

# **An algebraic integration for Mahler measure**

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## Mahler measure

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ , the (logarithmic) *Mahler measure* is :

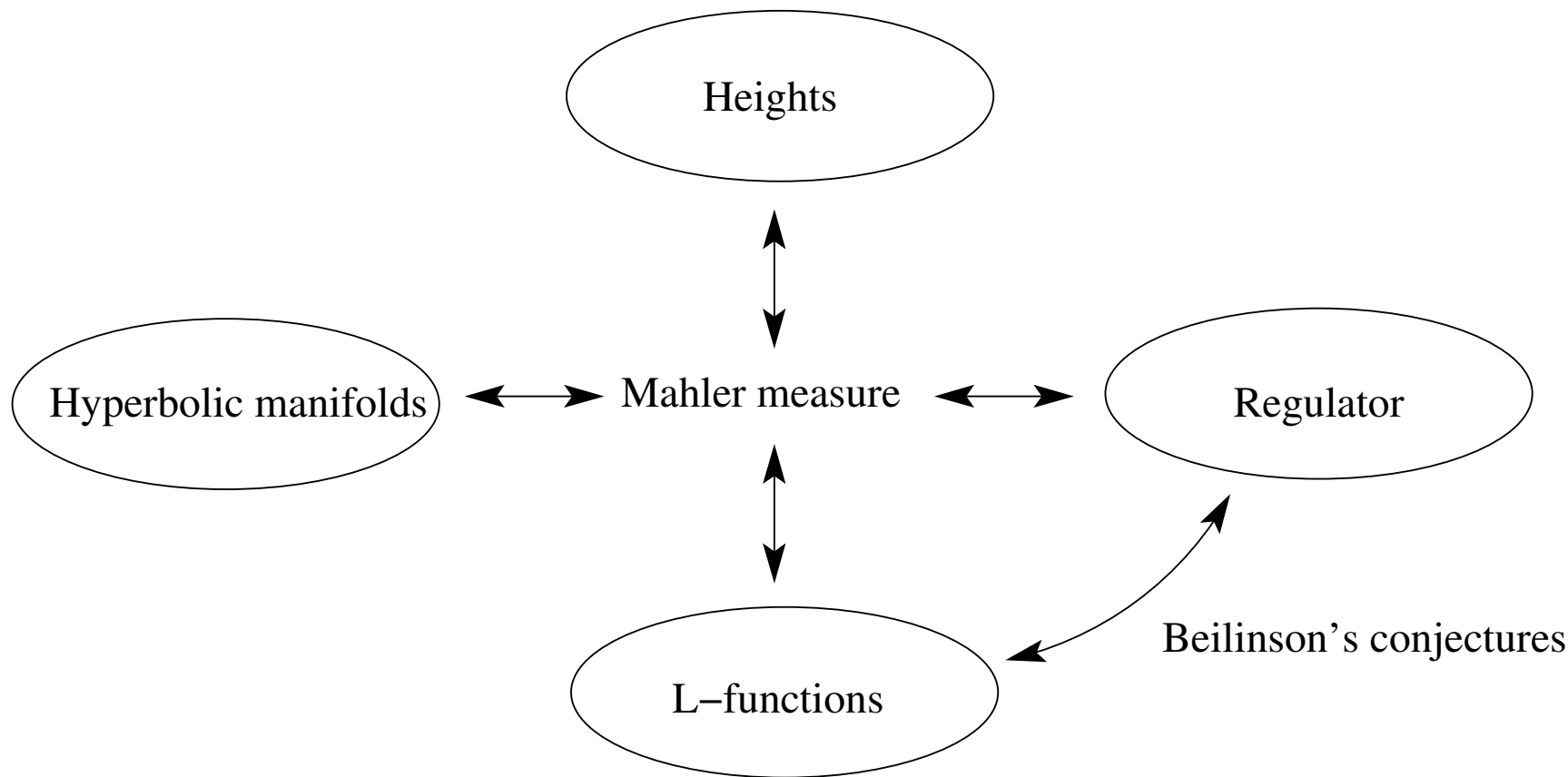
$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \end{aligned}$$

## Examples in several variables

Smyth (1981)

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

$$m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3)$$



## Examples in several variables

L (2003)

$$\pi^{2n} m \left( 1 + \left( \frac{1 - x_1}{1 + x_1} \right) \cdots \left( \frac{1 - x_{2n}}{1 + x_{2n}} \right) z \right)$$

$$= \sum_{h=1}^n c_{n,h} \pi^{2n-2h} \zeta(2h+1)$$

$$= \pi^{2n} \sum_{h=1}^n c'_{n,h} \zeta'(-2h)$$

$$c_{n,h} = \frac{2^{n-2}}{(2n-1)!} s_{n-h}(1^2, \dots, (n-1)^2) (2h)! (2^{2h+1} - 1)$$

## Examples

$$\begin{aligned} \pi^3 m \left( 1 + \left( \frac{1-x_1}{1+x_1} \right) \left( \frac{1-x_2}{1+x_2} \right) \left( \frac{1-x_3}{1+x_3} \right) z \right) \\ = 24L(\chi_{-4}, 4) + \pi^2 L(\chi_{-4}, 2) \end{aligned}$$

$$\begin{aligned} \pi^4 m \left( 1 + \left( \frac{1-x_1}{1+x_1} \right) \cdots \left( \frac{1-x_4}{1+x_4} \right) z \right) \\ = 62\zeta(5) + \frac{14}{3}\pi^2\zeta(3) \end{aligned}$$

$$\pi^4 m \left( 1 + x + \left( \frac{1-x_1}{1+x_1} \right) \left( \frac{1-x_2}{1+x_2} \right) (1+y)z \right) = 93\zeta(5)$$

## Polylogarithms

The  $k$ th polylogarithm is

$$\text{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to  $\mathbb{C} \setminus [1, \infty)$ .

Zagier:

$$P_k(x) := \text{Re}_k \left( \sum_{j=0}^k \frac{2^j B_j}{j!} (\log |x|)^j \text{Li}_{k-j}(x) \right)$$

$B_j$  is  $j$ th Bernoulli number,  $\text{Li}_0(x) \equiv -\frac{1}{2}$ ,

$\text{Re}_k = \text{Re}$  or  $\text{Im}$  if  $k$  is odd or even.

One-valued, real analytic in  $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ ,  
continuous in  $\mathbb{P}^1(\mathbb{C})$ .

$P_k$  satisfies lots of functional equations

$$P_k\left(\frac{1}{x}\right) = (-1)^{k-1} P_k(x) \quad P_k(\bar{x}) = (-1)^{k-1} P_k(x)$$

Bloch–Wigner dilogarithm ( $k = 2$ )

$$D(x) := \operatorname{Im}(\operatorname{Li}_2(x)) + \arg(1 - x) \log |x|$$

Five-term relation

$$D(x) + D(1 - xy) + D(y) + D\left(\frac{1 - y}{1 - xy}\right) + D\left(\frac{1 - x}{1 - xy}\right) = 0$$



## Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object  $X$  (for instance,  $X = \mathcal{O}_F$ ,  $F$  number field)
- L-function ( $L_X = \zeta_F$ )
- Finitely-generated abelian group  $K$  ( $K = \mathcal{O}_F^*$ )
- Regulator map

$$r : K \rightarrow \text{smooth differential forms}$$
$$(r = \log |\cdot|)$$

Conjecture: special value of  $L_X \sim_{\mathbb{Q}^*} \int_{\gamma} r(\xi)$

(E.g. Dirichlet class number formula,  $F$  real quadratic,  $\zeta'_F(0) \sim_{\mathbb{Q}^*} \log |\epsilon|$   $\epsilon \in \mathcal{O}_F^*$ )

## An algebraic integration for Mahler measure

Deninger (1997) : General framework.

Rodriguez-Villegas (1997) :  $P(x, y) \in \mathbb{C}[x, y]$

$$m(P) = m(P^*) - \frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$

$$\eta(x, y) = \log |x| d \arg y - \log |y| d \arg x$$

$$\eta(x, 1 - x) = dD(x)$$

In  $\wedge^2(\mathbb{C}(C)^*) \otimes \mathbb{Q}$ ,

$$x \wedge y = \sum_j r_j z_j \wedge (1 - z_j).$$

$\{x, y\} = 0$  in  $K_2(\mathbb{C}(C)) \otimes \mathbb{Q}$ .

$$\int_{\gamma} \eta(x, y) = \sum r_j \int_{\gamma} \eta(z_j, 1 - z_j) = \sum r_j D(z_j)|_{\partial\gamma}.$$

Big picture

$$\cdots \rightarrow (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \rightarrow K_2(C, \partial\gamma) \rightarrow K_2(C) \rightarrow \cdots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

- $\eta(x, y)$  is exact, then  $\{x, y\} \in K_3(\partial\gamma)$ . We have  $\partial\gamma \neq \emptyset$  and we use Stokes' Theorem.

$\rightsquigarrow$  dilogarithms  $\rightsquigarrow$  zeta function of a number field.

- $\partial\gamma = \emptyset$ , then  $\{x, y\} \in K_2(C)$ . We have  $\eta(x, y)$  is not exact.

$\rightsquigarrow$  L-series of a curve.

We may get combinations of both situations.

## The three-variable case

We follow Smyth's example

$$P(x, y, z) = (1-x) + (1-y)z \quad S = \{P(x, y, z) = 0\}$$

$$\begin{aligned} m(P) &= m(1-y) + \frac{1}{(2\pi i)^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{dx dy dz}{x y z} \\ &= \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{dx dy}{x y} \\ &= -\frac{1}{(2\pi)^2} \int_{\Gamma} \log |z| \frac{dx dy}{x y} \\ &= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x, y, z) \end{aligned}$$

$$\Gamma = S \cap \{|x| = |y| = 1, |z| \geq 1\}$$

$$\begin{aligned}
\eta(x, y, z) = & \log |x| \left( \frac{1}{3} d \log |y| d \log |z| - d \arg y d \arg z \right) \\
& + \log |y| \left( \frac{1}{3} d \log |z| d \log |x| - d \arg z d \arg x \right) \\
& + \log |z| \left( \frac{1}{3} d \log |x| d \log |y| - d \arg x d \arg y \right)
\end{aligned}$$

$$d\eta(x, y, z) = \operatorname{Re} \left( \frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z} \right)$$

### **Theorem 1**

$$\eta(x, 1-x, y) = d\omega(x, y)$$

where

$$\omega(x, y) = -D(x) d \arg y$$

$$+ \frac{1}{3} \log |y| (\log |1-x| d \log |x| - \log |x| d \log |1-x|)$$

$$\eta(x, y, z) = -\eta(x, 1 - x, y) - \eta(y, 1 - y, x)$$

$$m((1 - x) + (1 - y)z) = \frac{1}{4\pi^2} \int_{\gamma} \omega(x, y) + \omega(y, x)$$

## Theorem 2

$$\omega(x, x) = dP_3(x)$$

Want to apply Stokes' Theorem again.

Maillot: if  $P \in \mathbb{R}[x, y, z]$ ,

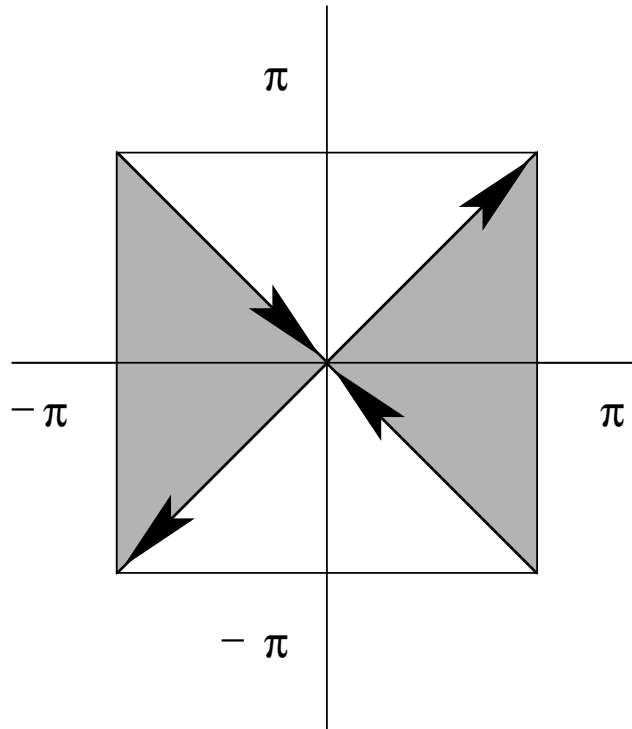
$$\partial\Gamma = \gamma = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$$

$\omega$  defined in

$$C = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\}$$

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$



$$m(P) = \frac{1}{4\pi^2} 8(P_3(1) - P_3(-1)) = \frac{7}{2\pi^2} \zeta(3)$$

In general

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x, y, z)$$

In  $\wedge^3(\mathbb{C}(S)^*) \otimes \mathbb{Q}$ ,

$$x \wedge y \wedge z = \sum r_i x_i \wedge (1 - x_i) \wedge y_i$$

$\{x, y, z\} = 0$  in  $K_3^M(\mathbb{C}(S)) \otimes \mathbb{Q}$ .

Then

$$\begin{aligned} \int_{\Gamma} \eta(x, y, z) &= \sum r_i \int_{\Gamma} \eta(x_i, 1 - x_i, y_i) \\ &= \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i) \end{aligned}$$



Need

$$\{x\}_2 \otimes y = \sum r_i \{x_i\}_2 \otimes x_i$$

in  $(\mathbf{B}_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$ , where

$$\mathbf{B}_2(F) := \mathbb{Z}[\mathbb{P}_F^1]/\text{five term relation}$$

Then

$$\int_{\gamma} \omega(x, y) = \sum r_i \int_{\gamma} \omega(x_i, x_i) = \sum r_i P_3(x_i)|_{\partial\gamma}$$

The condition is  $\{x_i\}_2 \otimes y_i$  is 0 "in"  
 $K_4^{\{3\}}(\mathbb{C}(C))_{\mathbb{Q}}$ .

Big picture II

$$\cdots \rightarrow K_4(\partial\Gamma) \rightarrow K_3(S, \partial\Gamma) \rightarrow K_3(S) \rightarrow \cdots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

$$\cdots \rightarrow (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \rightarrow K_4(C, \partial\gamma) \rightarrow K_4(C) \rightarrow \cdots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

In each step, we have the same two options as before.

## More about regulators

$$r_{\mathcal{D}} : K_{2j-i}^{\{j\}}(X) \rightarrow H_{\mathcal{D}}^i(X, \mathbb{R}(j))$$

"Hard to compute"

Example:

$$K_1^{\{1\}}(X) = \mathcal{O}_X^* \otimes \mathbb{Q} \xrightarrow{r_{\mathcal{D}} = \log |\cdot|} H_{\mathcal{D}}^1(X, \mathbb{R}(1)).$$

Special case  $i = j$ :

$$H_{\mathcal{D}}^i(X, \mathbb{R}(i)) = \{\varphi \in \mathcal{A}^{i-1}(X)(i-1) \mid d\varphi = \pi_{i-1}(\omega), \omega \in F^i(X)\} / d\mathcal{A}^{i-2}(X)(i-1).$$

$\mathcal{A}^i(X)(j)$  smooth  $i$ -forms with values in  $(2\pi i)^j \mathbb{R}$ .

$F^i(X)$  holomorphic  $i$ -forms with at most log singularities.

## Polylogarithmic motivic complexes

(Goncharov)

"Explicit construction" of  $r_{\mathcal{D}}$  and  $K_{2j-i}^{\{j\}}(X)$ .

$F$  field, define  $\mathcal{R}_n(F) \subset \mathbb{Z}[\mathbb{P}_F^1]$  and

$$\mathbf{B}_n(F) := \mathbb{Z}[\mathbb{P}_F^1] / \mathcal{R}_n(F)$$

$$\mathcal{R}_1(F) := \langle \{x\} + \{y\} - \{xy\}, \quad x, y \in F^*, \{0\}, \{\infty\} \rangle$$

$$\mathbb{Z}[\mathbb{P}_F^1] \xrightarrow{\delta_n} \begin{cases} \mathbf{B}_{n-1}(F) \otimes F^* & \text{if } n \geq 3 \\ \wedge^2 F^* & \text{if } n = 2 \end{cases}$$

$$\delta_n(\{x\}) = \begin{cases} \{x\}_{n-1} \otimes x & \text{if } n \geq 3 \\ (1-x) \wedge x & \text{if } n = 2 \\ 0 & \text{if } \{x\} = \{0\}, \{1\}, \{\infty\} \end{cases}$$

$$\mathcal{A}_n(F) := \ker \delta_n$$

$$\mathcal{R}_n(F) := \langle \alpha(0) - \alpha(1), \alpha(t) \in \mathcal{A}_n(F(t)) \rangle$$

### Proposition 3

$$\delta_n(\mathcal{R}_n(F)) = 0$$

$\mathbf{B}_F(n)$  :

$$\mathbf{B}_n(F) \xrightarrow{\delta} \mathbf{B}_{n-1}(F) \otimes F^* \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathbf{B}_2(F) \otimes \wedge^{n-2} F^* \xrightarrow{\delta} \wedge^n F^*$$

$$\delta : \{x\}_p \otimes \wedge_{i=1}^{n-p} y_i \rightarrow \delta_p(\{x\}_p) \wedge \wedge_{i=1}^{n-p} y_i.$$

$$\begin{aligned} H^1(\mathbf{B}_F(1)) &\cong K_1(F) \\ H^1(\mathbf{B}_F(2))_{\mathbb{Q}} &\cong K_3^{\text{ind}}(F)_{\mathbb{Q}} \\ H^2(\mathbf{B}_F(2)) &\cong K_2(F) \\ H^n(\mathbf{B}_F(n)) &\cong K_n^M(F) \end{aligned}$$

### Conjecture 4

$$H^i(\mathbf{B}_F(n) \otimes \mathbb{Q}) \cong K_{2n-i}^{\{n\}}(F)_{\mathbb{Q}}$$

(Goncharov)  $X$  complex algebraic variety. There exist  $\eta_n(m)$  inducing a homomorphism of complexes

$$\begin{array}{ccccccc}
 \mathbf{B}_n(\mathbb{C}(X)) & \xrightarrow{\delta} & \mathbf{B}_{n-1}(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & \wedge^n \mathbb{C}(X)^* \\
 \downarrow \eta_n(1) & & \downarrow \eta_n(2) & & & & \downarrow \eta_n(n) \\
 \mathcal{A}^0(X)(n-1) & \xrightarrow{d} & \mathcal{A}^1(X)(n-1) & \xrightarrow{d} & \dots & \xrightarrow{d} & \mathcal{A}^{n-1}(X)(n-1) \\
 (\mathcal{A}^i(X)(j) = \text{smooth } i\text{-forms with values in } (2\pi i)^j \mathbb{R}) & & & & & & \text{such} \\
 \text{that} & & & & & & 
 \end{array}$$

- $\eta_n(\mathbf{1})(\{x\}_n) = \widehat{P}_n(x)$ .
- $d\eta_n(n)(x_1 \wedge \dots \wedge x_n) = \pi_n \left( \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n} \right)$ .
- $\eta_n(m)$  compatible with residues (residues are given by tame symbols).

**Conjecture 5** *The image of*

$$\tilde{\eta}_n(i) : H^i(\mathbf{B}_X(n)) \rightarrow H_{\mathcal{D}}^i(X, \mathbb{R}(n))$$

*coincides with the image of Beilinson's regulator.*

## Examples we studied

Smyth(1981):

$$\pi^2 m(1 + x + y + z) = \frac{7}{2}\zeta(3)$$

Smyth(2002):

$$\pi^2 m(1 + x + y^{-1} + (1 + x + y)z) = \frac{14}{3}\zeta(3)$$

L (2003):

$$\pi^2 m \left( 1 + \left( \frac{1 - x_1}{1 + x_1} \right) \left( \frac{1 - x_2}{1 + x_2} \right) z \right) = 7\zeta(3)$$

$$\begin{aligned} \pi^2 m \left( 1 + \left( \frac{1 - x_1}{1 + x_1} \right) x + \left( 1 - \left( \frac{1 - x_1}{1 + x_1} \right) \right) y \right) \\ = \frac{7}{2}\zeta(3) + \frac{\pi^2 \log 2}{2} \end{aligned}$$



Condon (2003):

$$\pi^2 m \left( z - \left( \frac{1-x}{1+x} \right) (1+y) \right) = \frac{28}{5} \zeta(3)$$

D'Andrea & L (2003):

$$\begin{aligned} & \pi^2 m \left( z(1-xy)^{m+n} - (1-x)^m (1-y)^n \right) \\ &= 2n(P_3(\phi_1^m) - P_3(-\phi_2^m)) + 2m(P_3(\phi_2^n) - P_3(\phi_1^n)) \end{aligned}$$

$$\pi^2 m ((1-x)(1-y) - (1-w)(1-z)) = \frac{9}{2} \zeta(3)$$

L (2003):

$$\begin{aligned} & \pi^2 m \left( 1 + \left( \frac{1-x_1}{1+x_1} \right) \left( \frac{1-x_2}{1+x_2} \right) x \right. \\ & \left. + \left( 1 - \left( \frac{1-x_1}{1+x_1} \right) \left( \frac{1-x_2}{1+x_2} \right) \right) y \right) \\ &= \frac{21}{4} \zeta(3) + \frac{\pi^2 \log 2}{2} \end{aligned}$$