tag{25}
\end{equation*}
$$

Consider the change of variables $u=v-\ell, k=v-\ell+j$. Then the main term in (25) becomes

$$
\begin{align*}
& \left.\frac{\zeta_{q}(h)}{\log (n)^{\frac{v}{2}}} \sum_{k=0}^{v}\left(\frac{\mathcal{B}\left(1 / q, e^{t}\right)}{\Gamma\left(e^{t}\right)}\right)^{(v-k)}\right|_{t=0} \sum_{u=0}^{k}\binom{v}{u}\binom{v-u}{k-u} T_{k-u}(\log (n))(-1)^{u} \log (n)^{u} \\
& =\left.\frac{\zeta_{q}(h)}{\log (n)^{\frac{v}{2}}} \sum_{k=0}^{v}\binom{v}{k}\left(\frac{\mathcal{B}\left(1 / q, e^{t}\right)}{\Gamma\left(e^{t}\right)}\right)^{(v-k)}\right|_{t=0} \sum_{u=0}^{k}\binom{k}{u} T_{k-u}(\log (n))(-1)^{u} \log (n)^{u} \tag{26}
\end{align*}
$$

Since the generating function for the Touchard polynomials [Tou39] is

$$
e^{x\left(e^{t}-1\right)}=\sum_{k=0}^{\infty} \frac{T_{k}(x)}{k!} t^{k}
$$

one can see that the generating function for the inner sum in 26 is given by

$$
e^{x\left(e^{t}-1-t\right)}=\sum_{k=0}^{\infty} \frac{t^{k}}{k!} \sum_{u=0}^{k}\binom{k}{u} T_{k-u}(x)(-1)^{u} x^{u} .
$$

Notice that the coefficient of $x^{u}$ in the power series of $e^{x\left(e^{t}-1-t\right)}$ is given by $\frac{\left(e^{t}-1-t\right)^{u}}{u!}$, whose lowest power of $t$ is $t^{2 u}$. Therefore, thinking of $e^{x\left(e^{t}-1-t\right)}$ as a power series in $t$, we have that the coefficient of $t^{v}$ is a polynomial in $x$ of degree at most $\left\lfloor\frac{v}{2}\right\rfloor$.

Now suppose that $v$ is even. Then the coefficient of $x^{\frac{v}{2}} t^{v}$ in $e^{x\left(e^{t}-1-t\right)}$ is given by $\frac{1}{2^{\frac{v}{2}}\left(\frac{v}{2}\right)!}$. Back to the inner sum in (26), this gives a leading coefficient of $\frac{k!}{2^{\frac{k}{2}}\left(\frac{k}{2}\right)!}$ for $\log (n)^{\frac{k}{2}}$ when $k$ is even. Incorporating this information in (26), we get, for $v$ even,

$$
\begin{aligned}
& \frac{1}{\log (n)^{\frac{v}{2}}} \sum_{\ell=0}^{v}\binom{v}{\ell} \mathbb{E}\left(\Omega(f)^{\ell}\right)(-1)^{v-\ell} \log (n)^{v-\ell} \\
& =\frac{\zeta_{q}(h)}{\log (n)^{\frac{v}{2}}} \frac{\mathcal{B}(1 / q, 1)}{\Gamma(1)} \sum_{u=0}^{v}\binom{v}{u} T_{v-u}(\log (n))(-1)^{u} \log (n)^{u}+O\left(\frac{1}{\log n}\right) \\
& =\frac{v!}{2^{\frac{v}{2}}\left(\frac{v}{2}\right)!}+O\left(\frac{1}{\log n}\right),
\end{aligned}
$$

while for $v$ odd we get

$$
O\left(\frac{1}{\sqrt{\log n}}\right)
$$

as desired.

## 4. The distribution of $\omega$ over $h$-Free polynomials

In this short section we state, without proof, analogous results for $\omega$, which can be proven using the same techniques as the results for $\Omega$.

Theorem 4.1. For any $\varepsilon>0$ and as $n \rightarrow \infty$, the first moment of $\omega$ over the $h$-free polynomials of degree $n$ is given by

$$
\sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} \omega(f)=\frac{q^{n} \log (n)}{\zeta_{q}(h)}+\frac{q^{n}}{\zeta_{q}(h)}\left(B_{1}-\sum_{P} \frac{|P|-1}{|P|\left(|P|^{h}-1\right)}\right)+O\left(\frac{q^{n}}{n^{1-\varepsilon}}\right)
$$

while the second moment is given by

$$
\begin{aligned}
\sum_{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}} \omega(f)^{2}= & \frac{q^{n}(\log (n))^{2}}{\zeta_{q}(h)}+\frac{q^{n}(\log (n))}{\zeta_{q}(h)}\left[2\left(B_{1}-\sum_{P} \frac{|P|-1}{|P|\left(|P|^{h}-1\right)}\right)+1\right] \\
& +\frac{q^{n}}{\zeta_{q}(h)}\left[B_{1}^{2}+B_{1}-\zeta(2)-\sum_{P}\left(\frac{|P|^{h-1}-1}{|P|^{h}-1}\right)^{2}+\left(\sum_{P} \frac{|P|-1}{|P|\left(|P|^{h}-1\right)}\right)^{2}\right. \\
& \left.-\left(2 B_{1}+1\right) \sum_{P} \frac{|P|-1}{|P|\left(|P|^{h}-1\right)}\right]+O\left(\frac{q^{n}}{n^{1-\varepsilon}}\right)
\end{aligned}
$$

Finally, the variance is given by

$$
\operatorname{Var}_{h-\mathrm{free}, n}(\omega)=\log (n)+B_{1}-\zeta(2)-\sum_{P}\left(\frac{|P|^{h-1}-1}{|P|^{h}-1}\right)^{2}-\sum_{P} \frac{|P|-1}{|P|\left(|P|^{h}-1\right)}+O\left(\frac{1}{n^{1-\varepsilon}}\right)
$$

To prove the above results, one works with the Euler product

$$
\begin{aligned}
\mathcal{B}(u, z) & :=\prod_{P} \mathcal{Z}_{q}(u)^{-z}\left(1+z\left(u^{\operatorname{deg}(P)}+\cdots+u^{(h-1) \operatorname{deg}(P)}\right)\right) \\
& =\prod_{P}\left(1-u^{\operatorname{deg}(P)}\right)^{z}\left(1+z \frac{u^{\operatorname{deg}(P)}-u^{h \operatorname{deg}(P)}}{1-u^{\operatorname{deg}(P)}}\right)
\end{aligned}
$$

Moreover, working with $\mathcal{B}\left(u, e^{t}\right)$, we also obtain
Theorem 4.2. As $n \rightarrow \infty, \omega(f)$ with $f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}$ approaches a normal distribution, namely, for $\alpha \leq \beta$,

$$
\frac{1}{\left|\mathcal{S}_{h} \cap \mathcal{M}_{n}\right|}\left|\left\{f \in \mathcal{S}_{h} \cap \mathcal{M}_{n}: \alpha \leq \frac{\omega(f)-\log (n)}{\sqrt{\log (n)}} \leq \beta\right\}\right| \rightarrow \frac{1}{\sqrt{2 \pi}} \int_{\alpha}^{\beta} e^{-t^{2} / 2} d t
$$

## 5. The distribution of $\Omega$ over $h$-Full polynomials

Recall that $h$ is a fixed positive integer and that we denote by $\mathcal{N}_{h}$ the set of $h$-full monic polynomials. Let $f \in \mathcal{N}_{h}$. We proceed to regroup the primes in the factorization of $f$ according to the congruence modulo $h$ of their exponent. More precisely, we write

$$
f=\underbrace{P_{1,0}^{h t_{1,0}} \cdots P_{\ell_{0}, 0}^{h t_{\ell_{0}, 0}}}_{\exp \equiv 0(\bmod h)} \underbrace{P_{1,1}^{h t_{1,1}+1} \cdots P_{\ell_{1}, 1}^{h t_{\ell_{1}, 1}+1}}_{\exp \equiv 1(\bmod h)} \cdots \underbrace{P_{1, h-1}^{h t_{1, h-1}+h-1} \cdots P_{\ell_{h-1}, h-1}^{h t_{\ell_{h-1}, h-1}+h-1}}_{\exp \equiv h-1(\bmod h)}
$$

By setting

$$
\begin{aligned}
m & =P_{1,0}^{t_{1,0}} \cdots P_{\ell_{0}, 0}^{t_{\ell_{0}, 0}} P_{1,1}^{t_{1,1}-1} \cdots P_{\ell_{1}, 1}^{t_{\ell_{1}, 1}-1} \cdots P_{1, h-1}^{t_{1, h-1}-1} \cdots P_{\ell_{h-1}, h-1}^{t_{\ell_{h-1}, h-1}-1} \\
r_{j} & =P_{1, j} \cdots P_{\ell_{j}, j}, \quad(j=1, \ldots, h-1)
\end{aligned}
$$

we arrive at

$$
f=m^{h} r_{1}^{h+1} \cdots r_{h-1}^{2 h-1}
$$

and we remark that the $r_{j}$ are square-free and coprime in pairs.
5.1. The first moment. After the above observation, the first moment over the $h$-full polynomials is given by

$$
\begin{equation*}
\sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}} \Omega(f)=\sum_{m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1} \in \mathcal{M}_{n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) \Omega\left(m^{h} r_{1}^{h+1} \cdots r_{h-1}^{2 h-1}\right) \tag{27}
\end{equation*}
$$

Since $\Omega$ is completely additive and the $r_{j}$ are square-free, we have that

$$
\begin{aligned}
\Omega\left(m^{h} r_{1}^{h+1} \cdots r_{h-1}^{2 h-1}\right) & =h \Omega(m)+(h+1) \Omega\left(r_{1}\right)+\cdots+(2 h-1) \Omega\left(r_{h-1}\right) \\
& =h \Omega(m)+(h+1) \omega\left(r_{1}\right)+\cdots+(2 h-1) \omega\left(r_{h-1}\right) .
\end{aligned}
$$

Using this in (27), we obtain

$$
\begin{align*}
\sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}} \Omega(f)= & \sum_{\substack{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M} \\
\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{h-1}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) \Omega(m) \\
& +(h+1) \sum_{\substack{m, r_{1}, r_{2}, \ldots, r_{h-1} \in \mathcal{M} \\
\operatorname{deg}\left(m^{h} r_{2}^{h+2} \ldots r_{h-1}\right.}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) \omega\left(r_{1}\right) \\
& \vdots \\
& +(2 h-1) \sum_{\substack{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M} \\
\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-2}^{2 h-2}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) \omega\left(r_{h-1}\right) \\
& =h T_{0}(n)+(h+1) T_{1}(n)+\cdots+(2 h-1) T_{h-1}(n) . \tag{28}
\end{align*}
$$

Eventually we will see that $T_{0}(n)$ gives the main term, while the $T_{j}(n)$ give secondary terms.
In order to estimate the $T_{j}(n)$, as in the case of the $h$-free polynomials, we construct a generating function which we will later differentiate using Theorem 2.3.
5.1.1. The sum over $\Omega(m)$. We now consider a generating function for $T_{0}(n)$. Define

$$
\begin{aligned}
\mathcal{T}_{0}(u, z) & =: \sum_{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) z^{\Omega(m)} u^{\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)} \\
& =\prod_{P}\left(1-z u^{\operatorname{deg}(P) h}\right)^{-1}\left(1+u^{\operatorname{deg}(P)(h+1)}+u^{\operatorname{deg}(P)(h+2)}+\cdots+u^{\operatorname{deg}(P)(2 h-1)}\right) .
\end{aligned}
$$

We remark that $\frac{\partial}{\partial z} \mathcal{T}_{0}(u, 1)$ yields a generating series for $T_{0}$.
As usual we extract the singularities at $u^{h}=\frac{1}{q}$ and consider

$$
\begin{align*}
\mathcal{B}_{0}(u, z) & :=\mathcal{Z}_{q}\left(u^{h}\right)^{-z} \mathcal{T}_{0}(u, z) \\
& =\prod_{P} \frac{\left(1-u^{\operatorname{deg}(P) h}\right)^{z}}{1-z u^{\operatorname{deg}(P) h}}\left(1+u^{\operatorname{deg}(P)(h+1)}+u^{\operatorname{deg}(P)(h+2)}+\cdots+u^{\operatorname{deg}(P)(2 h-1)}\right) . \tag{29}
\end{align*}
$$

Our goal is to apply Theorem 2.3 to find the coefficients of $\mathcal{T}_{0}(u, z)$. Before doing that, we must verify that we can, indeed, apply the result.

Lemma 5.1. Set $A<\frac{3}{2}$. For $|z| \leq A, n \geq 2$, and $\sigma>\frac{1}{h+1}$,

$$
\sum_{0 \leq a \leq n} \frac{\left|\mathcal{B}_{0, z}(a)\right|}{q^{\sigma a}} \leq c_{A, \sigma},
$$

where $c_{A, \sigma}$ is a constant depending on $A$ and $\sigma$.
Proof. Our proof follows very similar steps to those in the proof of AP19, Proposition 2.5] and Lemma 3.1. Let $b_{z}(f)$ be a multiplicative function on $\mathcal{M}$ defined for prime powers by

$$
1+\sum_{k \geq 1} b_{z}\left(P^{k}\right) u^{k}=\left(1+u^{h+1}+\cdots+u^{2 h-1}\right) \frac{\left(1-u^{h}\right)^{z}}{1-z u^{h}}
$$

Then $\mathcal{B}_{0, z}(n)=\sum_{f \in \mathcal{M}_{n}} b_{z}(f)$. We see that $b_{z}\left(P^{k}\right)=0$ for $k \leq h$. By Cauchy's formula integrating over $|u|=\left(\frac{2}{3}\right)^{\frac{1}{h}}$,

$$
\left|b_{z}\left(P^{k}\right)\right| \leq\left(\frac{3}{2}\right)^{\frac{k}{h}} M_{A}
$$

for $k \geq h+1$, where

$$
M_{A}:=\sup _{|z| \leq A,|u| \leq\left(\frac{2}{3}\right)^{\frac{1}{h}}}\left|\left(1+u^{h+1}+\cdots+u^{2 h-1}\right) \frac{\left(1-u^{h}\right)^{z}}{1-z u^{h}}\right|
$$ is a constant dependent on $A$. Therefore,

$$
\begin{aligned}
\sum_{0 \leq a \leq n} \frac{\mathcal{B}_{0, z}(a)}{q^{a \sigma}} & \leq \sum_{f \in \mathcal{M}_{\leq n}} \frac{\left|b_{z}(f)\right|}{q^{\sigma \operatorname{deg}(f)}} \\
& \leq \prod_{P \in \mathcal{P}_{\leq \frac{n}{h}}}\left(1+\sum_{k \geq h+1} \frac{\left|b_{z}\left(P^{k}\right)\right|}{q^{\sigma \operatorname{deg}\left(P^{k}\right)}}\right) \\
& \leq \prod_{P \in \mathcal{P}_{\leq \frac{n}{h}}}\left(1+M_{A} \sum_{k \geq h+1}\left(\frac{(3 / 2)^{\frac{1}{h}}}{q^{\sigma \operatorname{deg}(P)}}\right)^{k}\right) \\
& =\prod_{P \in \mathcal{P}_{\leq \frac{n}{h}}}\left(1+\frac{M_{A}(3 / 2)^{\frac{h+1}{h}}}{q^{h \sigma \operatorname{deg}(P)}\left(q^{\sigma \operatorname{deg}(P)}-(3 / 2)^{\frac{1}{h}}\right)}\right) .
\end{aligned}
$$

Taking the logarithm, noticing that $q^{\sigma \operatorname{deg}(P)}-\left(\frac{3}{2}\right)^{\frac{1}{h}} \geq q^{\sigma \operatorname{deg}(P)}\left(1-\frac{(3 / 2)^{\frac{1}{h}}}{q^{\frac{1}{h+1}}}\right)$ for $\operatorname{deg}(P) \geq 1$, and applying the Prime Polynomial Theorem [Ros02, Theorem 2.2], we have

$$
\begin{aligned}
\sum_{P \in \mathcal{P}_{\leq \frac{n}{h}}} \log \left(1+\frac{M_{A}(3 / 2)^{\frac{h+1}{h}}}{q^{h \sigma \operatorname{deg}(P)}\left(q^{\sigma \operatorname{deg}(P)}-(3 / 2)^{\frac{1}{h}}\right)}\right) & \ll M_{A} \sum_{1 \leq d \leq \frac{n}{h}} \frac{q^{d(1-(h+1) \sigma)}}{d} \\
& \leq \frac{M_{A}}{q^{(h+1) \sigma-1}-1}
\end{aligned}
$$

The statement follows by applying the exponential function.
As in the $h$-free case, one can deduce from Lemma 5.1 that Theorem 2.3 can be applied to $\mathcal{B}_{0}$ in a neighborhood of $z=1$ to find $T_{0}$. Differentiating and evaluating $\mathcal{T}_{0}$ at $z=1$, we get

$$
\begin{aligned}
T_{0}(n)= & \sum_{\substack{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M} \\
\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) \Omega(m) \\
= & \left.\frac{\partial}{\partial z}\left(\frac{q^{\frac{n}{h}} n^{z-1}}{h^{z}} \sum_{k=0}^{h-1} \frac{\xi_{h}^{k n}}{\Gamma(z)} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right)\right|_{z=1}+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right) \\
= & \frac{q^{\frac{n}{h}}}{h} \log \left(\frac{n}{h}\right) \sum_{k=0}^{h-1} \xi_{h}^{k n} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, 1\right)+\left.\frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \frac{\partial}{\partial z}\left(\frac{\xi_{h}^{k n}}{\Gamma(z)} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right)\right|_{z=1} \\
& +O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right)
\end{aligned}
$$

where we have applied (14) to compute the error term.
We remark that

$$
\begin{equation*}
\sum_{k=0}^{h-1} \xi_{h}^{k n} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, 1\right)=\sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \tag{30}
\end{equation*}
$$

while
$\begin{aligned}\left.\sum_{k=0}^{h-1} \frac{\partial}{\partial z}\left(\frac{\xi_{h}^{k n}}{\Gamma(z)} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right)\right|_{z=1} & =\sum_{k=0}^{h-1} \xi_{h}^{k n} \frac{\left.\frac{\partial}{\partial z} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right|_{z=1} \Gamma(1)-\mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, 1\right) \Gamma^{\prime}(1)}{\Gamma(1)^{2}} \\ & =\sum_{k=0}^{h-1} \xi_{h}^{k n}\left(\left.\frac{\partial}{\partial z} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right|_{z=1}+\mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, 1\right) \gamma\right) .\end{aligned}$

Now the logarithmic derivative gives

$$
\begin{equation*}
\frac{\left.\frac{\partial}{\partial z} \mathcal{B}_{0}\left(\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{-1}, z\right)\right|_{z=1}}{\mathcal{B}_{0}\left(\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{-1}, 1\right)}=\sum_{P}\left(\log \left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}\right)=B_{2}-\gamma \tag{32}
\end{equation*}
$$

Combining all of the above, we obtain,

$$
\begin{align*}
T_{0}(n)= & \frac{q^{\frac{n}{h}}}{h}\left(\log \left(\frac{n}{h}\right)+B_{2}\right) \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& +O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right) \tag{33}
\end{align*}
$$

5.1.2. The sums over $\omega\left(r_{j}\right)$. We proceed similarly to calculate the $T_{j}(n)$ in (28). To this end, we consider the function

$$
\begin{align*}
\mathcal{T}_{j}(u, z) & =: \sum_{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) z^{\omega\left(r_{j}\right)} u^{\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)} \\
& =\prod_{P}\left(1-u^{\operatorname{deg}(P) h}\right)^{-1}\left(1+u^{\operatorname{deg}(P)(h+1)}+\cdots+z u^{\operatorname{deg}(P)(h+j)}+\cdots+u^{\operatorname{deg}(P)(2 h-1)}\right) \\
& =\mathcal{Z}_{q}\left(u^{h}\right) \mathcal{B}_{j}(u, z) \tag{34}
\end{align*}
$$

and we remark that $\frac{\partial}{\partial z} \mathcal{T}_{j}(u, 1)$ yields the generating series for $T_{j}$. Also remark that $\mathcal{B}_{j}(u, z)$ is absolutely convergent for $|z| \leq A$ (for any $A>0$ ) and $|u|<q^{-\frac{1}{h+1}}$.

By applying Perron's formula (Theorem 2.1), we have

$$
\sum_{\substack{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M} \\ \operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) z^{\omega\left(r_{j}\right)}=\frac{1}{2 \pi i} \oint_{|u|=\delta} \frac{\mathcal{B}_{j}(u, z)}{1-q u^{h}} \frac{d u}{u^{n+1}},
$$

THE NUMBER OF PRIME FACTORS IN $h$-FREE AND $h$-FULL POLYNOMIALS OVER FUNCTION FIELDS 23 where $\delta>0$ small. We move the above integral to the circle $|u|=q^{-\varepsilon-\frac{1}{h+1}}$ and we obtain the residues at the poles $u=\left(q^{\frac{1}{n}} \xi_{h}^{j}\right)^{-1}$. Thus we get

$$
\begin{aligned}
& \sum_{\substack{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M} \\
\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) z^{\omega\left(r_{j}\right)} \\
&=-\sum_{k=0}^{h-1} \operatorname{Res}_{u=\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}} \frac{\mathcal{B}_{j}(u, z)}{u^{n+1}\left(1-q u^{h}\right)}+\frac{1}{2 \pi i} \oint_{|u|=q^{-\varepsilon-\frac{1}{h+1}}} \frac{\mathcal{B}_{j}(u, z)}{1-q u^{h}} \frac{d u}{u^{n+1}} \\
&=\sum_{k=0}^{h-1}\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{n+1} \mathcal{B}_{j}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right) \frac{1}{q^{\frac{1}{h}} \xi_{h}^{k} \prod_{m \neq k}\left(1-\xi_{h}^{m-k}\right)}+O_{z}(1) O\left(q^{\frac{n}{h+1}+\varepsilon n}\right) \\
&= \frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \mathcal{B}_{j}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)+O_{z}(1) O\left(q^{\frac{n}{h+1}+\varepsilon n}\right) .
\end{aligned}
$$

Now we have

$$
\begin{aligned}
T_{j}(n) & =\sum_{\substack{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M} \\
\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) \omega\left(r_{j}\right) \\
& =\left.\frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \frac{\partial}{\partial z} \mathcal{B}_{j}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right|_{z=1}+O\left(q^{\frac{n}{h+1}+\varepsilon n}\right) .
\end{aligned}
$$

Computing the logarithmic derivative,

$$
\frac{\left.\frac{\partial}{\partial z} \mathcal{B}_{j}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right|_{z=1}}{\mathcal{B}_{j}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, 1\right)}=\sum_{P} \frac{\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}} .
$$

In sum, we get

$$
\begin{align*}
T_{j}(n)= & \frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& \times \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}}+O\left(q^{\frac{n}{n+1}+\varepsilon n}\right) . \tag{35}
\end{align*}
$$

5.1.3. Conclusion of the first moment computation. Finally, by combining (33) and (35) with (28), we obtain

$$
\begin{aligned}
& \sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}} \Omega(f) \\
= & q^{\frac{n}{h}}\left(\log \left(\frac{n}{h}\right)+B_{2}\right) \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
+ & \frac{q^{\frac{n}{h}}}{h} \sum_{j=1}^{h-1}(h+j) \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
\times & \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}}+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right) .
\end{aligned}
$$

### 5.2. The second moment. Recall that we have written

$$
f=m^{h} r_{1}^{h+1} \cdots r_{h-1}^{2 h-1}
$$

where the $r_{j}$ are square-free and coprime in pairs. As in the first moment case, we have

$$
\Omega\left(m^{h} r_{1}^{h+1} \cdots r_{h-1}^{2 h-1}\right)^{2}=h^{2} \Omega(m)^{2}+2 \sum_{j=1}^{h-1} h(h+j) \Omega(m) \omega\left(r_{j}\right)+\sum_{\ell=1}^{h-1} \sum_{j=1}^{h-1}(h+j)(h+\ell) \omega\left(r_{j}\right) \omega\left(r_{\ell}\right)
$$

Using this, we obtain

$$
\begin{aligned}
\sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}} \Omega(f)^{2} & =h^{2} \sum_{\substack{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M} \\
\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) \Omega(m)^{2} \\
& +2 \sum_{j=1}^{h-1} h(h+j) \sum_{\substack{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M} \\
\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) \Omega(m) \omega\left(r_{j}\right) \\
& +\sum_{j=1}^{h-1}(h+j)^{2} \sum_{\substack{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M}}}^{\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)=n} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) \omega\left(r_{j}\right)^{2} \\
& +\sum_{1 \leq j, \ell \leq h-1}^{j \neq \ell}(h+j)(h+\ell) \sum_{\substack{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M} \\
\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) \omega\left(r_{j}\right) \omega\left(r_{\ell}\right) \\
& =h^{2} T_{0,0}(n)+2 \sum_{j=1}^{h-1} h(h+j) T_{0, j}(n)+\sum_{j=1}^{h-1}(h+j)^{2} T_{j, j}(n) \\
& +\sum_{\substack{1 \leq j, \ell \leq h-1 \\
j \neq \ell}}(h+j)(h+\ell) T_{j, \ell}(n) .
\end{aligned}
$$

In this section we show how to obtain the formula for the second moment. We will see that the main term comes from $T_{0,0}(n)$.
5.2.1. The sum over $\Omega^{2}(m)$. For the sum over $\Omega^{2}(m)$, we apply Theorem 2.3 to $\mathcal{T}_{0}$ and $\mathcal{B}_{0}$ and obtain the following.

$$
\begin{align*}
T_{0,0}(n)= & \sum_{\substack{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M} \\
\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{h-1}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) \Omega(m)^{2} \\
= & \left.\frac{\partial}{\partial z}\left(z \frac{\partial}{\partial z}\left(\frac{q^{\frac{n}{h}} n^{z-1}}{h^{z}} \sum_{k=0}^{h-1} \frac{\xi_{h}^{k n}}{\Gamma(z)} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right)\right)\right|_{z=1}+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right) \\
= & \frac{q^{\frac{n}{h}}}{h}\left(\log \left(\frac{n}{h}\right)^{2}+\log \left(\frac{n}{h}\right)\right) \sum_{k=0}^{h-1} \xi_{h}^{k n} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, 1\right) \\
& +\left.\frac{q^{\frac{n}{h}}}{h}\left(2 \log \left(\frac{n}{h}\right)+1\right) \sum_{k=0}^{h-1} \frac{\partial}{\partial z}\left(\frac{\xi_{h}^{k n}}{\Gamma(z)} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right)\right|_{z=1} \\
& +\left.\frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \frac{\partial^{2}}{\partial z^{2}}\left(\frac{\xi_{h}^{k n}}{\Gamma(z)} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right)\right|_{z=1}+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right), \tag{37}
\end{align*}
$$

where we have applied (15) to compute the error term.
We now proceed to compute the second derivative of $\mathcal{B}_{0}$.

$$
\begin{aligned}
& \left.\sum_{k=0}^{h-1} \frac{\partial^{2}}{\partial z^{2}}\left(\frac{\xi_{h}^{k n}}{\Gamma(z)} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right)\right|_{z=1} \\
= & \sum_{k=0}^{h-1} \xi_{h}^{k n}\left(\frac{\left.\frac{\partial^{2}}{\partial z^{2}} \mathcal{B}_{0}\left(\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{-1}, z\right)\right|_{z=1} \Gamma(1)-\left.2 \frac{\partial}{\partial z} \mathcal{B}_{0}\left(\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{-1}, z\right)\right|_{z=1} \Gamma^{\prime}(1)}{\Gamma(1)^{2}}\right. \\
& \left.+\frac{\mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, 1\right)\left(2 \Gamma^{\prime}(1)^{2}-\Gamma(1) \Gamma^{\prime \prime}(1)\right)}{\Gamma(1)^{3}}\right) \\
= & \sum_{k=0}^{h-1} \xi_{h}^{k n}\left(\left.\frac{\partial^{2}}{\partial z^{2}} \mathcal{B}_{0}\left(\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{-1}, z\right)\right|_{z=1}+\left.2 \gamma \frac{\partial}{\partial z} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right|_{z=1}\right. \\
& +\mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, 1\right)\left(\gamma^{2}-\zeta(2)\right),
\end{aligned}
$$

where we have used the fact that $\Gamma^{\prime \prime}(1)=\gamma^{2}+\zeta(2)$. A computation via the logarithmic derivative finally yields

$$
\begin{aligned}
\frac{\left.\frac{\partial^{2}}{\partial z^{2}} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right|_{z=1}}{\left.\frac{\partial}{\partial z} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right|_{z=1}} & =\frac{\left(\sum_{P}\left(\log \left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}\right)\right)^{2}+\sum_{P} \frac{1}{(|P|-1)^{2}}}{\sum_{P}\left(\log \left(1-\frac{1}{|P|}\right)+\frac{1}{|P|-1}\right)} \\
& =\frac{\left(B_{2}-\gamma\right)^{2}+\sum_{P} \frac{1}{(|P|-1)^{2}}}{B_{2}-\gamma} .
\end{aligned}
$$

Finally, the result follows from applying the above as well as (30), (31), and (32) to (37):

$$
\begin{aligned}
T_{0,0}(n)= & \frac{q^{\frac{n}{h}}}{h}\left(\log \left(\frac{n}{h}\right)^{2}+\left(2 B_{2}+1\right) \log \left(\frac{n}{h}\right)+B_{2}^{2}+B_{2}-\zeta(2)+\sum_{P} \frac{1}{(|P|-1)^{2}}\right) \\
& \times \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& +O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right) .
\end{aligned}
$$

5.2.2. The sums over $\Omega(m) \omega\left(r_{j}\right)$. We proceed to compute these terms by considering the generating function

$$
\begin{aligned}
\mathcal{T}_{0, j}(u, w, z)= & : \sum_{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) w^{\omega\left(r_{j}\right)} z^{\Omega(m)} u^{\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)} \\
& =\prod_{P}\left(1-z u^{\operatorname{deg}(P) h}\right)^{-1}\left(1+u^{\operatorname{deg}(P)(h+1)}+\cdots+w u^{\operatorname{deg}(P)(h+j)}+\cdots+u^{\operatorname{deg}(P)(2 h-1)}\right)
\end{aligned}
$$

We remark that $\frac{\partial^{2}}{\partial w \partial z} \mathcal{T}_{0, j}(u, 1,1)$ yields a generating series for the $T_{0, j}(n)$.
Consider

$$
\begin{aligned}
\mathcal{B}_{0, j}(u, w, z): & =\mathcal{Z}_{q}\left(u^{h}\right)^{-z} \mathcal{T}_{0, j}(u, w, z) \\
& =\prod_{P} \frac{\left(1-u^{\operatorname{deg}(P) h}\right)^{z}}{1-z u^{\operatorname{deg}(P) h}}\left(1+u^{\operatorname{deg}(P)(h+1)}+\cdots+w u^{\operatorname{deg}(P)(h+j)}+\cdots+u^{\operatorname{deg}(P)(2 h-1)}\right)
\end{aligned}
$$

Fix $w$ in a neighborhood of 1 , Theorem 2.3 applies to $\mathcal{T}_{0, j}(u, w, z)$. Indeed, we have the following result, whose proof is almost identical to the proof of Lemma 5.1, and thus we will not repeat here.

Lemma 5.2. Set $A<\frac{3}{2}$. For $|z| \leq A,|w-1| \leq \delta, n \geq 2$, and $\sigma>\frac{1}{h+1}$,

$$
\sum_{0 \leq a \leq n} \frac{\left|\mathcal{B}_{0, j, w, z}(a)\right|}{q^{\sigma a}} \leq c_{A, \sigma, \delta}
$$

where $c_{A, \sigma, \delta}$ is a constant depending on $A, \sigma$ and $\delta$.
By applying Theorem 2.3, we obtain

$$
\mathcal{T}_{0, j, z}(n, w)=\frac{q^{\frac{n}{n}} n^{z-1}}{h^{z} \Gamma(z)} \sum_{k=0}^{h-1} \xi_{h}^{k n} \mathcal{B}_{0, j}\left(\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{-1}, w, z\right)+O_{w}(1) O_{A}\left(q^{\frac{n}{n}} n^{\operatorname{Re}(z)-2}\right),
$$

where the error term is obtained by considering the factors involving $w$ separated from the factors involving $z$.

By a similar calculation to the one presented in the $T_{0}(n)$ term of the first moment yields

$$
\begin{aligned}
T_{0, j}(n)= & \sum_{\substack{m_{, r_{1}, \ldots, r_{h-1} \in \mathcal{M}} \\
\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{h-1}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) \omega\left(r_{j}\right) \Omega(m) \\
& =\left.\frac{\partial^{2}}{\partial w \partial z}\left(\frac{q^{\frac{n}{h}} n^{z-1}}{h^{z} \Gamma(z)} \sum_{k=0}^{h-1} \xi_{h}^{k n} \mathcal{B}_{0, j}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, w, z\right)\right)\right|_{w=z=1}+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right) \\
& =\left.\frac{q^{\frac{n}{h}}}{h} \log \left(\frac{n}{h}\right) \sum_{k=0}^{h-1} \xi_{h}^{k n} \frac{\partial}{\partial w} \mathcal{B}_{0, j}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, w, 1\right)\right|_{w=1} \\
& +\left.\frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \frac{\partial^{2}}{\partial z \partial w} \mathcal{B}_{0, j}\left(\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{-1}, w, z\right)\right|_{w=z=1} \Gamma(1)-\left.\frac{\partial}{\partial w} \mathcal{B}_{0, j}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, w, 1\right)\right|_{w=1} \Gamma^{\prime}(1) \\
& \Gamma(1)^{2} \\
& +O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right) .
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\left.\frac{\partial}{\partial w} \mathcal{B}_{0, j}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, w, z\right)\right|_{w=1}= & \prod_{P} \frac{\left(1-\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P) h}\right)^{z}}{1-z\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P) h}} \\
& \times\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& \times \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}}
\end{aligned}
$$

and, by a similar computation to (32),

$$
\begin{aligned}
\left.\frac{\partial^{2}}{\partial z \partial w} \mathcal{B}_{0, j}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, w, z\right)\right|_{w=z=1}= & \left(B_{2}-\gamma\right) \\
& \times \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& \times \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}} .
\end{aligned}
$$

Finally we get

$$
\begin{aligned}
T_{0, j}(n) & =\frac{q^{\frac{n}{h}}}{h}\left(\log \left(\frac{n}{h}\right)+B_{2}\right) \\
& \times \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& \times \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}} \\
& +O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right) .
\end{aligned}
$$

5.2.3. The sums over $\omega\left(r_{j}\right) \omega\left(r_{\ell}\right)$ for $j \neq \ell$. We proceed to calculate these terms by considering the generating function

$$
\begin{aligned}
& \mathcal{T}_{j, l}(u, z, w) \\
& :=\sum_{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) w^{\omega\left(r_{j}\right)} z^{\omega\left(r_{\ell}\right)} u^{\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)} \\
& =\prod_{P}\left[\left(1-u^{h \operatorname{deg}(P)}\right)^{-1}\right. \\
& \left.\times\left(1+u^{\operatorname{deg}(P)(h+1)}+\cdots+w u^{\operatorname{deg}(P)(h+j)}+\cdots+z u^{\operatorname{deg}(P)(h+\ell)}+\cdots+u^{\operatorname{deg}(P)(2 h-1)}\right)\right] \\
& =\mathcal{Z}_{q}\left(u^{h}\right) \mathcal{B}_{j, \ell}(u, w, z)
\end{aligned}
$$

Remark that $\mathcal{B}_{j, \ell}(u, w, z)$ is absolutely convergent for $|w|,|z| \leq A$ (for any $A>0$ ) and $|u|<q^{-\frac{1}{h+1}}$.
By Perron's formula (Theorem 2.1), we have

$$
\sum_{\substack{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M} \\ \operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) w^{\omega\left(r_{j}\right)} z^{\omega\left(r_{\ell}\right)}=\frac{1}{2 \pi i} \oint_{|u|=\delta} \frac{\mathcal{B}_{j, \ell}(u, w, z)}{1-q u^{h}} \frac{d u}{u^{n+1}}
$$

with $\delta>0$ small. We move the above integral to the circle $|u|=q^{-\varepsilon-\frac{1}{h+1}}$ and we obtain the residues at the poles $u=\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{-1}$.

Thus we get

$$
\begin{aligned}
T_{j, \ell}(n) & =\sum_{\substack{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M} \\
\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) w^{\omega\left(r_{j}\right)} z^{\omega\left(r_{\ell}\right)} \\
& =-\sum_{k=0}^{h-1} \operatorname{Res}_{u=\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}} \frac{\mathcal{B}_{j, \ell}(u, w, z)}{u^{n+1}\left(1-q u^{h}\right)}+\frac{1}{2 \pi i} \oint_{|u|=q^{-\varepsilon-\frac{1}{h+1}}} \frac{\mathcal{B}_{j, \ell}(u, w, z)}{1-q u^{h}} \frac{d u}{u^{n+1}} \\
& =\sum_{k=0}^{h-1}\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{n+1} \mathcal{B}_{j, \ell}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, w, z\right) \frac{1}{q^{\frac{1}{h}} \xi_{h}^{k} \prod_{m \neq k}\left(1-\xi_{h}^{m-k}\right)}+O_{w, z}(1) O\left(q^{\frac{n}{h+1}+\varepsilon n}\right) \\
& =\frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \mathcal{B}_{j, \ell}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, w, z\right)+O_{w, z}(1) O\left(q^{\frac{n}{h+1}+\varepsilon n}\right)
\end{aligned}
$$

Now we have

$$
\begin{aligned}
T_{j, \ell}(n) & =\sum_{\substack{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M} \\
\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) \omega\left(r_{j}\right) \omega\left(r_{\ell}\right) \\
& =\left.\frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \frac{\partial^{2}}{\partial w \partial z} \mathcal{B}_{\ell}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, w, z\right)\right|_{w=z=1}+O\left(q^{\frac{n}{h+1}+\varepsilon n}\right) .
\end{aligned}
$$

Using the usual techniques with the logarithmic derivative, we arrive at

$$
\begin{aligned}
\left.\frac{\partial^{2}}{\partial z \partial w} \mathcal{B}_{j, \ell}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, w, z\right)\right|_{w=z=1} & =\prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& \times\left(\sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)-\operatorname{deg}(P)(2 h-1)}\right. \\
& \times \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+\ell)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}} \\
& \left.-\sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+\ell)}}{\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1))^{2}}\right)}\right)
\end{aligned}
$$

In sum, we get

$$
\begin{aligned}
T_{j, \ell}(n)= & \frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& \times\left(\sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}}\right. \\
& \times \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+\ell)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}} \\
& \left.-\sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+\ell)}}{\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)-\operatorname{deg}(P)(2 h-1)^{2}\right.}\right)+O\left(q^{\frac{n}{h+1}+\varepsilon n}\right) .
\end{aligned}
$$

5.2.4. The sums over $\omega^{2}\left(r_{j}\right)$. We remark that the sum over $\omega^{2}\left(r_{j}\right)$ is given by $\frac{\partial^{2}}{\partial z^{2}} \mathcal{T}_{j}(u, 1)$, where $\mathcal{T}_{j}(u, z)$ was defined in (34). Indeed, we have

$$
\begin{aligned}
T_{j, j}(n) & =\sum_{\substack{m, r_{1}, \ldots, r_{h}, 1 \in \mathcal{M} \\
\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)=n}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) \omega^{2}\left(r_{j}\right) \\
& =\left.\frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \frac{\partial}{\partial z}\left(z \frac{\partial}{\partial z} \mathcal{B}_{j}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right)\right|_{z=1}+O\left(q^{\frac{n}{h+1}+\varepsilon n}\right) \\
& =\left.\frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \frac{\partial^{2}}{\partial z^{2}} \mathcal{B}_{j}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right|_{z=1}+\left.\frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \frac{\partial}{\partial z} \mathcal{B}_{j}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)\right|_{z=1}+O\left(q^{\frac{n}{h+1}+\varepsilon n}\right) .
\end{aligned}
$$

By following a similar computation as in the other cases, we get

$$
\begin{aligned}
T_{j, j}(n)= & \frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& \times\left(\sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}}\right. \\
& +\left(\sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}}\right)^{2} \\
& \left.-\sum_{P}\left(\frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}}\right)^{2}\right)+O\left(q^{\frac{n}{h+1}+\varepsilon n}\right)
\end{aligned}
$$

5.2.5. Conclusion of the second moment computation. By combining the results from the previous subsections into (36), we finally obtain

$$
\begin{aligned}
& \frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& \times\left[h^{2}\left(\log \left(\frac{n}{h}\right)^{2}+\left(2 B_{2}+1\right) \log \left(\frac{n}{h}\right)+B_{2}^{2}+B_{2}-\zeta(2)+\sum_{P} \frac{1}{(|P|-1)^{2}}\right)\right. \\
& +\sum_{j=1}^{h-1}(h+j)\left(2 h\left(\log \left(\frac{n}{h}\right)+B_{2}\right)+(h+j)\right. \\
& \left.+\sum_{\ell=1}^{h-1}(h+\ell) \sum_{P} \frac{\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+\ell)}}{1+\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}}\right) \\
& \times \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}} \\
& \left.-\sum_{1 \leq j, \ell \leq h-1}(h+j)(h+\ell) \sum_{P} \frac{\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+\ell)}}{\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right)^{2}}\right]+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right)
\end{aligned}
$$

We can compute the variance from Theorem 1.4. Before proceeding, we need the following auxiliary result.

Lemma 5.3. We have

$$
\sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}} 1=\frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right)+O\left(q^{\frac{n}{h+1}+\varepsilon n}\right)
$$

$$
\begin{aligned}
\sum_{f \in \mathcal{N}_{h}} u^{\operatorname{deg}(f)} & =\sum_{m, r_{1}, \ldots, r_{h-1} \in \mathcal{M}} \mu^{2}\left(r_{1} \cdots r_{h-1}\right) u^{\operatorname{deg}\left(m^{h} r_{1}^{h+1} \ldots r_{h-1}^{2 h-1}\right)} \\
& =\prod_{P}\left(1-u^{\operatorname{deg}(P) h}\right)^{-1}\left(1+u^{\operatorname{deg}(P)(h+1)}+\cdots+u^{\operatorname{deg}(P)(2 h-1)}\right) \\
& =\mathcal{Z}_{q}\left(u^{h}\right) \mathcal{B}_{0}(u, 1)
\end{aligned}
$$

where we recall that $\mathcal{B}_{0}(u, 1)$ converges absolutely for $|u|<q^{-\frac{1}{h+1}}$.
We apply Perron's formula (Lemma 2.1), first integrating over $\delta>0$ small, and then moving the integral to the circle $|u|=q^{-\varepsilon-\frac{1}{h+1}}$ and collecting the residues at $u=\left(q^{\frac{1}{h}} \xi_{h} j\right)^{-1}$. This gives

$$
\begin{aligned}
\sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}} 1 & =\frac{1}{2 \pi i} \oint_{|u|=\delta} \frac{\mathcal{B}_{0}(u, 1)}{1-q u^{h}} \frac{d u}{u^{n+1}} \\
& =-\sum_{k=0}^{h-1} \operatorname{Res}_{u=\left(q^{\frac{1}{k}} \xi_{h}^{k}\right)^{-1}} \frac{\mathcal{B}_{0}(u, 1)}{u^{n+1}\left(1-q u^{h}\right)}+\frac{1}{2 \pi i} \oint_{|u|=q^{-\varepsilon-\frac{1}{h+1}}} \frac{\mathcal{B}_{0}(u, 1)}{1-q u^{h}} \frac{d u}{u^{n+1}} \\
& =\sum_{k=0}^{h-1}\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{n+1} \mathcal{B}_{0}\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, 1\right) \frac{1}{q^{\frac{1}{h}} \xi_{h}^{k} \prod_{m \neq k}\left(1-\xi_{h}^{m-k}\right)}+O\left(q^{\frac{n}{h+1}+\varepsilon n}\right) \\
& =\frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right)+O\left(q^{\frac{n}{h+1}+\varepsilon n}\right) .
\end{aligned}
$$

Proof of Theorem 1.5. By combining the results of Theorem 1.4 and Lemma 5.3, we have

$$
\begin{aligned}
& \mathbb{E}_{h, n}\left(\Omega^{2}\right)-\left(\mathbb{E}_{h, n}(\Omega)\right)^{2} \\
= & {\left[h^{2}\left(\log \left(\frac{n}{h}\right)+B_{2}-\zeta(2)+\sum_{P} \frac{1}{(|P|-1)^{2}}\right)\right.} \\
& +\left(\sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right)\right)^{-1} \\
& \times \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& \times\left[\sum_{j=1}^{h-1}(h+j)\left((h+j)+\sum_{\ell=1}^{h-1}(h+\ell) \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{\left.1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right)}\right)\right. \\
& \times \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+\ell)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}} \\
& -\sum_{1 \leq j, \ell \leq h-1}(h+j)\left(h+\ell \sum_{P} \frac{\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+\ell)}}{\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right)^{2}}\right] \\
& -\left[\left(\sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right)\right)^{-1}\right. \\
& \times \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}\right) \\
& \left.\times \sum_{j=1}^{h-1}(h+j) \sum_{P} \frac{\left(q^{\frac{1}{n}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+j)}}{1+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(h+1)}+\cdots+\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-\operatorname{deg}(P)(2 h-1)}}\right]^{2}+O\left(\frac{1}{n^{1-\varepsilon}}\right) .
\end{aligned}
$$

5.3. A discussion on the proof strategy for Theorem 1.4. The starting point in our proof of Theorem 1.4 was to consider the decomposition

$$
\begin{equation*}
f=m^{h} r_{1}^{h+1} \cdots r_{h-1}^{2 h-1} \tag{38}
\end{equation*}
$$

Then we used crucially that $\Omega$ is completely multiplicative. Another possible strategy would have been to start from the generating series

$$
\mathcal{C}(u, z):=\prod_{P}\left(1+z^{h} u^{h \operatorname{deg}(P)}+z^{h+1} u^{(h+1) \operatorname{deg}(P)}+\cdots\right)=\sum_{f \in \mathcal{N}_{h}} z^{\Omega(f)} u^{\operatorname{deg}(f)}
$$

and to apply an extension of Theorem 2.3 to

$$
\begin{align*}
\mathcal{B}(u, z) & :=\mathcal{Z}_{q}\left(u^{h}\right)^{-z^{h}} \mathcal{C}(u, z) \\
& =\prod_{P}\left(1-u^{h \operatorname{deg}(P)}\right)^{z^{h}}\left(1+\frac{z^{h} u^{h \operatorname{deg}(P)}}{1-z u^{\operatorname{deg}(P)}}\right) . \tag{39}
\end{align*}
$$

In fact, the extension of Theorem 2.3 that we need is the following.
Theorem 5.1 (Generalized Theorem 2.2). Let $C(u, z)=\sum_{n \geq 0} C_{z}(n) u^{n}$ and $B(u, z)=\sum_{n \geq 0} B_{z}(n) u^{n}$ be power series with coefficients depending on $z$ satisfying $C(u, z)=B(u, z) \mathcal{Z}_{q}\left(u^{h}\right)^{z^{h}}$. Suppose also that, uniformly for $\left|z^{h}\right| \leq A$,

$$
\sum_{n \geq 0} \frac{\left|B_{z}(n)\right|}{q^{\frac{n}{h}}} n^{2 A+2} \ll A_{A} 1
$$

Then, uniformly for $\left|z^{h}\right| \leq A$ and $n \geq 1$, we have

$$
C_{z}(n)=\frac{q^{\frac{n}{h}} n^{z^{h}-1}}{h^{z^{h}} \Gamma\left(z^{h}\right)} \sum_{k=0}^{h-1} \xi_{h}^{k n} B\left(\left(q^{\frac{1}{h}} \xi_{h}^{k}\right)^{-1}, z\right)+O_{A}\left(q^{\frac{n}{h}} n^{\mathrm{Re}\left(z^{h}\right)-2}\right) .
$$

This method leads to equivalent results. We have opted to follow the decomposition (38) to prove Theorem 1.4 because the proof is more clear, it shows the origin of the main term, and it leads to cleaner and more symmetric formulas for the first and second moment and the variance.

A crucial point that makes working with (39) particularly cumbersome has to do with the mixture of $z$ and $z^{h}$, and their different behavior under differentiation. This does not make the analysis of the Erdős-Kac theorem more complicated, and therefore we will use Theorem 5.1 in Section 5.4 for that.

It should be remarked however, that a proof that depends on the decomposition (38) can not be applied to $\omega$. Nevertheless, in the case of $\omega$, the direct generating series is simpler and does not involve the mixture of $z$ and $z^{h}$. We state the results obtained in this way for $\omega$ in Section 6 .
5.4. Higher moments of $\Omega$ and an Erdős-Kac result for $h$-full polynomials. The goal of this section is to prove Theorem 1.6 , which is the Erdős-Kac theorem for $\Omega$ over the $h$-full polynomials. To this end we will prove that

$$
\frac{1}{\left|\mathcal{N}_{h} \cap \mathcal{M}_{n}\right|} \sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}}\left(\frac{\Omega(f)-h \log \left(\frac{n}{h}\right)}{h \sqrt{\log \left(\frac{n}{h}\right)}}\right)^{v} \rightarrow C_{v}
$$

where the $C_{v}$ 's are given by (21).
As before, we will consider

$$
\frac{1}{\left|\mathcal{N}_{h} \cap \mathcal{M}_{n}\right|} \sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}}\left(\frac{\Omega(f)-h \log \left(\frac{n}{h}\right)}{h \sqrt{\log \left(\frac{n}{h}\right)}}\right)^{v}=\frac{1}{\left(h^{2} \log \left(\frac{n}{h}\right)\right)^{\frac{v}{2}}} \sum_{\ell=0}^{v}\binom{v}{\ell} \mathbb{E}\left(\Omega^{\ell}\right)(-1)^{v-\ell}\left(h \log \left(\frac{n}{h}\right)\right)^{v-\ell}
$$

By substituting $z=e^{t}$ into $\mathcal{B}(u, z)$ given by (39), we have

$$
\begin{aligned}
\mathcal{B}\left(u, e^{t}\right): & =\mathcal{Z}_{q}\left(u^{h}\right)^{-e^{h t}} \mathcal{C}\left(u, e^{t}\right) \\
& =\prod_{P}\left(1-u^{\operatorname{deg}(P) h}\right)^{e^{h t}}\left(1+\frac{e^{h t} u^{h \operatorname{deg}(P)}}{1-e^{t} u^{\operatorname{deg}(P)}}\right) .
\end{aligned}
$$

Theorem 5.1 applies to $\mathcal{B}\left(u, e^{t}\right)$ at $t=0$, and therefore,

$$
\sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}} e^{\Omega(f) t}=\frac{q^{\frac{n}{\hbar}} n^{e^{h t}-1}}{h^{h t}} \sum_{s=0}^{h-1} \frac{\xi_{h}^{s n}}{\Gamma\left(e^{h t}\right)} \mathcal{B}\left(\left(q^{\frac{1}{n}} \xi_{h}^{s}\right)^{-1}, e^{t}\right)+O_{A, \delta}\left(q^{\frac{n}{h}} n^{\operatorname{Re}\left(e^{t}\right)-2}\right)
$$

By the property of the moment generating function (22) combined with (24), we have

$$
\begin{aligned}
\mathbb{E}\left(\Omega^{\ell}\right)= & \left.\left(\sum_{s=0}^{h-1} \xi_{h}^{s n} \mathcal{B}\left(\left(q^{\frac{1}{h}} \xi_{h}^{s}\right)^{-1}, 1\right)\right)^{-1} \sum_{j=0}^{\ell}\binom{\ell}{j} h^{j} T_{j}\left(\log \left(\frac{n}{h}\right)\right) \sum_{s=0}^{h-1} \xi_{h}^{s n}\left(\frac{\mathcal{B}\left(\left(q^{\frac{1}{h}} \xi_{h}^{s}\right)^{-1}, e^{t}\right)}{\Gamma\left(e^{h t}\right)}\right)^{(\ell-j)}\right|_{t=0} \\
& +O\left(\frac{1}{n^{1-\varepsilon}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{\left(\sum_{s=0}^{h-1} \xi_{h}^{s n} \mathcal{B}\left(\left(q^{\frac{1}{h}} \xi_{h}^{s}\right)^{-1}, 1\right)\right)^{-1}}{h^{v} \log \left(\frac{n}{h}\right)^{\frac{v}{2}}} \sum_{\ell=0}^{v}\binom{v}{\ell} \mathbb{E}\left(\Omega^{\ell}\right)(-1)^{v-\ell}\left(h \log \left(\frac{n}{h}\right)\right)^{v-\ell} \\
& =\left.\frac{\left(\sum_{s=0}^{h-1} \xi_{h}^{s n} \mathcal{B}\left(\left(q^{\frac{1}{n}} \xi_{h}^{s}\right)^{-1}, 1\right)\right)^{-1}}{h^{v} \log \left(\frac{n}{h}\right)^{\frac{v}{2}}} \sum_{\ell=0}^{v}\binom{v}{\ell} \sum_{j=0}^{\ell}\binom{\ell}{j} h^{j} T_{j}\left(\log \left(\frac{n}{h}\right)\right) \sum_{s=0}^{h-1} \xi_{h}^{s n}\left(\frac{\mathcal{B}\left(\left(q^{\frac{1}{h}} \xi_{h}^{s}\right)^{-1}, e^{t}\right)}{\Gamma\left(e^{h t}\right)}\right)^{(\ell-j)}\right|_{t=0}
\end{aligned}
$$

$$
\begin{equation*}
\times(-1)^{v-\ell}\left(h \log \left(\frac{n}{h}\right)\right)^{v-\ell}+O\left(\frac{1}{n^{1-\varepsilon}}\right) . \tag{40}
\end{equation*}
$$

Consider the change of variables $u=v-\ell, k=v-\ell+j$. Then the main term in (40) becomes

$$
\begin{aligned}
& \left.\frac{\left(\sum_{s=0}^{h-1} \xi_{h}^{s n} \mathcal{B}\left(\left(q^{\frac{1}{h}} \xi_{h}^{s}\right)^{-1}, 1\right)\right)^{-1}}{h^{v} \log \left(\frac{n}{h}\right)^{\frac{v}{2}}} \sum_{k=0}^{v} \sum_{s=0}^{h-1} \xi_{h}^{s n}\left(\frac{\mathcal{B}\left(\left(q^{\frac{1}{h}} \xi_{h}^{s}\right)^{-1}, e^{t}\right)}{\Gamma\left(e^{h t}\right)}\right)^{(v-k)}\right|_{t=0} \\
& \times \sum_{u=0}^{k}\binom{v}{u}\binom{v-u}{k-u} h^{k-u} T_{k-u}\left(\log \left(\frac{n}{h}\right)\right)(-1)^{u}\left(h \log \left(\frac{n}{h}\right)\right)^{u} \\
& =\left.\frac{\left(\sum_{s=0}^{h-1} \xi_{h}^{s n} \mathcal{B}\left(\left(q^{\frac{1}{h}} \xi_{h}^{s}\right)^{-1}, 1\right)\right)^{-1}}{h^{v} \log \left(\frac{n}{h}\right)^{\frac{v}{2}}} \sum_{k=0}^{v}\binom{v}{k} \sum_{s=0}^{h-1} \xi_{h}^{s n}\left(\frac{\mathcal{B}\left(\left(q^{\frac{1}{h}} \xi_{h}^{s}\right)^{-1}, e^{t}\right)}{\Gamma\left(e^{h t}\right)}\right)^{(v-k)}\right|_{t=0} \\
& \times \sum_{u=0}^{k}\binom{k}{u} h^{k-u} T_{k-u}\left(\log \left(\frac{n}{h}\right)\right)(-1)^{u}\left(h \log \left(\frac{n}{h}\right)\right)^{u}
\end{aligned}
$$

Similarly to the $h$-free case we get, for $v$ even, that the main term should come from setting $k=v$, which leads to

$$
\begin{aligned}
& \frac{1}{h^{v} \log \left(\frac{n}{h}\right)^{\frac{v}{2}}} \sum_{\ell=0}^{v}\binom{v}{\ell} \mathbb{E}\left(\Omega(f)^{\ell}\right)(-1)^{v-\ell}\left(h \log \left(\frac{n}{h}\right)\right)^{v-\ell} \\
& =\frac{\left(\sum_{s=0}^{h-1} \xi_{h}^{s n} \mathcal{B}\left(\left(q^{\frac{1}{h}} \xi_{h}^{s}\right)^{-1}, 1\right)\right)^{-1}}{h^{v} \log \left(\frac{n}{h}\right)^{\frac{v}{2}}} \sum_{s=0}^{h-1} \xi_{h}^{s n}\left(\frac{\mathcal{B}\left(\left(q^{\frac{1}{h}} \xi_{h}^{s}\right)^{-1}, 1\right)}{\Gamma(1)}\right) \\
& \times \sum_{u=0}^{v}\binom{v}{u} h^{v-u} T_{v-u}\left(\log \left(\frac{n}{h}\right)\right)(-1)^{u}\left(h \log \left(\frac{n}{h}\right)\right)^{u}+O\left(\frac{1}{\log n}\right) \\
& =\frac{v!}{2^{\frac{v}{2}}\left(\frac{v}{2}\right)!}+O\left(\frac{1}{\log n}\right)
\end{aligned}
$$

while for $v$ odd we get

$$
O\left(\frac{1}{\sqrt{\log n}}\right)
$$

## 6. The distribution of $\omega$ OVER $h$-full polynomials

In this short section we state, without proof, analogous results for $\omega$. The proofs are quite direct from the corresponding generating function.
Theorem 6.1. For any $\varepsilon>0$ and as $n \rightarrow \infty$, the first moment of $\omega$ over the $h$-full polynomials of degree $n$ is given by

$$
\begin{aligned}
\sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}} \omega(f)= & \frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|-|P|^{1-\frac{1}{h}} \xi_{h}^{-\operatorname{deg}(P) k}}\right) \\
& \times\left(\log \left(\frac{n}{h}\right)+B_{1}+\sum_{P}\left(\frac{1}{|P|-|P|^{1-\frac{1}{h}} \xi_{h}^{-\operatorname{deg}(P) k}+1}-\frac{1}{|P|}\right)\right)+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right)
\end{aligned}
$$

while the second moment is given by

$$
\begin{aligned}
\sum_{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}} \omega(f)^{2}= & \frac{q^{\frac{n}{h}}}{h} \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|-|P|^{1-\frac{1}{h}} \xi_{h}^{-\operatorname{deg}(P) k}}\right) \\
& \times\left[\log \left(\frac{n}{h}\right)^{2}+\log \left(\frac{n}{h}\right)\left[2 B_{1}+1+2 \sum_{P}\left(\frac{1}{|P|-|P|^{1-\frac{1}{h}} \xi_{h}^{-\operatorname{deg}(P) k}+1}-\frac{1}{|P|}\right)\right]\right. \\
& +B_{1}^{2}+B_{1}-\zeta(2)-\sum_{P} \frac{1}{\left(|P|-|P|^{1-\frac{1}{h}} \xi_{h}^{-\operatorname{deg}(P) k}+1\right)^{2}} \\
& +\left(\sum _ { P } \left(\frac{1}{\left.\left.|P|-|P|^{1-\frac{1}{h} \xi_{h}^{-\operatorname{deg}(P) k}+1}-\frac{1}{|P|}\right)\right)^{2}}\right.\right. \\
& \left.+\left(2 B_{1}+1\right) \sum_{P}\left(\frac{1}{|P|-|P|^{1-\frac{1}{h}} \xi_{h}^{-\operatorname{deg}(P) k}+1}-\frac{1}{|P|}\right)\right]+O\left(\frac{q^{\frac{n}{h}}}{n^{1-\varepsilon}}\right) .
\end{aligned}
$$

Finally, the variance is given by

$$
\begin{aligned}
& \operatorname{Var}_{h-\mathrm{full}, n}(\omega)=\log \left(\frac{n}{h}\right)+B_{1}-\zeta(2)+\left(\sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|-|P|^{1-\frac{1}{h}} \xi_{h}^{-\operatorname{deg}(P) k}}\right)\right)^{-1} \\
& \times \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|-|P|^{1-\frac{1}{h}} \xi_{h}^{-\operatorname{deg}(P) k}}\right)\left[-\sum_{P} \frac{1}{\left(|P|-|P|^{1-\frac{1}{h}} \xi_{h}^{-\operatorname{deg}(P) k}+1\right)^{2}}\right. \\
& +\left(\sum_{P}\left(\frac{1}{|P|-|P|^{1-\frac{1}{h}} \xi_{h}^{-\operatorname{deg}(P) k}+1}-\frac{1}{|P|}\right)\right)^{2}+\sum_{P}\left(\frac{1}{\left.\left.|P|-|P|^{1-\frac{1}{h} \xi_{h}^{-\operatorname{deg}(P) k}+1}-\frac{1}{|P|}\right)\right]}\right. \\
& -\left[\left(\sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|-|P|^{1-\frac{1}{h}} \xi_{h}^{-\operatorname{deg}(P) k}}\right)\right)^{-1}\right. \\
& \times \sum_{k=0}^{h-1} \xi_{h}^{k n} \prod_{P}\left(1-\frac{1}{|P|}\right)\left(1+\frac{1}{|P|-|P|^{1-\frac{1}{h}} \xi_{h}^{-\operatorname{deg}(P) k}}\right) \sum_{P}\left(\frac{1}{\left.\left.|P|-|P|^{1-\frac{1}{h} \xi_{h}^{-\operatorname{deg}(P) k}+1}-\frac{1}{|P|}\right)\right]^{2}}\right. \\
& +O\left(\frac{1}{n^{1-\varepsilon}}\right)
\end{aligned}
$$

To prove the above results, one works with the Euler product

$$
\begin{aligned}
\mathcal{B}(u, z) & :=\prod_{P} \mathcal{Z}_{q}\left(u^{h}\right)^{-z}\left(1+z\left(u^{h \operatorname{deg}(P)}+u^{(h+1) \operatorname{deg}(P)}+\cdots\right)\right) \\
& =\prod_{P}\left(1-u^{h \operatorname{deg}(P)}\right)^{z}\left(1+\frac{z u^{h \operatorname{deg}(P)}}{1-u^{\operatorname{deg}(P)}}\right)
\end{aligned}
$$

Moreover, working with $\mathcal{B}\left(u, e^{t}\right)$, we also obtain
Theorem 6.2. As $n \rightarrow \infty, \omega(f)$ with $f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}$ approaches a normal distribution, namely, for $\alpha \leq \beta$,

$$
\frac{1}{\left|\mathcal{N}_{h} \cap \mathcal{M}_{n}\right|}\left|\left\{f \in \mathcal{N}_{h} \cap \mathcal{M}_{n}: \alpha \leq \frac{\omega(f)-\log \left(\frac{n}{h}\right)}{\sqrt{\log \left(\frac{n}{h}\right)}} \leq \beta\right\}\right| \rightarrow \frac{1}{\sqrt{2 \pi}} \int_{\alpha}^{\beta} e^{-t^{2} / 2} d t
$$

## 7. Conclusion

In sum, we have studied Erdős-Kac type of theorems for $h$-free and $h$-full polynomials. These families differ in the following sense. As seen in Lemma 3.3, for certain constant $C_{h}$, there are $C_{h} q^{n}$ $h$-free polynomials in the set of monic polynomials of degree $n$, which has size $q^{n}$. In contrast, for certain constant $D_{h}$, Lemma 5.3 shows that there are $D_{h} \frac{q \frac{n}{h}}{h} h$-full polynomials. Thus, while the $h$-free polynomials correspond to a considerable proportion of the polynomials in $\mathbb{F}_{q}[T]$, the $h$-full polynomials are a sparse set. These differences are reflected at the level of the moments, as the results for $h$-free polynomials are closer to the results for the full family than the results for $h$-full polynomials. However, in both cases we obtain Erdős-Kac type of theorems, as in the full family case. One could then argue that an Erdős-Kac result over the $h$-full polynomials is less predictable than in the case of the $h$-free polynomials.

In [EL21], the authors consider the function $\omega_{k}(n)$, which counts the number of prime factors of $n$ with multiplicity $k$. An interesting feature of this function lies in the fact that $\omega_{1}$ satisfies an Erdős-Kac type of theorem, but $\omega_{k}$ for $k \geq 2$ does not. This work has been recently extended to functions fields by the same authors and their collaborators [DEKL23]. It would be interesting to investigate $\omega_{k}$ in the context of $h$-free and $h$-full polynomials.

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