THE AREAL MAHLER MEASURE UNDER A POWER CHANGE OF VARIABLES

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Abstract. The Mahler measure of a multivariable polynomial is given by an integral over the unit torus, while the areal Mahler measure, defined by Pritsker [12], is given by an integral over the product of unit disks. It is well-known that the classical Mahler measure is invariant under the change of variables $x \mapsto x^r$, where r is an integer, but this is not the case for the areal Mahler measure. In this note we investigate how the areal Mahler measure changes with this transformation and provide some specific examples.

1. Introduction

The (logarithmic) Mahler measure of a multivariable non-zero rational function $P \in \mathbb{C}(x_1, \dots, x_n)^\times$ is given by

$$\mathrm{m}(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where $\mathbb{T}^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : |x_1| = \cdots = |x_n| = 1\}$ is the n-dimensional unit torus and the integration is taken with respect to the Haar measure.

When P is a single variable polynomial, $\mathrm{m}(P)$ can be expressed in terms of the roots of P by means of Jensen's formula. In the multivariable case, we do not know a general formula for $\mathrm{m}(P)$ but many examples are known where $\mathrm{m}(P)$ is related to particular values of special functions including the Riemann zeta function, L-functions, etc. The first such formula was given by Smyth [14, 2]:

$$\mathrm{m}(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2), \tag{1}$$

where $L(\chi_{-3}, s)$ is the Dirichlet L-function associated to the primitive character χ_{-3} of conductor 3, given by $\chi_{-3}(n) = \left(\frac{-3}{n}\right)$. The appearance of such special values has been explained in terms of evaluations of regulators by Deninger [6], Boyd [3], and Rodriguez-Villegas [13] (see also the book of Brunault and Zudilin [4] for a more detailed exposition).

In this article we consider the (logarithmic) areal Mahler measure defined by Pritsker [12] for $P \in \mathbb{C}(x_1, \dots, x_n)^\times$ as

$$\mathrm{m}_{\mathbb{D}}(P) = \frac{1}{\pi^n} \int_{\mathbb{D}^n} \log |P(x_1, \dots, x_n)| dA(x_1) \cdots dA(x_n),$$

where

$$\mathbb{D}^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : |x_1|, \dots, |x_n| \leq 1\}$$

is the product of n unit disks, and the measure is the natural measure in the A^0 Bergman space. The basic properties of this object (particularly for the one-variable polynomial case) have been studied by Pritsker [12], Choi and Samuels [5], and Flammang [8]. In recent work [11] the authors

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of this note started a systematic study of examples of areal Mahler measure in the multivariable case and they computed, among several cases,

$$\mathrm{m}_{\mathbb{D}}(1+x+y)=\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)+\frac{1}{6}-\frac{11\sqrt{3}}{16\pi}, \tag{2}$$

which should be compared to (1).

A property of the standard Mahler measure is its invariance respect to the following transformation. Let us write

$$P(\mathbf{x})=\sum_{\mathbf{m}} c_{\mathbf{m}}\mathbf{x}^{\mathbf{m}}\in\mathbb{C}[x_1,\dots,x_n],$$

where $\mathbf{x}^{\mathbf{m}}=x_1^{m_1}\cdots x_n^{m_n}$. Let A be an $n\times n$ integer matrix with non-zero determinant, and define

$$P^{(A)}(\mathbf{x}):=\sum_{\mathbf{m}} c_{\mathbf{m}}\mathbf{x}^{A\mathbf{m}}.$$

Then

$$\mathrm{m}(P)=\mathrm{m}\left(P^{(A)}\right). \tag{3}$$

(See Exercise 3.1 in [7].) While the above transformation has been described for polynomials, it is straightforward to generalize it to rational functions.

The goal of this project is to investigate the simplest possible case of the above transformation, namely, when one of the variables, x, is replaced by a power of itself, x^r, where r is a positive integer, in the areal Mahler measure case. To illustrate this, we compute the areal Mahler measures of $1+x^r+y^s$, where r and s are positive integers and we obtain results that are different from (2), which corresponds to the case $r=s=1$. More precisely, we prove the following statement.

Theorem 1. *Let r,s be positive integers. We have*

$$\begin{aligned}
&\mathrm{m}_{\mathbb{D}}\left(1+x^r+y^s\right)\\
&=\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)-\frac{r}{6}+\frac{\sqrt{3}r}{12\pi}\left[\zeta\left(1,\frac{r+2}{3r}\right)-\zeta\left(1,\frac{2r+2}{3r}\right)+\zeta\left(1,\frac{r+1}{3r}\right)-\zeta\left(1,\frac{2r+1}{3r}\right)\right]\\
&-\frac{2}{\pi}\sum_{1\le k}\sum_{h=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2h}\frac{(-1)^{h-1}{}_2F_1\left[\frac12-h,k-h+\frac1r+\frac12;k-h+\frac1r+\frac32;\frac14\right]}{2^{k-2h+1}k(kr+2)\left(2k+\frac2r-2h+1\right)}+\frac{s}{6}\sum_{1\le k}\binom{\frac1s}{k}^2\frac{1}{kr+1}\\
&-\frac{s\sqrt{3}}{\pi}\sum_{0\le j<k}\binom{\frac1s}{k}\binom{\frac1s}{j}\frac{\chi_{-3}(k-j)}{((k+j)r+2)(k-j)}+\frac{s}{4\pi}\sum_{1\le k}\binom{\frac1s}{k}^2\frac{{}_2F_1\left[\frac12,k+\frac1r+\frac12;k+\frac1r+\frac32;\frac14\right]}{(kr+1)\left(2k+1+\frac2r\right)}\\
&+\frac{s}{\pi}\sum_{0\le j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\frac1s}{k}\binom{\frac1s}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left[\frac12-h,k-h+\frac1r+\frac12;k-h+\frac1r+\frac32;\frac14\right]}{2^{k-j-2h}\left((k+j)r+2\right)\left(2k+\frac2r-2h+1\right)},
\end{aligned}$$

where $\zeta(s,x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^s}$ is the Hurwitz zeta-function and ${}_2F_1[a,b;c;z]$ is the hypergeometric function given in (7).

Remark 2. *For the case $r=1$, the formula from Theorem 1 should be interpreted as a regularization, namely the divergent terms $\zeta(1,1)$ with opposite signs cancel each other. More precisely, for $r=1$, the line*

$$\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)-\frac{r}{6}+\frac{\sqrt{3}r}{12\pi}\left[\zeta\left(1,\frac{r+2}{3r}\right)-\zeta\left(1,\frac{2r+2}{3r}\right)+\zeta\left(1,\frac{r+1}{3r}\right)-\zeta\left(1,\frac{2r+1}{3r}\right)\right]$$

should be replaced by

$$\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)-\frac{1}{4}+\frac{\sqrt{3}}{4\pi}.$$

Remark 3. *The result of Theorem 1 should be symmetric respect to r and s, which is certainly not obvious to guess from the formula itself! This phenomenon is observed numerically, but we do not have a direct proof of it.*

We also compute the areal Mahler measure of a similar family, namely, $(1+x)^r+y^s$ and obtain interesting results depending on s.

Theorem 4. *Let r,s be positive integers. We have*

$$\begin{aligned}
&\mathrm{m}_{\mathbb{D}}((1+x)^r+y^s)\\
&=r\left(\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)+\frac{1}{6}-\frac{\sqrt{3}}{2\pi}\right)-\frac{s}{6}+\frac{s}{6}\frac{\Gamma\left(\frac{2r}{s}+2\right)}{\Gamma\left(\frac{r}{s}+2\right)^2}\\
&-\frac{s\sqrt{3}}{\pi}\sum_{0\le j<k}\binom{\frac{r}{s}}{k}\binom{\frac{r}{s}}{j}\frac{\chi_{-3}(k-j)}{(k+j+2)(k-j)}+\frac{s}{4\pi}\sum_{1\le k}\binom{\frac{r}{s}}{k}^2\frac{{}_2F_1\left[\frac{1}{2},k+\frac{3}{2};k+\frac{5}{2};\frac{1}{4}\right]}{(k+1)(2k+3)}\\
&+\frac{s}{\pi}\sum_{0\le j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\frac{r}{s}}{k}\binom{\frac{r}{s}}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left[\frac{1}{2}-h,k-h+\frac{3}{2};k-h+\frac{5}{2};\frac{1}{4}\right]}{2^{k-j-2h}(k+j+2)(2k-2h+3)}.
\end{aligned}$$

Remark 5. *We remark that Theorem 1 and Theorem 4 should coincide in the case of $r=1$. This results in the identities*

$$\sum_{1\le k}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2j}\frac{(-1)^{j-1}{}_2F_1\left[\frac{1}{2}-j,k-j+\frac{3}{2};k-j+\frac{5}{2};\frac{1}{4}\right]}{2^{k-2j}k(k+2)\,(2k-2j+3)}=\frac{3\sqrt{3}}{4}-\frac{5\pi}{12}\tag{4}$$

and

$$\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{k+1}=-1+\frac{\Gamma\left(\frac{2}{s}+2\right)}{\Gamma\left(\frac{1}{s}+2\right)^2}.\tag{5}$$

While equation (5) is proven in Corollary 10, we do not know how to prove equation (4), which can be seen to be numerically true.

If, in addition, we set $s=1$, we recover formula (2) by employing the evaluation of ${}_2F_1\left[\frac{1}{2},\frac{5}{2};\frac{7}{2};\frac{1}{4}\right]$, given in (10).

We also prove the following result, which explains the effect of the change $x\mapsto x^r$ in general, as $r\to\infty$.

Theorem 6. *Let $P(x_1,\dots,x_n)\in\mathbb{C}(x_1,\dots,x_n)^\times$ and let $P(0,x_2,\dots,x_n)\in\mathbb{C}(x_2,\dots,x_n)^\times$ be the rational function resulting from P by setting $x_1=0$. Let r be a positive integer. Then we have*

$$\lim_{r\to\infty}\mathrm{m}_{\mathbb{D}}(P(x_1^r,x_2,\dots,x_n))=\mathrm{m}_{\mathbb{D}}(P(0,x_2,\dots,x_n)).$$

This article is organized as follows. We start with Section 2, where we compute the areal Mahler measures of polynomials with two terms, namely $\mathrm{m}_{\mathbb{D}}(x^r-a)$ and $\mathrm{m}_{\mathbb{D}}(x^r+y^s)$. We review some necessary background on hypergeometric functions in Section 3. Theorem 4 is proven in Section 4, while Theorem 1 is proven in Section 5. The order is reversed because the proof of Theorem 1 is considerably more involved. Finally, we close the article with the proof of Theorem 6, which is a result of a different flavour than the others, in Section 6.

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2. The case of two terms

In this section we consider the effect of the transformation $x \mapsto x^r$ in the simplest possible polynomials, namely those with only two monomials. Before proceeding with the discussion, we recall the formula for the areal Mahler measure of a linear polynomial, which follows from [12, Theorem 1.1] (see also equation (9) in [11]).

$$\mathrm{m}_{\mathbb{D}}(x-a) = \begin{cases} \log^+|a| & |a| \geq 1, \\ \frac{1}{2}(|a|^2-1) & |a| \leq 1. \end{cases} \tag{6}$$

For the linear case of one variable we have the following result.

Proposition 7. *Let r be a positive integer. We have*

$$\mathrm{m}_{\mathbb{D}}(x^r-a) = \begin{cases} \log^+|a| & |a| \geq 1, \\ \frac{r}{2}\left(|a|^{\frac{2}{r}}-1\right) & |a| \leq 1. \end{cases}$$

Proof. Let ξ_r denote a primitive rth root of unity and let $a^{\frac{1}{r}}$ denote any rth root of a. By multiplicativity we have

$$\mathrm{m}_{\mathbb{D}}(x^r-a) = \mathrm{m}_{\mathbb{D}}\left(\prod_{j=0}^{r-1}\left(x-a^{\frac{1}{r}}\xi_r^j\right)\right) = \sum_{j=0}^{r-1}\mathrm{m}_{\mathbb{D}}\left(x-a^{\frac{1}{r}}\xi_r^j\right).$$

The conclusion follows immediately from equation (6). □

Now we consider the case of x^r+y^s.

Theorem 8. *Let r, s be positive integers. We have*

$$\mathrm{m}_{\mathbb{D}}(x^r+y^s) = -\frac{rs}{2(r+s)}.$$

Proof. By definition and by Proposition 7,

$$\begin{aligned} \mathrm{m}_{\mathbb{D}}(x^r+y^s) =& \frac{1}{\pi^2}\int_{\mathbb{D}^2}\log|x^r+y^s|dA(x)dA(y) \\ =& \frac{r}{2\pi}\int_{\mathbb{D}}\left(|y|^{\frac{2s}{r}}-1\right)dA(y) \\ =& r\int_0^1\left(\rho^{\frac{2s}{r}}-1\right)\rho d\rho \\ =& -\frac{rs}{2(r+s)}. \end{aligned}$$

□

3. Background on hypergeometric functions

In this section we recall some standard results of hypergeometric functions which will be needed for the proofs of Theorems 1 and 4. Recall that hypergeometric series are given by

$$ {}_2F_1[a,b;c;z] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{7}$$

where $(a)_n$ denotes the Pochhammer symbol given by $(a)_0 = 1$, and for $n \geq 1$,

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1).$$

We will need the following result.

Theorem 9. *[Gauss Hypergeometric Theorem, Eq. 15.1.20 in [1]] Let $a, b, c \in \mathbb{C}$ such that $c \notin \mathbb{Z}_{\leq 0}$ and $\mathrm{Re}(c-a-b) > 0$. Then*

$$ {}_2F_1[a,b;c;1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Corollary 10. *Let $t > 0$. Then*

$$\sum_{0\leq k} \binom{t}{k}^2 \frac{1}{k+1} = \frac{\Gamma(2(t+1))}{\Gamma(t+2)^2}.$$

Proof. We apply Theorem 9 with $a = b = -t$ and $c = 2$ together with the fact that $\Gamma(2) = 1$ to obtain

$$\frac{\Gamma(2(t+1))}{\Gamma(t+2)^2} = {}_2F_1[-t,-t;2;1] = \sum_{0\leq k} \frac{(-t)_k^2}{(2)_k k!} = \sum_{0\leq k} \frac{[t(t-1)\cdots(t-k+1)]^2}{(k+1)!k!} = \sum_{0\leq k}\binom{t}{k}^2\frac{1}{k+1}.$$

$\square$

Lemma 11. *Let $\beta > -1$ be a real number and n be a non negative integer (possibly 0). Then*

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (-2\cos\theta)^\beta \cos(n\theta)d\theta =&(-1)^n \int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}} (2\cos\tau)^\beta\cos(n\tau)d\tau \\ =&\sum_{h=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2h}\frac{(-1)^{h+n}{}_2F_1\left[\frac{\beta+n+1}{2}-h,\frac{1}{2}-h;\frac{\beta+n+3}{2}-h;\frac{1}{4}\right]}{2^{n-2h+1}(\beta+n+1-2h)}.\end{aligned}$$

Proof. We first notice that the equality between the integrals follows from the change of variables $\theta + \pi = \tau$. We remark that $\cos(n\theta) = T_n(\cos\theta)$, where $T_n(x)$ is the Chebyshev polynomial of the first kind. By using this, together with the change of variables $t = -2\cos\theta$, we have

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (-2\cos\theta)^\beta\cos(n\theta)d\theta = \int_0^1 t^\beta T_n\left(-\frac{t}{2}\right)\frac{dt}{\sqrt{4-t^2}}. \tag{8}$$

The Chebyshev polynomials can be explicitly expressed as

$$T_n(x) = \frac{1}{2}\left[\left(x-\sqrt{x^2-1}\right)^n + \left(x+\sqrt{x^2-1}\right)^n\right] = \sum_{h=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2h}\left(x^2-1\right)^h x^{n-2h}.$$

Then, using this, we can evaluate the integral in (8) as

$$\begin{aligned}\int_0^1 t^\beta T_n\left(-\frac{t}{2}\right)\frac{dt}{(4-t^2)^{\frac{1}{2}}} &= \sum_{h=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2h}\int_0^1 t^\beta\left(\frac{t^2}{4}-1\right)^h\left(-\frac{t}{2}\right)^{n-2h}\frac{dt}{(4-t^2)^{\frac{1}{2}}}\\ &= \sum_{h=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2h}\frac{(-1)^{n+h}}{2^{n-2h+1}}\int_0^1 t^{\beta+n-2h}\left(1-\frac{t^2}{4}\right)^{h-\frac{1}{2}}dt\\ &= \sum_{h=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2h}\frac{(-1)^{n+h}{}_2F_1\left[\frac{\beta+n+1}{2}-h,\frac{1}{2}-h;\frac{\beta+n+3}{2}-h;\frac{1}{4}\right]}{2^{n-2h+1}(\beta+n+1-2h)},\end{aligned}$$

where the last identity follows from making the change $u=t^2$ and then applying equation 15.3.1 in [1]. $\square$

Applications of Lemma 11 will naturally lead to evaluations of the hypergeometric function at $z=\frac{1}{4}$. Here we record two identities that will be useful for simplifying some formulas:

$$ {}_2F_1\left[\frac{1}{2},\frac{3}{2};\frac{5}{2};\frac{1}{4}\right]=2\pi-3\sqrt{3} \tag{9}$$

and

$$ {}_2F_1\left[\frac{1}{2},\frac{5}{2};\frac{7}{2};\frac{1}{4}\right]=10\pi-\frac{35\sqrt{3}}{2}. \tag{10}$$

Equation (9) follows from the more general formula 07.23.03.2888.01 in [9] :

$$ {}_2F_1\left[\frac{1}{2},\frac{3}{2};\frac{5}{2};z\right]=\frac{3}{2z^{3/2}}(\arcsin(\sqrt{z})-\sqrt{z(1-z)}),$$

by setting $z=\frac{1}{4}$, while equation (10) follows from formula 07.23.03.2933.01 in [10]:

$$ {}_2F_1\left[\frac{1}{2},\frac{5}{2};\frac{7}{2};z\right]=\frac{5}{8z^{5/2}}(3\arcsin(\sqrt{z})-\sqrt{z(1-z)}(3+2z)),$$

by setting $z=\frac{1}{4}$.

4. The areal Mahler measure of $(1+x)^r+y^s$

In this section we prove Theorem 4, which is simpler than Theorem 1. To place the result in perspective, we first consider the classical case.

Lemma 12. *Let r,s be positive integers. We have*

$$\mathrm{m}((1+x)^r+y^s)=r\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2).$$

Proof. First notice that the left-hand side is completely independent of s, since a particular case of equation (3) implies

$$\mathrm{m}((1+x)^r+y^s)=\mathrm{m}((1+x)^r+y).$$

Let ξ_r denote a primitive rth root of unity. We have

$$\mathrm{m}((1+x)^r+y)=\mathrm{m}((1+x)^r-y^r)=\mathrm{m}\left(\prod_{j=0}^{r-1}(1+x-\xi_r^jy)\right)=\sum_{j=0}^{r-1}\mathrm{m}(1+x-\xi_r^jy)=r\mathrm{m}(1+x+y),$$

since $\mathrm{m}(1+x-\xi_r^j y)=\mathrm{m}(1+x+y)$ for any j. The result follows from equation (1). □

Proof of Theorem 4. Let ξ_s be a primitive s root of unity and let $\alpha=\frac{r}{s}$. When $x\in\mathbb{D}$, $(1+x)^\alpha$ is well-defined, and we can consider the principal branch of the α-th power. By multiplicativity we have

$$\mathrm{m}_{\mathbb{D}}((1+x)^r+y^s)=\mathrm{m}_{\mathbb{D}}((1+x)^r-y^s)=\sum_{j=0}^{s-1}\mathrm{m}_{\mathbb{D}}\left((1+x)^\alpha-\xi_s^j y\right),$$

where we have extended the definition of $\mathrm{m}_{\mathbb{D}}$ to the algebraic functions $(1+x)^\alpha-\xi_s^j y$ in the natural way using the integral.

By definition and by application of equation (6),

$$\begin{aligned}\mathrm{m}_{\mathbb{D}}((1+x)^\alpha-\xi_s y)=&\frac{1}{\pi^2}\int_{\mathbb{D}^2}\log|(1+x)^\alpha-\xi_s y|\,dA(y)dA(x)\\ =&\frac{\alpha}{\pi}\int_{\mathbb{D}\cap\{|1+x|\geq 1\}}\log|1+x|dA(x)+\frac{1}{2\pi}\int_{\mathbb{D}\cap\{|1+x|\leq 1\}}\left(|1+x|^{2\alpha}-1\right)dA(x).\end{aligned}\tag{11}$$

The first integral was already computed in the proof of the case of $1+x+y$ (see the proof of [11, Theorem 1.1]) and is given by

$$\frac{\alpha}{\pi}\int_{\mathbb{D}\cap\{|1+x|\geq 1\}}\log|1+x|dA(x)=\alpha\left(\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)+\frac{1}{6}-\frac{\sqrt{3}}{2\pi}\right).$$

We now treat the second integral in (11). Writing $x=\rho e^{i\theta}$, we have

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathbb{D}\cap\{|1+x|\leq 1\}}\left(|1+x|^{2\alpha}-1\right)dA(x)\\ =&\frac{1}{\pi}\int_{\frac{2\pi}{3}}^{\pi}\int_0^1((1+\rho e^{i\theta})^\alpha(1+\rho e^{-i\theta})^\alpha-1)\rho d\rho d\theta+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_0^{-2\cos\theta}((1+\rho e^{i\theta})^\alpha(1+\rho e^{-i\theta})^\alpha-1)\rho d\rho d\theta\\ =&\frac{1}{\pi}\sum_{0\leq j,k}\binom{\alpha}{k}\binom{\alpha}{j}\left[\int_{\frac{2\pi}{3}}^{\pi}\int_0^1\rho^{k+j+1}e^{i(k-j)\theta}d\rho d\theta+\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_0^{-2\cos\theta}\rho^{k+j+1}e^{i(k-j)\theta}d\rho d\theta\right]\\ &-\frac{1}{\pi}\left[\int_{\frac{2\pi}{3}}^{\pi}\int_0^1\rho d\rho d\theta+\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_0^{-2\cos\theta}\rho d\rho d\theta\right]\\ =&\frac{1}{\pi}\sum_{0\leq j,k}\binom{\alpha}{k}\binom{\alpha}{j}\left[\int_{\frac{2\pi}{3}}^{\pi}\int_0^1\rho^{k+j+1}e^{i(k-j)\theta}d\rho d\theta+\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_0^{-2\cos\theta}\rho^{k+j+1}e^{i(k-j)\theta}d\rho d\theta\right]\\ &-\frac{1}{6}+\frac{1}{\pi}\left(\frac{\sqrt{3}}{4}-\frac{\pi}{6}\right).\end{aligned}\tag{12}$$

We treat the integrals in (12) separately for the cases $k=j$ and $k\neq j$. When $k=j$, we have

$$\begin{aligned}
&\frac{1}{\pi}\sum_{0\leq k}\binom{\alpha}{k}^2\left[\int_{\frac{2\pi}{3}}^{\pi}\int_0^1 \rho^{2k+1}d\rho d\theta+\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_0^{-2\cos\theta}\rho^{2k+1}d\rho d\theta\right]\\
=&\frac{1}{\pi}\sum_{0\leq k}\binom{\alpha}{k}^2\frac{1}{2(k+1)}\left[\frac{\pi}{3}+\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}(-2\cos\theta)^{2(k+1)}d\theta\right]\\
=&\frac{\Gamma(2\alpha+2)}{6\Gamma(\alpha+2)^2}+\frac{1}{4\pi}\sum_{0\leq k}\binom{\alpha}{k}^2\frac{{}_2F_1\left[\frac{1}{2},k+\frac{3}{2};k+\frac{5}{2};\frac{1}{4}\right]}{(k+1)(2k+3)}\\
=&\frac{\Gamma(2\alpha+2)}{6\Gamma(\alpha+2)^2}+\frac{1}{6}-\frac{\sqrt{3}}{4\pi}+\frac{1}{4\pi}\sum_{1\leq k}\binom{\alpha}{k}^2\frac{{}_2F_1\left[\frac{1}{2},k+\frac{3}{2};k+\frac{5}{2};\frac{1}{4}\right]}{(k+1)(2k+3)} \qquad (13)
\end{aligned}$$

where we have applied Corollary 10, Lemma 11, and equation (9).

Note that the expression in (12) is conjugated under the change $(k,j)\to(j,k)$, when $k\neq j$. Therefore, when $k\neq j$, we derive that

$$\begin{aligned}
&\frac{1}{\pi}\sum_{\substack{0\leq j,k\\k\neq j}}\binom{\alpha}{k}\binom{\alpha}{j}\int_{\frac{2\pi}{3}}^{\pi}\int_0^1\rho^{k+j+1}e^{i(k-j)\theta}d\rho d\theta\\
=&\frac{1}{\pi}\sum_{\substack{0\leq j,k\\k\neq j}}\binom{\alpha}{k}\binom{\alpha}{j}\frac{1}{k+j+2}\int_{\frac{2\pi}{3}}^{\pi}e^{i(k-j)\theta}d\theta\\
=&\frac{2}{\pi}\sum_{0\leq j<k}\binom{\alpha}{k}\binom{\alpha}{j}\frac{1}{k+j+2}\int_{\frac{2\pi}{3}}^{\pi}\cos((k-j)\theta)d\theta\\
=&-\frac{2}{\pi}\sum_{0\leq j<k}\binom{\alpha}{k}\binom{\alpha}{j}\frac{1}{(k+j+2)(k-j)}\sin\left(\frac{2(k-j)\pi}{3}\right)\\
=&-\frac{\sqrt{3}}{\pi}\sum_{0\leq j<k}\binom{\alpha}{k}\binom{\alpha}{j}\frac{\chi_{-3}(k-j)}{(k+j+2)(k-j)}, \qquad (14)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\pi}\sum_{\substack{0\leq j,k\\k\neq j}}\binom{\alpha}{k}\binom{\alpha}{j}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_0^{-2\cos\theta}\rho^{k+j+1}e^{i(k-j)\theta}d\rho d\theta\\
=&\frac{2}{\pi}\sum_{0\leq j<k}\binom{\alpha}{k}\binom{\alpha}{j}\frac{1}{k+j+2}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}(-2\cos\theta)^{k+j+2}\cos((k-j)\theta)d\theta\\
=&\frac{2}{\pi}\sum_{0\leq j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\alpha}{k}\binom{\alpha}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left[\frac{1}{2}-h,k-h+\frac{3}{2};k-h+\frac{5}{2};\frac{1}{4}\right]}{2^{k-j-2h+1}(k+j+2)(2k-2h+3)}, \qquad (15)
\end{aligned}$$

by Lemma 11.

Combining (12), (13), (14), and (15), we obtain

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathbb{D}\cap\{|1+x|\leq 1\}}(|1+x|^{2\alpha}-1)dA(x)\\ =&\frac{\Gamma(2\alpha+2)}{6\Gamma(\alpha+2)^2}-\frac{1}{6}+\frac{1}{4\pi}\sum_{1\leq k}\binom{\alpha}{k}^2\frac{{}_2F_1\left[\frac{1}{2},k+\frac{3}{2};k+\frac{5}{2};\frac{1}{4}\right]}{(k+1)(2k+3)}-\frac{\sqrt{3}}{\pi}\sum_{0\leq j<k}\binom{\alpha}{k}\binom{\alpha}{j}\frac{\chi_{-3}(k-j)}{(k+j+2)(k-j)}\\ &+\frac{1}{\pi}\sum_{0\leq j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\alpha}{k}\binom{\alpha}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left[\frac{1}{2}-h,k-h+\frac{3}{2};k-h+\frac{5}{2};\frac{1}{4}\right]}{2^{k-j-2h}(k+j+2)(2k-2h+3)}.\end{aligned}$$

By recalling that $\alpha=\frac{r}{s}$, we obtain the result. $\square$

5. The areal Mahler measure of $1+x^r+y^s$

In this section we prove Theorem 1, our main result. Before proceeding to its proof, we show the following auxiliary statement.

Lemma 13. *For $r>1$, we have*

$$\begin{aligned}\sum_{1\leq k}\frac{(-1)^{k-1}\chi_{-3}(k)}{k^2(kr+2)}=&\frac{3}{4}L(\chi_{-3},2)-\frac{\pi r}{6\sqrt{3}}\\ &+\frac{r}{12}\left[\zeta\left(1,\frac{r+2}{3r}\right)-\zeta\left(1,\frac{2r+2}{3r}\right)+\zeta\left(1,\frac{r+1}{3r}\right)-\zeta\left(1,\frac{2r+1}{3r}\right)\right],\end{aligned}$$

and for $r=1$,

$$\sum_{1\leq k}\frac{(-1)^{k-1}\chi_{-3}(k)}{k^2(k+2)}=\frac{3}{4}L(\chi_{-3},2)-\frac{\pi}{4\sqrt{3}}+\frac{1}{4}.$$

Proof. From the identity

$$\frac{1}{k^2(kr+2)}=\frac{1}{2k^2}-\frac{r}{4k}+\frac{r^2}{4(kr+2)},$$

we have that

$$\sum_{1\leq k}\frac{(-1)^{k-1}\chi_{-3}(k)}{k^2(kr+2)}=\frac{1}{2}\sum_{1\leq k}\frac{(-1)^{k-1}\chi_{-3}(k)}{k^2}-\frac{r}{4}\sum_{1\leq k}\frac{(-1)^{k-1}\chi_{-3}(k)}{k}+\frac{r^2}{4}\sum_{1\leq k}\frac{(-1)^{k-1}\chi_{-3}(k)}{kr+2}. \tag{16}$$

We consider the different terms on the right-hand side of (16). They are

$$\sum_{1\leq k}\frac{(-1)^{k-1}\chi_{-3}(k)}{k^2}=\sum_{1\leq k}\frac{\chi_{-3}(k)}{k^2}-2\sum_{1\leq k}\frac{\chi_{-3}(2k)}{(2k)^2}=\frac{3}{2}\sum_{1\leq k}\frac{\chi_{-3}(k)}{k^2}=\frac{3}{2}L(\chi_{-3},2), \tag{17}$$

$$\sum_{1\leq k}\frac{(-1)^{k-1}\chi_{-3}(k)}{k}=\sum_{1\leq k}\frac{\chi_{-3}(k)}{k}-2\sum_{1\leq k}\frac{\chi_{-3}(2k)}{2k}=2\sum_{1\leq k}\frac{\chi_{-3}(k)}{k}=\frac{2\pi}{3\sqrt{3}}, \tag{18}$$

and finally

$$\begin{aligned}\sum_{1\le k}\frac{(-1)^{k-1}\chi_{-3}(k)}{kr+2} &= \sum_{1\le k}\frac{\chi_{-3}(k)}{kr+2} - 2\sum_{1\le k}\frac{\chi_{-3}(2k)}{2kr+2} = \sum_{1\le k}\frac{\chi_{-3}(k)}{kr+2} + \sum_{1\le k}\frac{\chi_{-3}(k)}{kr+1}\\ &= \sum_{0\le j}\frac{1}{3jr+r+2} - \sum_{0\le j}\frac{1}{3jr+2r+2} + \sum_{0\le j}\frac{1}{3jr+r+1} - \sum_{0\le j}\frac{1}{3jr+2r+1}\\ &= \frac{1}{3r}\left[\zeta\left(1,\frac{r+2}{3r}\right) - \zeta\left(1,\frac{2r+2}{3r}\right) + \zeta\left(1,\frac{r+1}{3r}\right) - \zeta\left(1,\frac{2r+1}{3r}\right)\right]. \qquad (19)\end{aligned}$$

By combining (17), (18), and (19) with (16), we get the result for $r>1$.

When $r=1$, (19) becomes

$$\frac{1}{3}\left[\zeta\left(1,\frac{2}{3}\right) - \zeta\left(1,\frac{4}{3}\right)\right] = 1 - \frac{\pi}{3\sqrt{3}}.$$

$\square$

Proof of Theorem 1. Our goal is to calculate $\mathrm{m}_{\mathbb{D}}\left(1+x^r+y^s\right)$. Since $1+x^r+y^s=\prod_{j=0}^{s-1}\left(\sqrt[s]{1+x^r}+\xi_s^j y\right)$ and for any $k\neq \ell$ we have $\mathrm{m}_{\mathbb{D}}\left(\sqrt[s]{1+x^r}+\xi_s^k y\right)=\mathrm{m}_{\mathbb{D}}\left(\sqrt[s]{1+x^r}+\xi_s^\ell y\right)$, we can write

$$\mathrm{m}_{\mathbb{D}}\left(1+x^r+y^s\right) = \sum_{j=0}^{s-1}\mathrm{m}_{\mathbb{D}}\left(\sqrt[s]{1+x^r}+\xi_s^j y\right) = s\mathrm{m}_{\mathbb{D}}\left(\sqrt[s]{1+x^r}+y\right).$$

Here we note that the function $\sqrt[s]{1+x^r}$ is well-defined when $x\in\mathbb{D}$, and, from now on, we consider the principal branch of the s-th root.

By definition and by application of equation (6), we obtain

$$\begin{aligned}\mathrm{m}_{\mathbb{D}}\left(\sqrt[s]{1+x^r}+y\right) =&\frac{1}{\pi^2}\int_{\mathbb{D}^2}\log\left|\sqrt[s]{1+x^r}+y\right|dA(x)dA(y)\\ =&\frac{1}{s\pi}\int_{\mathbb{D}\cap\{|1+x^r|\ge 1\}}\log\left|1+x^r\right|dA(x) + \frac{1}{2\pi}\int_{\mathbb{D}\cap\{|1+x^r|\le 1\}}\left(\left|1+x^r\right|^{\frac{2}{s}}-1\right)dA(x). \qquad (20)\end{aligned}$$

For $x=\rho e^{i\theta}$ with $0\le\rho\le 1$ and θ defined modulo 2π, the condition $|1+x^r|\ge 1$ is equivalent to

$$1+\rho^{2r}+2\rho^r\cos(r\theta)\ge 1 \Longleftrightarrow \rho^r+2\cos(r\theta)\ge 0.$$

Therefore, for $\ell\in\mathbb{Z}\cap[0,r-1]$, the condition $|1+x^r|\ge 1$ holds when $\frac{(4\ell-1)\pi}{2r}\le\theta\le\frac{(4\ell+1)\pi}{2r}$ and $0\le\rho\le 1$, and, when $\frac{(4\ell+1)\pi}{2r}\le\theta\le\frac{(6\ell+2)\pi}{3r}$ as well as $\frac{(6\ell-2)\pi}{3r}\le\theta\le\frac{(4\ell-1)\pi}{2r}$ and $\sqrt[r]{-2\cos(r\theta)}\le\rho\le 1$.

Similarly, for $\ell\in\mathbb{Z}\cap[0,r-1]$, the condition $|1+x^r|\le 1$ implies that that the second integral needs to be evaluated when $\frac{(4\ell+1)\pi}{2r}\le\theta\le\frac{(6\ell+2)\pi}{3r}$ as well as $\frac{(6\ell+4)\pi}{3r}\le\theta\le\frac{(4\ell+3)\pi}{2r}$ and $0\le\rho\le\sqrt[r]{-2\cos(r\theta)}$, and when $\frac{(6\ell+2)\pi}{3r}\le\theta\le\frac{(6\ell+4)\pi}{3r}$ and $0\le\rho\le 1$.

We start by evaluating the first integral in (20). Following the above discussion, we have

$$\begin{aligned}\int_{\mathbb{D}\cap\{|1+x^r|\ge 1\}}\log\left|1+x^r\right|dA(x) = \sum_{\ell=0}^{r-1}\mathrm{Re}\Bigg[&\int_{\frac{(4\ell-1)\pi}{2r}}^{\frac{(4\ell+1)\pi}{2r}}\int_0^1\log\left(1+\rho^r e^{ir\theta}\right)\rho d\rho d\theta\\ &+\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_{\sqrt[r]{-2\cos(r\theta)}}^1\log\left(1+\rho^r e^{ir\theta}\right)\rho d\rho d\theta + \int_{\frac{(6\ell-2)\pi}{3r}}^{\frac{(4\ell-1)\pi}{2r}}\int_{\sqrt[r]{-2\cos(r\theta)}}^1\log\left(1+\rho^r e^{ir\theta}\right)\rho d\rho d\theta\Bigg].\end{aligned}$$

Since $\log\left(1+\rho^r e^{ir\theta}\right)=\sum_{1\le k}(-1)^{k-1}\frac{\rho^{kr}e^{ikr\theta}}{k}$, we have

$$\begin{aligned}\operatorname{Re}\left[\int_{\frac{(4\ell-1)\pi}{2r}}^{\frac{(4\ell+1)\pi}{2r}}\int_0^1\log\left(1+\rho^r e^{ir\theta}\right)\rho d\rho d\theta\right]&=\operatorname{Re}\left[\sum_{1\le k}\frac{(-1)^{k-1}}{k}\int_{\frac{(4\ell-1)\pi}{2r}}^{\frac{(4\ell+1)\pi}{2r}}\int_0^1\rho^{kr+1}e^{ikr\theta}d\rho d\theta\right]\\&=\operatorname{Re}\left[\sum_{1\le k}\frac{(-1)^{k-1}}{k(kr+2)}\int_{\frac{(4\ell-1)\pi}{2r}}^{\frac{(4\ell+1)\pi}{2r}}e^{ikr\theta}d\theta\right]\\&=\operatorname{Re}\left[\sum_{1\le k}\frac{(-1)^{k}i}{k^2r(kr+2)}\left(e^{\frac{ik(4\ell+1)\pi}{2}}-e^{\frac{ik(4\ell-1)\pi}{2}}\right)\right]\\&=\frac{2}{r}\sum_{1\le k}\frac{1}{k^2(kr+2)}\sin\left(\frac{k\pi}{2}\right).\end{aligned}$$

Now, using the fact that $-2\cos(r\theta)=2\cos(r\theta+\pi)$, we have

$$\begin{aligned}&\operatorname{Re}\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_{\sqrt[r]{2\cos(r\theta+\pi)}}^1\log\left(1+\rho^r e^{ir\theta}\right)\rho d\rho d\theta\right]=\operatorname{Re}\left[\sum_{1\le k}\frac{(-1)^{k-1}}{k}\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_{\sqrt[r]{2\cos(r\theta+\pi)}}^1\rho^{kr+1}e^{ikr\theta}d\rho d\theta\right]\\&=\operatorname{Re}\left[\sum_{1\le k}\frac{(-1)^{k-1}}{k(kr+2)}\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\left(1-(2\cos(r\theta+\pi))^{k+\frac{2}{r}}\right)e^{ikr\theta}d\theta\right]\\&=\frac{1}{r}\operatorname{Re}\left[\sum_{1\le k}\frac{(-1)^{k-1}}{k(kr+2)}\int_{\frac{(4\ell+3)\pi}{2}}^{\frac{(6\ell+5)\pi}{3}}\left(1-(2\cos(\tau))^{k+\frac{2}{r}}\right)e^{ik(\tau-\pi)}d\tau\right]\\&=\frac{1}{r}\sum_{1\le k}\frac{(-1)^{k-1}}{k(kr+2)}\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}}\left(1-(2\cos(\tau))^{k+\frac{2}{r}}\right)\cos(k\tau-k\pi)d\tau\\&=\frac{1}{r}\sum_{1\le k}\frac{(-1)^{k-1}\cos(k\pi)}{k(kr+2)}\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}}\left(1-(2\cos(\tau))^{k+\frac{2}{r}}\right)\cos(k\tau)d\tau\\&=\frac{1}{r}\sum_{1\le k}\frac{(-1)^{k-1}\cos(k\pi)}{k^2(kr+2)}\left(\sin\left(\frac{5k\pi}{3}\right)-\sin\left(\frac{3k\pi}{2}\right)\right)-\frac{1}{r}\sum_{1\le k}\frac{(-1)^{k-1}\cos(k\pi)}{k(kr+2)}\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}}(2\cos(\tau))^{k+\frac{2}{r}}\cos(k\tau)d\tau,\end{aligned}$$

where we have set $\tau=r\theta+\pi$. By Lemma 11, we have

$$\begin{aligned}&\operatorname{Re}\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_{\sqrt[r]{2\cos(r\theta+\pi)}}^1\log\left(1+\rho^r e^{ir\theta}\right)\rho d\rho d\theta\right]\\=&\frac{1}{r}\sum_{1\le k}\frac{1}{k^2(kr+2)}\left(\sin\left(\frac{k\pi}{3}\right)-\sin\left(\frac{k\pi}{2}\right)\right)\\&-\frac{1}{r}\sum_{1\le k}\sum_{h=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2h}\frac{(-1)^{h-1}{}_2F_1\left[\frac12-h,k-h+\frac1r+\frac12;k-h+\frac1r+\frac32;\frac14\right]}{2^{k-2h+1}k(kr+2)\left(2k+\frac2r-2h+1\right)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\mathrm{Re}\left[\int_{\frac{(6\ell-2)\pi}{3r}}^{\frac{(4\ell-1)\pi}{2r}}\int_{\sqrt[r]{-2\cos(r\theta)}}^{1}\log\left(1+\rho^r e^{ir\theta}\right)\rho d\rho d\theta\right]\\
=&\frac{1}{r}\sum_{1\le k}\frac{1}{k^2(kr+2)}\left(\sin\left(\frac{k\pi}{3}\right)-\sin\left(\frac{k\pi}{2}\right)\right)\\
&-\frac{1}{r}\sum_{1\le k}\sum_{h=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2h}\frac{(-1)^{h-1}{}_2F_1\left[\frac{1}{2}-h,k-h+\frac{1}{r}+\frac{1}{2};k-h+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right]}{2^{k-2h+1}k(kr+2)\left(2k+\frac{2}{r}-2h+1\right)}.
\end{aligned}$$

Therefore, combining the above results we obtain

$$\begin{aligned}
&\int_{\mathbb{D}\cap\{|1+x^r|\ge 1\}}\log|1+x^r|\,dA(x)\\
=&2\sum_{1\le k}\frac{\sin\left(\frac{k\pi}{3}\right)}{k^2(kr+2)}-2\sum_{1\le k}\sum_{h=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2h}\frac{(-1)^{h-1}{}_2F_1\left[\frac{1}{2}-h,k-h+\frac{1}{r}+\frac{1}{2};k-h+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right]}{2^{k-2h+1}k(kr+2)\left(2k+\frac{2}{r}-2h+1\right)}\\
=&\sqrt{3}\sum_{1\le k}\frac{(-1)^{k-1}\chi_{-3}(k)}{k^2(kr+2)}-2\sum_{1\le k}\sum_{h=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2h}\frac{(-1)^{h-1}{}_2F_1\left[\frac{1}{2}-h,k-h+\frac{1}{r}+\frac{1}{2};k-h+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right]}{2^{k-2h+1}k(kr+2)\left(2k+\frac{2}{r}-2h+1\right)}.
\end{aligned}\tag{21}$$

The second integral in (20) yields

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{D}\cap\{|1+x^r|\le 1\}}\left(|1+x^r|^{\frac{2}{s}}-1\right)dA(x)=&\frac{1}{2}\sum_{\ell=0}^{r-1}\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_0^{\sqrt[r]{-2\cos(r\theta)}}\left(\left(1+\rho^r e^{ir\theta}\right)^{\frac{1}{s}}\left(1+\rho^r e^{-ir\theta}\right)^{\frac{1}{s}}-1\right)\rho d\rho d\theta\right.\\
&+\int_{\frac{(6\ell+4)\pi}{3r}}^{\frac{(4\ell+3)\pi}{2r}}\int_0^{\sqrt[r]{-2\cos(r\theta)}}\left(\left(1+\rho^r e^{ir\theta}\right)^{\frac{1}{s}}\left(1+\rho^r e^{-ir\theta}\right)^{\frac{1}{s}}-1\right)\rho d\rho d\theta\\
&\left.+\int_{\frac{(6\ell+2)\pi}{3r}}^{\frac{(6\ell+4)\pi}{3r}}\int_0^{1}\left(\left(1+\rho^r e^{ir\theta}\right)^{\frac{1}{s}}\left(1+\rho^r e^{-ir\theta}\right)^{\frac{1}{s}}-1\right)\rho d\rho d\theta\right]
\end{aligned}$$

$$=\frac{1}{2}\sum_{\ell=0}^{r-1}\sum_{\substack{0\le k,j\\(k,j)\neq(0,0)}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_0^{\sqrt[r]{-2\cos(r\theta)}}\rho^{(k+j)r+1}e^{ir(k-j)\theta}d\rho d\theta\right.\tag{22}$$

$$+\int_{\frac{(6\ell+4)\pi}{3r}}^{\frac{(4\ell+3)\pi}{2r}}\int_0^{\sqrt[r]{-2\cos(r\theta)}}\rho^{(k+j)r+1}e^{ir(k-j)\theta}d\rho d\theta\tag{23}$$

$$\left.+\int_{\frac{(6\ell+2)\pi}{3r}}^{\frac{(6\ell+4)\pi}{3r}}\int_0^{1}\rho^{(k+j)r+1}e^{ir(k-j)\theta}d\rho d\theta\right].\tag{24}$$

For $k=j$, the inner sum for integral (24) gives

$$\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\int_{\frac{(6\ell+2)\pi}{3r}}^{\frac{(6\ell+4)\pi}{3r}}\int_0^1\rho^{2kr+1}d\rho d\theta=\frac{\pi}{3r}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{kr+1}.\tag{25}$$

The combination of the inner sums of integrals (22) and (23) yields, when $k=j$,

$$\begin{aligned}
&\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_0^{\sqrt[r]{2\cos(r\theta+\pi)}}\rho^{2kr+1}d\rho d\theta+\int_{\frac{(6\ell+4)\pi}{3r}}^{\frac{(4\ell+3)\pi}{2r}}\int_0^{\sqrt[r]{2\cos(r\theta-\pi)}}\rho^{2kr+1}d\rho d\theta\right]\\
&=\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{2(kr+1)}\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}(2\cos(r\theta+\pi))^{2k+\frac{2}{r}}\,d\theta+\int_{\frac{(6\ell+4)\pi}{3r}}^{\frac{(4\ell+3)\pi}{2r}}(2\cos(r\theta-\pi))^{2k+\frac{2}{r}}\,d\theta\right]\\
&=\frac{1}{r}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{2(kr+1)}\left[\int_{\frac{(4\ell+3)\pi}{2}}^{\frac{(6\ell+5)\pi}{3}}(2\cos(\tau))^{2k+\frac{2}{r}}\,d\tau+\int_{\frac{(6\ell+1)\pi}{3}}^{\frac{(4\ell+1)\pi}{2}}(2\cos(\tau))^{2k+\frac{2}{r}}\,d\tau\right]\\
&=\frac{1}{r}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{2(kr+1)}\left[\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}}(2\cos(\tau))^{2k+\frac{2}{r}}\,d\tau+\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(2\cos(\tau))^{2k+\frac{2}{r}}\,d\tau\right]\\
&=\frac{1}{r}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{kr+1}\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}}(2\cos(\tau))^{2k+\frac{2}{r}}\,d\tau.
\end{aligned}$$

Applying Lemma 11, we get

$$\begin{aligned}
&\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_0^{\sqrt[r]{2\cos(r\theta+\pi)}}\rho^{2kr+1}d\rho d\theta+\int_{\frac{(6\ell+4)\pi}{3r}}^{\frac{(4\ell+3)\pi}{2r}}\int_0^{\sqrt[r]{2\cos(r\theta+\pi)}}\rho^{2kr+1}d\rho d\theta\right]\\
&=\frac{1}{r}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{{}_2F_1\left[\frac{1}{2},k+\frac{1}{r}+\frac{1}{2};k+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right]}{2(kr+1)\left(2k+1+\frac{2}{r}\right)}.
\end{aligned}\tag{26}$$

We now treat the case when $k\neq j$. For the inner sum of integral (24), the $k\neq j$ case yields

$$\begin{aligned}
&\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\int_{\frac{(6\ell+2)\pi}{3r}}^{\frac{(6\ell+4)\pi}{3r}}\int_0^1\rho^{(k+j)r+1}e^{ir(k-j)\theta}d\rho d\theta\\
&=\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{1}{(k+j)r+2}\int_{\frac{(6\ell+2)\pi}{3r}}^{\frac{(6\ell+4)\pi}{3r}}e^{ir(k-j)\theta}d\theta\\
&=\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{2(-1)^{k-j}}{r((k+j)r+2)(k-j)}\sin\left(\frac{(k-j)\pi}{3}\right)\\
&=-\frac{\sqrt{3}}{r}\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{\chi_{-3}(k-j)}{((k+j)r+2)(k-j)}.
\end{aligned}\tag{27}$$

Now, the inner sum for integral (22) in the $k\neq j$ case gives

$$\begin{aligned}
&\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_0^{\sqrt[r]{2\cos(r\theta+\pi)}}\rho^{r(k+j)+1}e^{ir(k-j)\theta}d\rho d\theta\\
=&\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{1}{(k+j)r+2}\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}(2\cos(r\theta+\pi))^{k+j+\frac{2}{r}}e^{ir(k-j)\theta}d\theta\\
=&\frac{1}{r}\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{1}{(k+j)r+2}\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}}(2\cos(\tau))^{k+j+\frac{2}{r}}e^{i(k-j)(\tau-\pi)}d\tau\\
=&\frac{1}{r}\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{(-1)^{k-j}}{(k+j)r+2}\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}}(2\cos(\tau))^{k+j+\frac{2}{r}}e^{i(k-j)\tau}d\tau.
\end{aligned}$$

The above expression gets conjugated under the change $(k,j)\to(j,k)$. That means that it suffices to take the real part, and therefore it suffices to find

$$\frac{2}{r}\sum_{0\le j<k}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{(-1)^{k-j}}{(k+j)r+2}\int_{\frac{3\pi}{2}}^{\frac{5\pi}{3}}(2\cos(\tau))^{k+j+\frac{2}{r}}\cos((k-j)\tau)d\tau.$$

By Lemma 11,

$$\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_0^{\sqrt[r]{2\cos(r\theta+\pi)}}\rho^{r(k+j)+1}e^{ir(k-j)\theta}d\rho d\theta$$

$$=\frac{1}{r}\sum_{0\le j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left[\frac{1}{2}-h,k-h+\frac{1}{r}+\frac{1}{2};k-h+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right]}{2^{k-j-2h}\left((k+j)r+2\right)\left(2k+\frac{2}{r}-2h+1\right)}. \tag{28}$$

Similarly, we have

$$\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\int_{\frac{(6\ell+4)\pi}{3r}}^{\frac{(4\ell+3)\pi}{2r}}\int_0^{\sqrt[r]{2\cos(r\theta+\pi)}}\rho^{r(k+j)+1}e^{ir(k-j)\theta}d\rho d\theta$$

$$=\frac{1}{r}\sum_{0\le j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left[\frac{1}{2}-h,k-h+\frac{1}{r}+\frac{1}{2};k-h+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right]}{2^{k-j-2h}\left((k+j)r+2\right)\left(2k+\frac{2}{r}-2h+1\right)}. \tag{29}$$

Collecting (25) and (27), we have that the integral in (24) yields

$$\begin{aligned}
&\frac{1}{2}\sum_{\ell=0}^{r-1}\sum_{\substack{0\le k,j\\(k,j)\neq(0,0)}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\int_{\frac{(6\ell+2)\pi}{3r}}^{\frac{(6\ell+4)\pi}{3r}}\int_0^1\rho^{(k+j)r+1}e^{ir(k-j)\theta}d\rho d\theta\\
=&\frac{\pi}{6}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{kr+1}-\frac{\sqrt{3}}{2}\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{\chi_{-3}(k-j)}{((k+j)r+2)(k-j)}.
\end{aligned}$$

Combining (26), (28) and (29), we derive that the integrals in (22) and (23) yield

$$\begin{aligned}
&\frac{1}{2}\sum_{\ell=0}^{r-1}\sum_{\substack{0\le k,j\\(k,j)\neq(0,0)}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\left[\int_{\frac{(4\ell+1)\pi}{2r}}^{\frac{(6\ell+2)\pi}{3r}}\int_0^{\sqrt[r]{-2\cos(r\theta)}}\rho^{(k+j)r+1}e^{ir(k-j)\theta}d\rho d\theta\right.\\
&\left.+\int_{\frac{(6\ell+4)\pi}{3r}}^{\frac{(4\ell+3)\pi}{2r}}\int_0^{\sqrt[r]{-2\cos(r\theta)}}\rho^{(k+j)r+1}e^{ir(k-j)\theta}d\rho d\theta\right]\\
&=\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{{}_2F_1\left[\frac{1}{2},k+\frac{1}{r}+\frac{1}{2};k+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right]}{4(kr+1)\left(2k+1+\frac{2}{r}\right)}\\
&+\sum_{0\le j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left[\frac{1}{2}-h,k-h+\frac{1}{r}+\frac{1}{2};k-h+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right]}{2^{k-j-2h}\left((k+j)r+2\right)\left(2k+\frac{2}{r}-2h+1\right)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{D}\cap\{|1+x^r|\le 1\}}\left(|1+x^r|^{\frac{2}{s}}-1\right)dA(x)\\
=&\frac{\pi}{6}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{kr+1}-\frac{\sqrt{3}}{2}\sum_{\substack{0\le k,j\\k\neq j}}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{\chi_{-3}(k-j)}{((k+j)r+2)(k-j)}+\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{{}_2F_1\left[\frac{1}{2},k+\frac{1}{r}+\frac{1}{2};k+\frac{1}{r}+\frac{3}{2};\frac{1}{4}\right]}{4(kr+1)\left(2k+1+\frac{2}{r}\right)}\\
&+\sum_{0\le j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left[\frac{1}{2}-h,\frac{1}{2}-h+k+\frac{1}{r};\frac{3}{2}-h+k+\frac{1}{r};\frac{1}{4}\right]}{2^{k-j-2h}\left((k+j)r+2\right)\left(2k+\frac{2}{r}-2h+1\right)}.
\end{aligned}$$

By combining the above with the result of (21) and Lemma 13 in equation (20) we conclude the proof of the statement. □

6. A limiting property for the areal Mahler measure

In this section we prove Theorem 6, which sheds light on how the change of variables $x \mapsto x^r$ interacts with the areal Mahler measure as $r \to \infty$ in generality.

Proof of Theorem 6. Without loss of generality, we can consider the polynomial case. Let

$$P(x_1,\dots,x_n)=\sum_{m_1,\dots,m_n\ge 0}c_{m_1,\dots,m_n}x_1^{m_1}\cdots x_n^{m_n}\in\mathbb{C}[x_1,\dots,x_n]$$

r	s	$\mathrm{m}_{\mathbb{D}}$	r	s	$\mathrm{m}_{\mathbb{D}}$	r	s	$\mathrm{m}_{\mathbb{D}}$	r	s	$\mathrm{m}_{\mathbb{D}}$
1	1	0.111	2	1	0.074	5	1	0.037	10	1	0.020
1	2	0.074	2	2	0.049	5	2	0.024	10	2	0.013
1	3	0.056	2	3	0.036	5	3	0.018	10	3	0.010
1	4	0.045	2	4	0.029	5	4	0.014	10	4	0.008
1	5	0.037	2	5	0.024	5	5	0.011	10	5	0.006
1	10	0.020	2	10	0.013	5	10	0.006	10	10	0.003
1	20	0.011	2	20	0.006	5	20	0.003	10	20	0.002

TABLE 1. Values of $\mathrm{m}_{\mathbb{D}}(1+x^r+y^s)$ given by Theorem 1.

be a non-zero polynomial and recall that $P(0,x_2,\dots,x_n)$ denotes the polynomial resulting from P by setting $x_1=0$. Given $0\le R<1$, let $\mathbb{D}_R$ denote the disk at the origin of radius R. We have

$$\begin{aligned}
&\lim_{r\to\infty}\int_{\mathbb{D}^{n-1}}\int_{\mathbb{D}_R}\log|P(x_1^r,x_2,\dots,x_n)|\,dA(x_1)\cdots dA(x_n)\\
=&\lim_{r\to\infty}\int_{\mathbb{D}^{n-1}}\int_{\mathbb{D}_R}\log\left|\sum_{m_1,\dots,m_n\ge 0}c_{m_1,\dots,m_n}x_1^{rm_1}\cdots x_n^{m_n}\right|dA(x_1)\cdots dA(x_n)\\
=&\int_{\mathbb{D}^{n-1}}\int_0^{2\pi}\int_0^R\lim_{r\to\infty}\log\left|\sum_{m_1,\dots,m_n\ge 0}c_{m_1,\dots,m_n}\rho^{rm_1}e^{irm_1\theta}x_2^{m_2}\cdots x_n^{m_n}\right|\rho d\rho d\theta dA(x_2)\cdots dA(x_n)\\
=&\pi R^2\int_{\mathbb{D}^{n-1}}\log\left|\sum_{m_1=0,m_2,\dots,m_n\ge 0}c_{0,m_2,\dots,m_n}x_2^{m_2}\cdots x_n^{m_n}\right|dA(x_2)\cdots dA(x_n)\\
=&\pi R^2\int_{\mathbb{D}^{n-1}}\log|P(0,x_2,\dots,x_n)|\,dA(x_2)\cdots dA(x_n)=\pi^nR^2\mathrm{m}_{\mathbb{D}}\left(P(0,x_2,\dots,x_n)\right),
\end{aligned}$$

where the exchanges between integrals and limits follow from the fact that the integrand is bounded above by $\log\left(\sum_{m_1,\dots,m_n\ge 0}|c_{m_1,\dots,m_n}|\right)$. Then

$$\lim_{r\to\infty}\mathrm{m}_{\mathbb{D}}\left(P\left(x_1^r,x_2,\dots,x_n\right)\right)=\lim_{R\to 1^-}R^2\mathrm{m}_{\mathbb{D}}\left(P\left(0,x_2,\dots,x_n\right)\right)=\mathrm{m}_{\mathbb{D}}\left(P\left(0,x_2,\dots,x_n\right)\right).$$

This concludes the proof of Theorem 6. □

7. Conclusion

In this article we have explored how the areal Mahler measure varies under the change of variables $x\mapsto x^r$, where r is a positive integer. This change of variables does not affect the standard Mahler measure and it therefore represents a clear distinction between the standard definition and the areal version.

While it would be difficult to explore the result of these limits directly from the formulas given in Theorems 1 and 4, one can see Theorem 6 in action by doing some numerical experiments. This is illustrated in Table 1, where the values $\mathrm{m}_{\mathbb{D}}(1+x^r+y^r)$ are listed for some choices of r and s. We see, first of all, the symmetry resulting from exchanging r and s, and we also see that the value of $\mathrm{m}_{\mathbb{D}}(1+x^r+y^r)$ approaches zero when r or s grow, as they approach $\mathrm{m}_{\mathbb{D}}(1+y^s)=0$ or $\mathrm{m}_{\mathbb{D}}(1+x^r)=0$ respectively.

Similarly Table 2 illustrates the values of $\mathrm{m}_{\mathbb{D}}((1+x)^r+y^s)$ for some choices of r and s. We see again that as s grows, the value of $\mathrm{m}_{\mathbb{D}}((1+x)^r+y^s)$ approaches zero, the value of $\mathrm{m}_{\mathbb{D}}((1+x)^r)$.

r	s	$m_{\mathbb{D}}$	r	s	$m_{\mathbb{D}}$	r	s	$m_{\mathbb{D}}$
1	1	0.11069	2	1	0.29242	10	1	1.96069
1	2	0.07440	2	2	0.22139	10	2	1.80754
1	3	0.05600	2	3	0.17800	10	3	1.67597
1	4	0.04490	2	4	0.14880	10	4	1.56188
1	5	0.03746	2	5	0.12781	10	5	1.46209
1	10	0.02050	2	10	0.07493	10	10	1.10694
1	10^2	0.00224	2	10^2	0.00886	10	10^2	0.20495
1	10^3	0.00023	2	10^3	0.00090	10	10^3	0.02239

TABLE 2. Values of $m_{\mathbb{D}}((1+x)^r + y^s)$ given by Theorem 4.

The table also shows that the value of $m_{\mathbb{D}}((1+x)^r + y^s)$ grows when r grows. Presumably, the areal Mahler measure is multiplied by r.

It would be interesting to understand these phenomena in full generality, including to describe the difference between $m_{\mathbb{D}}(P)$ and $m_{\mathbb{D}}(P^{(A)})$ for A an $n \times n$ integer matrix with non-zero discriminant as in (3).

REFERENCES

[1] Milton Abramowitz and Irene A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. National Bureau of Standards Applied Mathematics Series, No. 55. U. S. Government Printing Office, Washington, D.C., 1964. For sale by the Superintendent of Documents.

[2] David W. Boyd. Speculations concerning the range of Mahler's measure. *Canad. Math. Bull.*, 24(4):453–469, 1981. URL: `http://dx.doi.org/10.4153/CMB-1981-069-5`, `doi:10.4153/CMB-1981-069-5`.

[3] David W. Boyd. Mahler's measure and special values of L-functions. *Experiment. Math.*, 7(1):37–82, 1998. URL: `http://projecteuclid.org/euclid.em/1047674271`.

[4] François Brunault and Wadim Zudilin. *Many variations of Mahler measures—a lasting symphony*, volume 28 of *Australian Mathematical Society Lecture Series*. Cambridge University Press, Cambridge, 2020.

[5] Kwok-Kwong Stephen Choi and Charles L. Samuels. Two inequalities on the areal Mahler measure. *Illinois J. Math.*, 56(3):825–834, 2012. URL: `http://projecteuclid.org/euclid.ijm/1391178550`.

[6] Christopher Deninger. Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbf{Z}^n$-actions. *J. Amer. Math. Soc.*, 10(2):259–281, 1997. URL: `http://dx.doi.org/10.1090/S0894-0347-97-00228-2`, `doi:10.1090/S0894-0347-97-00228-2`.

[7] Graham Everest and Thomas Ward. *Heights of polynomials and entropy in algebraic dynamics*. Universitext. Springer-Verlag London, Ltd., London, 1999. `doi:10.1007/978-1-4471-3898-3`.

[8] V. Flammang. The Mahler measure and its areal analog for totally positive algebraic integers. *J. Number Theory*, 151:211–222, 2015. `doi:10.1016/j.jnt.2014.12.023`.

[9] Wolfram Research, Inc. Hypergeometric ${}_2F_1$. URL: `http://functions.wolfram.com/07.23.03.2888.01`.

[10] Wolfram Research, Inc. Hypergeometric ${}_2F_1$. URL: `http://functions.wolfram.com/07.23.03.2933.01`.

[11] Matilde N. Lalin and Subham Roy. Evaluations of the areal Mahler measure of multivariable polynomials. *J. Number Theory*, 254:103–145, 2024. `doi:10.1016/j.jnt.2023.07.006`.

[12] Igor E. Pritsker. An areal analog of Mahler's measure. *Illinois J. Math.*, 52(2):347–363, 2008. URL: `http://projecteuclid.org/euclid.ijm/1248355339`.

[13] F. Rodriguez-Villegas. Modular Mahler measures. I. In *Topics in number theory (University Park, PA, 1997)*, volume 467 of *Math. Appl.*, pages 17–48. Kluwer Acad. Publ., Dordrecht, 1999.

[14] C. J. Smyth. On measures of polynomials in several variables. *Bull. Austral. Math. Soc.*, 23(1):49–63, 1981. URL: `http://dx.doi.org/10.1017/S0004972700006894`, `doi:10.1017/S0004972700006894`.

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