Mahler measure of multivariable polynomials and polylogarithms

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April 10th, 2006

Mahler measure for one-variable polynomials

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

$$\lim_{n\to\infty} \frac{|\alpha^{n+1}-1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

For

$$P(x) = a \prod_i (x - \alpha_i)$$

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}$$

$$m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|$$

Kronecker's Lemma

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

where Φ_{n_i} are cyclotomic polynomials

Lehmer's question

Lehmer (1933)

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1)$$

$= \log(1.176280818\ldots) = 0.162357612\ldots$

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

Does there exist $C > 0$, for all $P(x) \in \mathbb{Z}[x]$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

Is the above polynomial the best possible?

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Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|$$

recovers one-variable case.

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Properties

- α algebraic number, and P_α minimal polynomial over $\mathbb{Q}$,

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$$

  where h is the logarithmic Weil height.
- $m(P) \geq 0$ if P has integral coefficients.
- $m(P \cdot Q) = m(P) + m(Q)$

Boyd & Lawton Theorem

$P \in \mathbb{C}[x_1, \dots, x_n]$

$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) = m(P(x_1, x_2, \dots, x_n))$$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

Examples in several variables

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1)$$

- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

$$\mathrm{L}(\chi_{-3},s) = \sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right) \stackrel{?}{=} \frac{\mathrm{L}'(E_k,0)}{B_k} \qquad k\in\mathbb{N}, \quad k\neq 4$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\right) = 2\mathrm{L}'(\chi_{-4},-1)$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right) = \mathrm{L}'(A,0)$$

$$A: y^2 = x^3-44x+112$$

Cassaigne – Maillot (2000) for $a, b, c \in \mathbb{C}^*$,

$$\pi m(a + bx + cy)$$

$$= \begin{cases} D\left(\left|\frac{a}{b}\right| e^{i\gamma}\right) + \alpha \log|a| + \beta \log|b| + \gamma \log|c| & \triangle \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle \end{cases}$$

Bloch–Wigner dilogarithm $(k = 2)$

$$D(x) := \operatorname{Im}(\operatorname{Li}_2(x)) + \arg(1 - x) \log|x|$$

Five-term relation

$$D(x) + D(1 - xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0$$

Mahler measure and hyperbolic volumes

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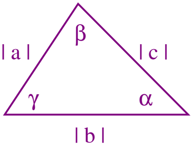
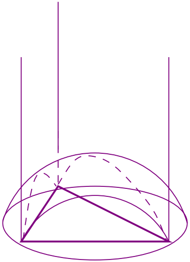
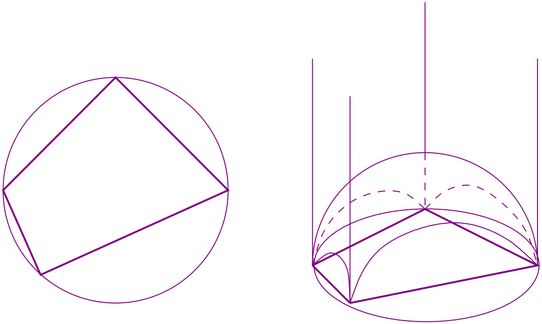
Ideal tetrahedron:





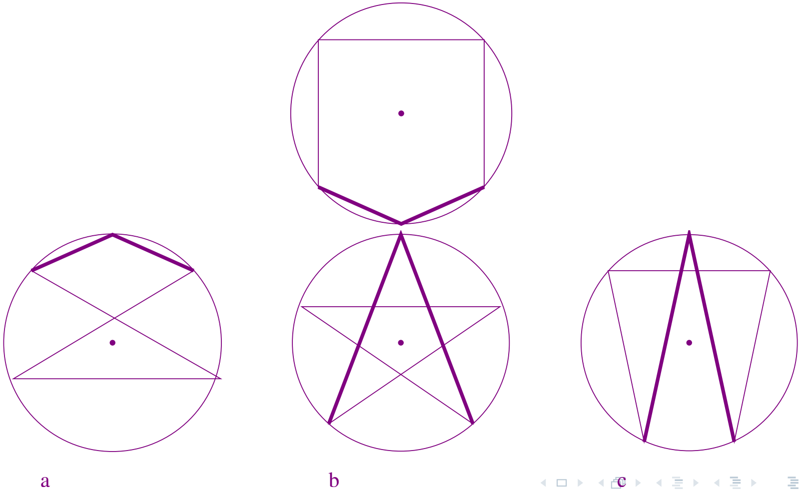
Vandervelde (2003)

$$y = \frac{bx+d}{ax+c} \qquad \text{quadrilateral}$$



L(2004)

$$y = \frac{x^n - 1}{t(x^m - 1)} = \frac{x^{n-1} + \cdots + 1}{t(x^{m-1} + \cdots + 1)} \quad \text{polyhedral}$$



M orientable, complete, one-cusped, hyperbolic manifold.

$$M = \bigcup_{j=1}^{k} \Delta(z_j)$$

$$\mathrm{Vol}(M) = \sum_{j=1}^{k} D(z_j)$$

Boyd (2000)

$$\pi m(A) = \sum V_i$$

A is the A-polynomial.

$V_0 = \mathrm{Vol}(M)$ and the other V_i are pseudovolumes.

- $\mathrm{Im}\, z_i > 0 \rightsquigarrow$ geometric solution to the Gluing and Completeness equations.
- Other solutions $\rightsquigarrow$ pseudovolumes.

Examples in three variables

- Condon (2003):

$$\pi^2 m\left(z - \left(\frac{1-x}{1+x}\right)(1+y)\right) = \frac{28}{5}\zeta(3)$$

- D'Andrea & L. (2003):

$$\pi^2 m\left(z(1-xy)^2 - (1-x)(1-y)\right) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)}$$

- Boyd & L. (2005):

$$\pi^2 m(x^2 + 1 + (x+1)y + (x-1)z) = \pi \mathrm{L}(\chi_{-4}, 2) + \frac{21}{8}\zeta(3)$$

Examples with more than three variables

L.(2003):

- $$\pi^3 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)=24\mathrm{L}(\chi_{-4},4)$$

- $$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right)=62\zeta(5)+\frac{14}{3}\pi^2\zeta(3)$$

- $$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=93\zeta(5)$$

Known fromulas for n.

Polylogarithms

The *kth polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Zagier:

$$\mathcal{L}_k(x) := \mathrm{Re}_k \left(\sum_{j=0}^{k-1} \frac{2^j B_j}{j!} (\log |x|)^j \mathrm{Li}_{k-j}(x) \right)$$

B_j is jth Bernoulli number

$\mathrm{Re}_k = \mathrm{Re}$ or Im if k is odd or even.

One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

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$\mathcal{L}_k$ satisfies lots of functional equations

$$\mathcal{L}_k\left(\frac{1}{x}\right) = (-1)^{k-1}\mathcal{L}_k(x) \qquad \mathcal{L}_k(\bar{x}) = (-1)^{k-1}\mathcal{L}_k(x)$$

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Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F a number field)
- L-function ($\mathrm{L}_F = \zeta_F$)
- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)
- Regulator map $\mathrm{reg} : K \to \mathbb{R}$ ($\mathrm{reg} = \log|\cdot|$)

$$(K \,\mathrm{rank}\, 1) \qquad \mathrm{L}'_X(0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

(Dirichlet class number formula, for F real quadratic, $\zeta'_F(0) \sim_{\mathbb{Q}^*} \log|\epsilon|$, $\epsilon \in \mathcal{O}_F^*$)

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An algebraic integration for Mahler measure

Deninger (1997): General framework

Rodriguez-Villegas (1997) : $P(x, y) \in \mathbb{C}[x, y]$

$$m(P) = m(P^*) - \frac{1}{2\pi} \int_\gamma \eta(x, y)$$

$$\eta(x, y) = \log |x| \mathrm{d} \arg y - \log |y| \mathrm{d} \arg x$$

$$\eta(x, 1 - x) = \mathrm{d}D(x) \qquad \mathrm{d}\eta(x, y) = \mathrm{Im}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}\right)$$

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The three-variable case

$$P(x,y,z) = (1-x) - (1-y)z \quad X = \{P(x,y,z) = 0\}$$

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} \frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \log|z| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x,y,z)$$

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$\Gamma = X \cap \{|x|=|y|=1, |z|\geq 1\}$

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$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x,y,z)$$

$$
\begin{aligned}
\eta(x,y,z) = \log|x| \left(\frac{1}{3}\mathrm{d}\log|y| \wedge \mathrm{d}\log|z| - \mathrm{d}\arg y \wedge \mathrm{d}\arg z\right)\\
+ \log|y| \left(\frac{1}{3}\mathrm{d}\log|z| \wedge \mathrm{d}\log|x| - \mathrm{d}\arg z \wedge \mathrm{d}\arg x\right)\\
+ \log|z| \left(\frac{1}{3}\mathrm{d}\log|x| \wedge \mathrm{d}\log|y| - \mathrm{d}\arg x \wedge \mathrm{d}\arg y\right)
\end{aligned}
$$

$$
\mathrm{d}\eta(x,y,z) = \mathrm{Re}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y} \wedge \frac{\mathrm{d}z}{z}\right)
$$

$$\eta(x, 1-x, y) = \mathrm{d}\,\omega(x, y)$$

where

$$\omega(x, y) = -D(x)\mathrm{d}\arg y$$

$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$z = \frac{1-x}{1-y}$$

$$\eta(x, y, z) = -\eta(x, 1-x, y) - \eta(y, 1-y, x)$$

$$m(P) = \frac{1}{(2\pi)^2}\int_\Gamma \eta(x, 1-x, y) + \eta(y, 1-y, x) = \frac{1}{(2\pi)^2}\int_{\partial\Gamma} \omega(x, y) + \omega(y, x)$$

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$$\omega(x,x) = \mathrm{d}\mathcal{L}_3(x)$$

$$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$$

Maillot: if $P \in \mathbb{R}[x,y,z]$,

$$\partial\Gamma = \gamma = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$$

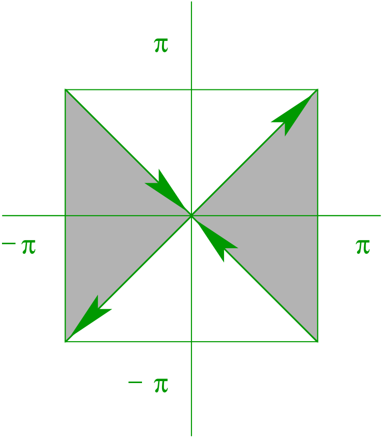
ω defined in

$$C = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\}$$

Want to apply Stokes' Theorem again.

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$



$$m((1-x)-(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y)+\omega(y,x)$$

$$\omega(x,x) = \mathrm{d}\mathcal{L}_3(x)$$

$$= \frac{1}{4\pi^2} 8(\mathcal{L}_3(1) - \mathcal{L}_3(-1)) = \frac{7}{2\pi^2}\zeta(3)$$

In general

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2}\int_\Gamma \eta(x,y,z)$$

Need

$$x \wedge y \wedge z = \sum r_i \;\; x_i \wedge (1-x_i) \wedge y_i$$

in $\bigwedge^3(\mathbb{C}(X)^*) \otimes \mathbb{Q}$,

($\{x,y,z\} = 0$ in $K_3^M(\mathbb{C}(X)) \otimes \mathbb{Q}$) then

$$\int_\Gamma \eta(x,y,z) = \sum r_i \int_\Gamma \eta(x_i, 1-x_i, y_i)$$
$$= \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i)$$

Let

$$R_2(x,y) = [x] + [y] + [1-xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right] = 0$$

in $\mathbb{Z}\left[\mathbb{P}^1_{\mathbb{C}(C)}\right]$.
F field,

$$B_2(F) := \mathbb{Z}[\mathbb{P}^1_F] / \langle [0], [\infty], R_2(x,y) \rangle$$

Need

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$.
Then

$$\int_\gamma \omega(x,y) = \sum r_i \, \mathcal{L}_3(x_i)|_{\partial\gamma}$$

Big picture in three variables

$$\cdots \to K_4(\partial\Gamma) \to K_3(X,\partial\Gamma) \to K_3(X) \to \dots$$

$$\partial\Gamma = X \cap \mathbb{T}^3$$

$$\cdots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C,\partial\gamma) \to K_4(C) \to \dots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

Example in the non-exact case

Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right) \stackrel{?}{=} \frac{\mathrm{L}'(E_k,0)}{B_k} \qquad k\in\mathbb{N}, \quad k\neq 4$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\right) = 2\mathrm{L}'(\chi_{-4},-1)$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right) = \mathrm{L}'(A,0)$$

$$A: y^2 = x^3 - 44x + 112$$

Identities

Boyd (1997), Rodriguez-Villegas (2000)

$$7m(y^2 + 2xy + y - x^3 - 2x^2 - x) = 5m(y^2 + 4xy + y - x^3 + x^2)$$

Rogers (2005)

$$m(4n^2) + m\left(\frac{4}{n^2}\right) = 2m\left(2n + \frac{2}{n}\right)$$

where

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} - k\right)$$

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Idea in the Elliptic Curve case

- For $\{x, y\} \in K_2(E)$:

$$r(\{x, y\}) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \eta(x, y)$$

γ generates $H_1(E, \mathbb{Z})^-$

- $$r(\{x, y\}) = D^E((x) \diamond (y))$$

if $(x), (y)$ supported on $E_{tors}(\bar{\mathbb{Q}})$.

- $$\pi D^E \sim L(E, 2)$$

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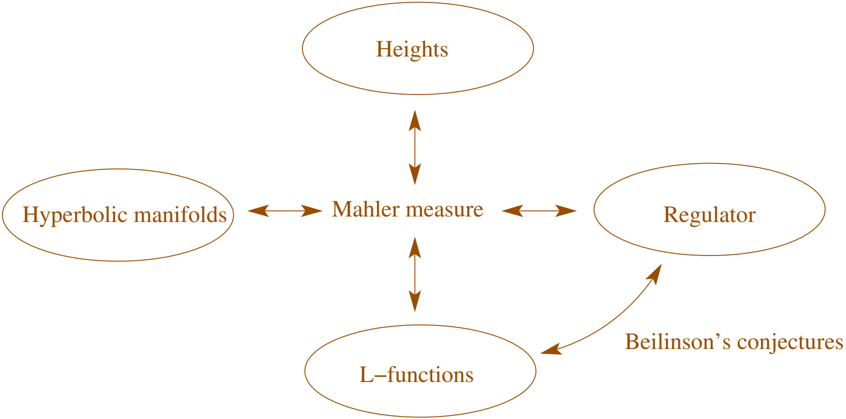
Heights
Hyperbolic manifolds
Mahler measure
Regulator
L–functions
Beilinson's conjectures