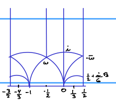
Modular Forms



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① Introduction

"There are five fundamental operations in mathematics: addition, substraction, multiplication, division, and modular forms." Eichler (1912–1992)

Let $Q(x_1,\dots,x_k)$ be a quadratic form of rank k over $\mathbb{Z}$. For example, $Q(x_1,\dots,x_k)=x_1^2+\dots+x_k^2$.

Questions ① What numbers does it represent? For which $n\in\mathbb{Z}$ does $Q(x_1,\dots,x_k)=n$ have a solution in $\mathbb{Z}$?

② Let $r_Q(n)$ denote the number of solutions in $\mathbb{Z}$ of $Q(x_1,\dots,x_k)=n$. What is $r_Q(n)$?

For the specific case of $Q(x_1,\dots,x_k)=x_1^2+\dots+x_k^2$, we write $r_k(n)$. Then Lagrange gave a positive answer to ① for $n>0$ and $k\geq 4$. The cases $k=2,3$ were studied by Fermat and Gauss.

Let us look at question ②. Jacobi defined the θ-function

$$\theta(z)=\sum_{n=-\infty}^{\infty} q^{n^2} \qquad q=e^{2\pi i z}$$

This function is well defined in $\{z\in\mathbb{C} \mid \operatorname{Im}(z)>0\}$

Then $\theta^2(z)=\sum_{l=-\infty}^{\infty} q^{l^2}\sum_{m=-\infty}^{\infty} q^{m^2}=\sum_{l,m} q^{l^2+m^2}=\sum_{n\geq 0} r_2(n)q^n$

Similarly,

$$\theta^k(z)=\sum_{n\geq 0} r_k(n)q^n.$$

We'll see that $\theta(z)$ satisfies

$$\theta^k(z+1)=\theta^k(z), \qquad \theta^k\left(\frac{-1}{4z}\right)=\left(\frac{2z}{i}\right)^{\frac{k}{2}}\theta^k(z).$$

It turns out that the space of functions satisfying these identities is a finite-dimensional $\mathbb{C}$-vector space and that we can give a basis for it: $\{G(z)-2G(2z),\ G(2z)-2G(4z)\}$, where $G(z)=-\frac{1}{24}+\sum_{n=1}^{\infty}\sigma(n)q^n$ is an Eisenstein series and $\sigma(n)=\sum_{d\mid n} d$ is the divisor function. Using this, one can prove that $\theta^4(z)=1+8\sum_{n\geq 1}\sigma(n)q^n-32\sum_{n\geq 1}\sigma(n)q^{4n}$ and

$$r_4(n)=\begin{cases} 8\sigma(n) & 4\nmid n \\ 8\sigma(n)-32\sigma(n/4) & 4\mid n \end{cases} = 8\left(2+(-1)^n\right)\sum_{\substack{d\mid n\\ 2\nmid d}} d > 0 \ !$$

Another example. The series $q\prod_{n\geq 1}(1-q^n)^{24}=\sum_{n\geq 1}\tau(n)q^n$

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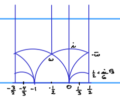

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defines the **Ramanujan τ-function**. It is related to the **partition function** $\sum_{n\geq 0} p(n)x^n = \prod_{n\geq 1}(1-x^n)^{-1}$, where

$$p(n) = \#\{\{a_1,\dots,a_k\} \subseteq \mathbb{Z}_{>0} \mid a_1+\dots+a_k=n\}$$

$\tau(n)$ has amazing properties...

Modular forms. Functions on the complex upper half plane satisfying certain transformation properties. They have applications in analysis, number theory, elliptic curves and representation theory.

<u>Motivation</u> The word **modular** comes from the **moduli space** of complex elliptic curves. A **complex elliptic curve** is given by an equation of the form $E: y^2 = 4x^3 - g_4x - g_6$, where $g_4, g_6 \in \mathbb{C}$ such that $\Delta = g_4^3 - 27g_6^2 \neq 0$

<u>Def:</u> A **lattice** (of full rank) in the complex plane is a subgroup $\Lambda \subseteq \mathbb{C}$ of the form $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where $\omega_1, \omega_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent.

<u>Def:</u> Two lattices Λ_1, Λ_2 are called **homothetic** if there is a $\lambda \in \mathbb{C}^\times$ such that $\Lambda_2 = \lambda\Lambda_1 = \{\lambda\omega \mid \omega \in \Lambda_1\}$.

We write $\Lambda_2 \simeq \Lambda_1$

There is a correspondence

$$\mathcal{L}/\simeq \; = \{\Lambda \subseteq \mathbb{C}\}/_{\text{homothety}} \longleftrightarrow \{E/\mathbb{C}\}/_{\text{isomorphism}}$$

$$\Lambda \longleftrightarrow \mathbb{C}/\Lambda$$

Consider the functions $G: \mathcal{L}/\simeq \longrightarrow \mathbb{C}\cup\{\infty\}$.

They come from $F: \mathcal{L} \longrightarrow \mathbb{C}$ such that

$$F(\lambda\Lambda) = F(\Lambda) \quad \forall \lambda \in \mathbb{C}^\times \; \forall \Lambda \in \mathcal{L}.$$

This is too restrictive. Instead, we consider

<u>Def:</u> A function $\mathcal{F}: \mathcal{L} \longrightarrow \mathbb{C}$ is called **homogeneous of weight** $k \in \mathbb{Z}$ if it satisfies

$$\mathcal{F}(\lambda\Lambda) = \lambda^{-k}\mathcal{F}(\Lambda) \quad \forall \lambda \in \mathbb{C}^\times \; \forall \Lambda \in \mathcal{L}.$$

<u>Ex:</u> If $k \in \mathbb{Z}$, $k > 2$, the **Eisenstein series**

$$G_k: \mathcal{L} \longrightarrow \mathbb{C}$$

is given by $G_k(\Lambda) = \sum_{\omega\in\Lambda-\{0\}} \frac{1}{\omega^k}$ We have

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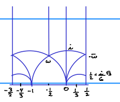

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③ $G_k(\lambda\Lambda)=\lambda^{-k}G_k(\Lambda)$ $\forall \lambda\in\mathbb{C}^\times$ $\forall \Lambda\in\mathcal{L}$

In the elliptic curve correspondence $g_4(\Lambda)=60\,G_4(\Lambda)$, $g_6(\Lambda)=140\,G_6(\Lambda)$

$\Lambda \longrightarrow \mathbb{C}/\Lambda$ given by $y^2=4x^3-g_4(\Lambda)x-g_6(\Lambda)$

The modular group

Def: The complex upper half-plane is given by

$$\mathbb{H}=\{z\in\mathbb{C} \mid \operatorname{Im}(z)>0\}=\{x+iy \mid x,y\in\mathbb{R},\ y>0\}$$

The full modular group is given by

$$SL_2(\mathbb{Z})=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \;\middle|\; a,b,c,d\in\mathbb{Z},\ ad-bc=1\right\}$$

We will see that $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$

These objects appear naturally in the study of homogeneous functions on lattices. Before, we consider more generally $\mathbb{C}\setminus\mathbb{R}$, which is acted upon by

$GL_2(\mathbb{Z})=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \;\middle|\; a,b,c,d\in\mathbb{Z},\ ad-bc=\pm 1\right\}$.

For $z\in\mathbb{C}\setminus\mathbb{R}$, consider $\Lambda_z=\mathbb{Z}z+\mathbb{Z}$.

For any lattice $\Lambda\subseteq\mathbb{C}$, we have

$\mathbb{Z}\omega_1+\mathbb{Z}\omega_2=\omega_2\Lambda_z$ with $z=\omega_1/\omega_2\in\mathbb{C}\setminus\mathbb{R}$.

Swapping ω_1 and ω_2 if necessary, we can assume $z\in\mathbb{H}$

Thus we have $F:\mathcal{L}\longrightarrow\mathbb{C}$ homogeneous is determined by $f:\mathbb{H}\longrightarrow\mathbb{C}$ given by $f(z):=F(\Lambda_z)$.

To study the properties of $f(z)$ we consider an action on $\mathbb{C}\setminus\mathbb{R}$ which restricts to an action on $\mathbb{H}$.

Let $\Lambda=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$, $\omega_1,\omega_2\in\mathbb{C}$, $\operatorname{Im}(\omega_1/\omega_2)>0$ If (ω_1',ω_2') is another basis, there must be $a,b,c,d\in\mathbb{Z}$ such that $\omega_1'=a\omega_1+b\omega_2$, $\omega_2'=c\omega_1+d\omega_2$, or $\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix}=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$

For (ω_1',ω_2') to be a basis, $\gamma:=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ must be invertible, so $\det\gamma=\pm1$ and $\gamma\in GL_2(\mathbb{Z})$. Further, we get $z'=\gamma z=\dfrac{az+b}{cz+d}$.

The number $j(\gamma,z)=cz+d$ is called factor of automorphy.

Prop: Let $\gamma,\gamma'\in GL_2(\mathbb{R})$ and $z\in\mathbb{C}\setminus\mathbb{R}$. Then

(i) $\operatorname{Im}(\gamma z)=\dfrac{\det(\gamma)\operatorname{Im}(z)}{|j(\gamma,z)|^2}$ (ii) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}z=z$

(iii) $\gamma(\gamma' z) = (\gamma\gamma') z$

Proof: (i) $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\operatorname{Im}(\gamma z) = \operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \operatorname{Im}\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} =$$

$$= \operatorname{Im}\left(\frac{ac|z|^2 + bd + adz + bc\bar{z}}{|cz+d|^2}\right) = \frac{(ad-bc)\operatorname{Im}(z)}{|cz+d|^2} = \frac{\det(\gamma)\operatorname{Im}(z)}{|j(\gamma,z)|^2}$$

(ii) & (iii) are excercises. #

Coro: Let $\omega_1/\omega_2 \in \mathbb{H}$. Then $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ and $\omega_1'/\omega_2' \in \mathbb{H}$ if and only if

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \gamma \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \quad \text{for some } \gamma \in SL_2(\mathbb{Z})$$

Consider

$$GL_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R},\ ad-bc>0 \right\}$$

Coro: (i) The map

$$GL_2(\mathbb{R}) \times \mathbb{C}\setminus\mathbb{R} \longrightarrow \mathbb{C}\setminus\mathbb{R}$$
$$(\gamma, z) \longrightarrow \gamma z$$

defines an action of $GL_2(\mathbb{R})$ on $\mathbb{C}\setminus\mathbb{R}$.

(ii) The map

$$GL_2^+(\mathbb{R}) \times \mathbb{H} \longrightarrow \mathbb{H}$$
$$(\gamma, z) \longrightarrow \gamma z$$

defines an action of $GL_2^+(\mathbb{R})$ on $\mathbb{H}$.

The actions above induce actions of $GL_2(\mathbb{Z})$ on $\mathbb{C}\setminus\mathbb{R}$ and of $SL_2(\mathbb{Z})$ on $\mathbb{H}$. (We will care mostly about this last action)

Prop: Let $F : \mathcal{L} \longrightarrow \mathbb{C}$ be a homogeneous function of weight $k \in \mathbb{Z}$ and let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be given by $f(z) = F(\Lambda_z)$. Then

$$f(\gamma z) = j(\gamma,z)^k f(z) \qquad \forall \gamma \in SL_2(\mathbb{Z})\ \forall z \in \mathbb{H}$$

Proof: Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathbb{H}$. We have

$$\mathbb{Z}(az+b) + \mathbb{Z}(cz+d) = \mathbb{Z}z + \mathbb{Z}. \quad \text{This gives}$$

$$\Lambda_{\gamma z} = \mathbb{Z}\frac{az+b}{cz+d} + \mathbb{Z} = (cz+d)^{-1}\left(\mathbb{Z}(az+b) + \mathbb{Z}(cz+d)\right) =$$

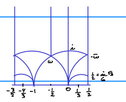
$$= (cz+d)^{-1}(\mathbb{Z}z + \mathbb{Z}) = j(\gamma,z)^{-1}\Lambda_z \Longrightarrow$$

$$f(\gamma z) = F(\Lambda_{\gamma z}) = F\left(j(\gamma,z)^{-1}\Lambda_z\right) = j(\gamma,z)^k F(\Lambda_z) = j(\gamma,z)^k f(z)$$

#

Rem: Some people work with the projective modular group

$$PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) / \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$



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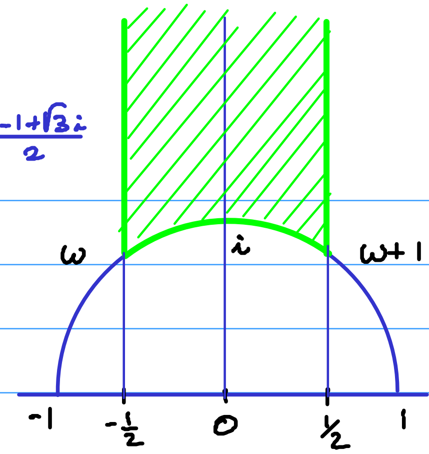
A fundamental domain

Let D be a closed subset of $\mathbb{H}$ given by

$\omega = \frac{-1+\sqrt{3}i}{2}$



$$D = \{z \in \mathbb{H} \mid -1/2 \le \operatorname{Re} z \le 1/2,\ |z| \ge 1\}$$

Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$

Notice that $S^4 = 1$ and $(ST)^3 = S^2$

Thm: Let $D \subseteq \mathbb{H}$ defined as above.

(i) Every point in $\mathbb{H}$ is equivalent, under the action of $SL_2(\mathbb{Z})$ to a point of D.

(ii) If $z, z' \in D$ are two distinct points in the same $SL_2(\mathbb{Z})$-orbit, then either $z' = z \pm 1$ (they are on the vertical lines) or $z' = -1/z$ (they are on the circular part of the boundary)

(iii) Let $z \in D$ and let H_z be the stabilizer of $SL_2(\mathbb{Z})$. Then H_z is

$$\begin{cases} \text{Cyclic of order } 6 \text{ generated by } ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} & \text{if } z = \omega, \\ \text{Cyclic of order } 6 \text{ generated by } TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} & \text{if } z = \omega+1, \\ \text{Cyclic of order } 4 \text{ generated by } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{if } z = i, \\ \text{Cyclic of order } 2 \text{ generated by } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \text{otherwise.} \end{cases}$$

(iv) The group $SL_2(\mathbb{Z})$ is generated by S and T.

Proof: (i) Let $z \in \mathbb{H}$. We have.

$$\operatorname{Im}(\gamma z) = \frac{\operatorname{Im}(z)}{|cz+d|^2} \quad \text{if} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \langle S, T \rangle$$

For a fixed z there are finitely many $(c,d) \in \mathbb{Z}^2$ with $|cz+d| < 1$

Thus we can find $\gamma \in \langle S, T \rangle$ such that

$$|cz+d| \le |c'z+d'| \quad \forall \gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \langle S, T \rangle$$ and this implies

$\operatorname{Im}(\gamma z) \ge \operatorname{Im}(\gamma' z)$ $\forall \gamma' \in \langle S, T \rangle$.

By multiplying on the left by a power of T, which has the effect of translating γz by an integer, we may choose $\gamma \in \langle S, T \rangle$ such that $-\frac{1}{2} \le \operatorname{Re}(\gamma z) \le 1/2$.

We have $\operatorname{Im}(\gamma z) \ge \operatorname{Im}(S\gamma z) = \operatorname{Im}(-1/\gamma z) = \frac{\operatorname{Im}(\gamma z)}{|\gamma z|^2}$

(6)

Thus $|\gamma z| \geq 1$ and $\gamma z \in \mathcal{D}$.

We have that for any $z \in \mathbb{H}$, there is $\gamma \in \langle S,T \rangle \subseteq SL_2(\mathbb{Z})$ such that $\gamma z \in \mathcal{D}$.

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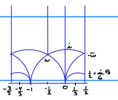

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(ii) (iii) Let $z, z' \in \mathcal{D}$ distinct points in the same $SL_2(\mathbb{Z})$-orbit.

Assume $\operatorname{Im}(z') \geq \operatorname{Im}(z)$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $\gamma z = z'$. Thus $\operatorname{Im}(z') = \dfrac{\operatorname{Im}(z)}{|cz+d|^2} \leq \dfrac{\operatorname{Im}(z')}{|cz+d|^2} \Rightarrow$ $|cz+d| \leq 1$.

By the identity $|cz+d|^2 = |cx+d|^2 + |cy|^2$ $(z = x+iy)$ and the fact that $y > \frac{\sqrt{3}}{2}$ since $z \in \mathcal{D}$, this is only possible if $|c| \leq 1$.

If $c=0$, since $ad-bc=1$, then $a=d=\pm 1$ and $z' = z \pm b$. Since $\operatorname{Re}(z), \operatorname{Re}(z') \in [-\frac{1}{2}, \frac{1}{2}]$, then $z = z' \pm 1$, $b = \pm 1$, and $\operatorname{Re}(z) = \pm \frac{1}{2}$.

If $c=1$, then $1 \geq |cz+d|^2 = |z+d|^2 = |x+d|^2 + |y|^2$ and $|x+d|^2 \leq 1 - |y|^2 \leq 1 - \frac{3}{4} = \frac{1}{4}$. Thus $|x+d| \leq \frac{1}{2}$ and $|d| \leq 1$.

If $d=0$, $|z|=1$. Also $b=-1$ and $z' = \dfrac{az-1}{z+0} = a - \dfrac{1}{z}$.

This is only possible if $a=0$, if $z=\omega$ and $a=-1$, and if $z = \omega+1$ and $a=1$.

If $d = \pm 1$, $x = \mp \frac{1}{2}$ and $|y|^2 \leq \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2} \Rightarrow z = \omega$ for $d=1$ or $z = \omega+1$ for $d=-1$.

If $d=1$, $a-b=1$ and $(a,b) = (1,0)$ or $(0,-1)$.

The case $c=-1$ is analogous, since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $-\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ act on $\mathbb{H}$ in the same way. Putting everything together,

γ	$\in \langle S,T \rangle$	z	$z' = \gamma z$	Fixed points
$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\pm I$	$\forall z \in \mathcal{D}$	z	$\forall z \in \mathcal{D}$.
$\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\pm T$	$\operatorname{Re}(z) = -1/2$	$z+1$	None
$\pm\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	$\pm T^{-1}$	$\operatorname{Re}(z) = 1/2$	$z-1$	None
$\pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\pm S$	$\|z\|=1$	$-1/z$	i
$\pm\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$\pm T^{-1}S$	ω	ω	ω
$\pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$\pm ST$	ω	ω	ω
$\pm\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$	$\pm TS$	$\omega+1$	$\omega+1$	$\omega+1$
$\pm\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$\pm ST^{-1}$	$\omega+1$	$\omega+1$	$\omega+1$.

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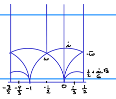

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(7) (iv) Let $z \in D^\circ$ (interior) and $\gamma \in SL_2(\mathbb{Z})$. By part (i) there is $\gamma_0 \in \langle S,T\rangle$ such that $\gamma_0(\gamma z) \in D$. Then both $z, \gamma_0(\gamma z) \in D$. Since z is not on the boundary, $z = \gamma_0(\gamma z)$ and $\gamma_0 \gamma = \pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Thus $\gamma = \pm \gamma_0^{-1} \in \langle S,T\rangle$.