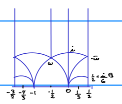
Modular Forms



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L-functions In modern number theory L-functions are fundamental tools for studying arithmetic objects and the relations among them.

Ex: Riemann ζ-function, Dirichlet L-functions, L-functions for elliptic curves.

Modular forms are a "source" of L-functions. Wiles proved in 1994 a relationship between L-functions of modular forms and of elliptic curves, which is the basis of the proof of Fermat's Last Theorem.

Also, all the partial results known for BSD rest on modular forms.

The Mellin transform

Lemma: Let $g:(0,\infty)\longrightarrow \mathbb{C}$ be a continuous function such that for some real numbers $a<b$ we have

$|g(t)| \ll t^{-a}$ as $t\longrightarrow 0$, and $|g(t)|\ll t^{-b}$ as $t\longrightarrow\infty$

Then the integral

$$\mathcal{M}g(s)=\int_0^\infty g(t)\,t^s\,\frac{dt}{t}$$

converges absolutely and uniformly on compact subsets of the strip $\{s\in\mathbb{C} \mid a<\operatorname{Re}s<b\}$.

Proof: Let $\alpha,\beta\in\mathbb{R}$, $a<\alpha<\beta<b$. Then

$$\int_0^\infty |g(t)\,t^s|\,\frac{dt}{t} \ll \int_0^1 t^{-a}\,t^{\operatorname{Re}s}\,\frac{dt}{t}+\int_1^\infty t^{-b}\,t^{\operatorname{Re}s}\,\frac{dt}{t}$$

$$\leq \int_0^1 t^{\alpha-a}\,\frac{dt}{t}+\int_1^\infty t^{\beta-b}\,\frac{dt}{t}=\frac{1}{\alpha-a}+\frac{1}{b-\beta}$$

This implies that the integral defining $\mathcal{M}g(s)$ converges absolutely and uniformly on the strip $\{s\in\mathbb{C} \mid \alpha\leq\operatorname{Re}s\leq\beta\}$ and the claim follows. #

Def: For a function g satisfying the assumptions of the Lemma, the function $\mathcal{M}g$ is called the Mellin transform of g.

Rem: The Mellin inversion formula expresses $g(t)$ in terms of $\mathcal{M}g(s)$ as

$$g(t)=\frac{1}{2\pi i}\int_{\operatorname{Re}(s)=c}\mathcal{M}g(s)\,t^{-s}\,ds,$$

where $c \in \mathbb{R}$, $a < c < b$, and the integral is taken over the vertical line $\mathrm{Re}(s) = c$ in the upward direction.

Ex: The Γ-function is defined as the Mellin transform of the function $t \longrightarrow \exp(-t)$:

$$\Gamma(s) = \int_0^\infty \exp(-t)\, t^s \,\frac{dt}{t},$$

where the integral converges absolutely for $\mathrm{Re}(s) > 0$. Using integration by parts one gets $\Gamma(s+1) = s\Gamma(s)$, $\mathrm{Re}(s) > 0$ and this relation can be used to extend the Γ-function to a meromorphic function on $\mathbb{C}$ with poles in the non-positive integers and no other poles.

The L-function of a modular form

Prop: Let Γ be a congruence subgroup, $k \in \mathbb{Z}_{>0}$, and $f \in M_k(\Gamma)$ with Fourier expansion at ∞ given by

$$f(z) = \sum_{n=0}^{\infty} a_n(f) \exp\left(\frac{2\pi i n z}{h}\right) \quad \left(\text{with } h = h_\Gamma([\infty]) \in \mathbb{Z}_{>0}\right)$$

Then there exists a $C \in \mathbb{R}_{>0}$ such that for all $n \in \mathbb{Z}_{>0}$,

$$|a_n(f)| \le C\, n^{k/2} \text{ if } f \in S_k(\Gamma),$$

$$|a_n(f)| \le C\, n^{k-1} \text{ if } f \in \mathcal{E}_k(\Gamma).$$

Proof: We will only prove the case $f \in S_k(\Gamma)$, which is what we need. Consider the associated function $\tilde{f}$ on $\mathbb{D}$

$$\tilde{f}(q_h) = \sum_{n=1}^{\infty} a_n(f)\, q_h^n \qquad \tilde{f}\left(\exp\left(\frac{2\pi i z}{h}\right)\right) = f(z).$$

Note that $a_n(f)$ is the coefficient of q_h^{-1} in the Laurent expansion of $\tilde{f}(q_h)/q_h^{n+1}$ around $q_h = 0$. By Cauchy's formula

$$a_n(f) = \frac{1}{2\pi i} \oint_{C_r} \frac{\tilde{f}(q_h)}{q_h^n} \frac{dq_h}{q_h},$$ where C_r is a positively oriented circle of radius r, $0 < r < 1$.

Write $r = \exp\left(-\frac{2\pi y}{h}\right)$ with $y > 0$. Then we parametrize C_r by $q_h = \exp\left(\frac{2\pi i (x + iy)}{h}\right)$, $0 \le x \le h$.

$$a_n = \frac{1}{h} \int_0^h f(x + iy) \exp\left(-\frac{2\pi i n (x+iy)}{h}\right) dx.$$

Letting $q = \frac{1}{n}$, $\quad a_n(f) = \frac{\exp(2\pi/n)}{n} \int_0^n f\left(x + \frac{i}{n}\right) \exp\left(\frac{-2\pi i n x}{n}\right) dx$

By Lemma from Notes 9, $z \longrightarrow |f(z)|^2 \operatorname{Im}(z)^k$ is bounded on $\mathbb{H}$. (for f a cusp) So there is $C' > 0$ such that

$$|f(x+i/n)| \leq C' n^{k/2}$$

Looking at the integral, we get

$$|a_n(f)| \leq \exp(2\pi/n)\, C' n^{k/2}$$

This proves the result for $f \in S_k(\Gamma)$.

For $f \in E_k(\Gamma)$, see Chapter 4 in Diamond-Shurman #.

Let Γ be a congruence subgroup. Let f be a Hecke eigenform normalized so that $a_1(f) = 1$.

We can define the L-function as follows. Suppose that:

$$|a_n(f)| \ll n^r \text{ as } n \longrightarrow \infty.$$

By the previous Prop, if f is any modular form of weight $k \in \mathbb{Z}_{>0}$, we can take $r = \max\{k-1, k/2\}$. If $f \in S_k(\Gamma)$ we can take $r = k/2$. Then for any $a > r+1$ the Dirichlet series $\displaystyle L(f,s) = \sum_{n \geq 1} \frac{a_n(f)}{n^s}$ converges uniformly on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) \geq a\}$

This therefore defines a holomorphic function of $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > r+1\}$

(Even though the constant term $a_0(f)$ may be non-zero, it does not appear in the definition of $L(f,s)$).

It is not immediately clear that this function can be analytically continued to a meromorphic function on all of $\mathbb{C}$ that satisfies a functional equation. These properties will follow from the transformation properties of f under the Atkin-Lehner operator ω_N.

Assume that there is a normalized Hecke eigenform f^* and a complex number η_f satisfying

$$f|_k \alpha_N = \eta_f f^* \quad (\star)$$

Since $\alpha_N^2 = \begin{pmatrix} -N & 0 \\ 0 & -N \end{pmatrix}$, we have $f^*|_k \alpha_N = \eta_{f^*} f$, where $\eta_f \eta_{f^*} = (-N)^k$

We define the completed L-function attached to f as

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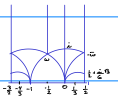

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$$\Lambda(f,s) = N^{s/2}\frac{\Gamma(s)}{(2\pi)^s} L(f,s).$$

Prop: Let $f \in S_k(\Gamma_1(N))$ be a primitive form. Then there exists a primitive form $f^* \in S_k(\Gamma_1(N))$ and a complex number $\eta_f \neq 0$ satisfying (★)

Proof: Let a_p and $\varepsilon(d)$ denote the eigenvalues of the Hecke operators T_p and the diagonal operators $\langle d\rangle$ on f. Write $g = f|_k \alpha_N$. For every prime p, we use that $\alpha_N = \begin{pmatrix}0&-1\\N&0\end{pmatrix}$ normalizes $\Gamma_1(N)$ and $\alpha_N\begin{pmatrix}1&0\\0&p\end{pmatrix}\alpha_N^{-1} = \begin{pmatrix}p&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&p\end{pmatrix}^t$ to deduce $T_p^+ = w_N^{-1} T_p w_N$.

If $p \nmid N$, then $T_p^+ = \langle p\rangle^{-1} T_p$.

This implies that g is an eigenform for T_p, $p\nmid N$, indeed:

$$T_p g = T_p(f|_k \alpha_N) = T_p T_{\alpha_N}(f) = T_p w_N(f) =$$
$$w_N(w_N^{-1}T_p w_N)(f) = w_N T_p^+(f) = w_N \langle p\rangle^{-1} T_p(f) =$$
$$= \varepsilon(p)^{-1} a_p w_N f = \varepsilon(p)^{-1} a_p g.$$

Therefore g is an eigenform for all Hecke operators T_m with $(m,N)=1$. Similarly, let $\begin{pmatrix}a&b\\c&d\end{pmatrix} \in \Gamma_0(N)$

$$\begin{pmatrix}0&-1\\N&0\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0&1/N\\-1&0\end{pmatrix} = \begin{pmatrix}d&-c/N\\Nb&a\end{pmatrix} \Rightarrow w_N^{-1}\langle d\rangle w_N = \langle d\rangle^{-1}$$

$$\langle d\rangle g = \langle d\rangle(f|_k \alpha_N) = \langle d\rangle w_N(f) = w_N(w_N^{-1}\langle d\rangle w_N)(f)$$

$= w_N\langle d\rangle^{-1} f = \varepsilon(d)^{-1} w_N f = \varepsilon(d)^{-1} g$ and g is an eigenform for $\langle d\rangle$

Thus we can write g as a scalar multiple of a primitive form.

And $g = \eta_f f^*$ #

Rem: In fact, one can prove that $f^*(z) = \overline{f(-\bar{z})}$. (Problem 9, HW 5)

Thm: Let f be a normalized Hecke eigenform of weight k for $\Gamma_1(N)$ and let f^* and η_f as before.

(i) The function $\Lambda(f,s)$ can be continued to a meromorphic function on $\mathbb{C}$ with at most simple poles at $s=0$ and $s=k$, and no other poles. If f is a cusp form, then $\Lambda(f,s)$ is holomorphic on $\mathbb{C}$.

(ii) The functions $\Lambda(f,s)$ and $\Lambda(f^*,s)$ are related by the functional equation

$$\Lambda(f,k-s) = \epsilon_f \Lambda(f^*,s),$$

where $\epsilon_f = i^k \eta_f N^{-k/2}$

Proof: We rewrite ☆ as $f\left(-\frac{1}{Nz}\right) = \eta_f z^k f^*(z) \quad \forall z \in \mathbb{H}$

Because $f^*(it)$ is bounded as $t \to \infty$, this formula implies $|f(it)| \ll t^{-k}$ as $t \to 0$. Moreover, we have $|f(it) - a_0(f)| \ll \exp(-2\pi t)$ as $t \to \infty$. This implies that the integral defining the Mellin transform of $f(it) - a_0(f)$ converges uniformly on compact subsets of $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > k\}$

If r is a real number such that $|a_n(f)| \ll n^{-r}$ as $n \to \infty$, then we can compute the Mellin transform for $\mathrm{Re}(s) > \max\{k, r+1\}$

$$\int_0^\infty (f(it) - a_0(f)) t^s \frac{dt}{t} = \int_0^\infty \sum_{n \geq 1} a_n(f) \exp(-2\pi n t) t^s \frac{dt}{t} =$$

$$= \sum_{n \geq 1} a_n(f) \int_0^\infty \exp(-2\pi n t) t^s \frac{dt}{t} = \sum_{n \geq 1} a_n(f) (2\pi n)^{-s} \int_0^\infty \exp(-u) u^s \frac{du}{u} =$$

$$= \frac{\Gamma(s)}{(2\pi)^s} \sum_{n \geq 1} a_n(f) n^{-s} = \frac{\Gamma(s)}{(2\pi)^s} L(f, s).$$

On the other hand, we can write for $\mathrm{Re}(s) > k$.

$$\int_0^\infty (f(it) - a_0(f)) t^s \frac{dt}{t} = \int_{1/\sqrt{N}}^\infty (f(it) - a_0(f)) t^s \frac{dt}{t} + \int_0^{1/\sqrt{N}} (f(it) - a_0(f)) t^s \frac{dt}{t}$$

The second term is

$$\int_0^{1/\sqrt{N}} (f(it) - a_0(f)) t^s \frac{dt}{t} = \int_0^{1/\sqrt{N}} f(it) t^s \frac{dt}{t} - a_0(f) N^{-s/2} \frac{1}{s}$$

☆

$$= i^k \eta_f N^{-s} \int_{1/\sqrt{N}}^\infty f^*(it) t^{k-s} \frac{dt}{t} - a_0(f) N^{-s/2} \frac{1}{s}$$

$$= i^k \eta_f N^{-s} \int_{1/\sqrt{N}}^\infty (f^*(it) - a_0(f^*)) t^{k-s} \frac{dt}{t} + i^k \eta_f N^{-s} a_0(f^*) N^{\frac{s-k}{2}} \frac{1}{s-k} - a_0(f) N^{-s/2} \frac{1}{s}$$

Putting everything together,

$$\int_0^\infty (f(it) - a_0(f)) t^s \frac{dt}{t} = \int_{1/\sqrt{N}}^\infty (f(it) - a_0(f)) t^s \frac{dt}{t}$$

$$+ i^k \eta_f N^{-s} \int_{1/\sqrt{N}}^\infty (f^*(it) - a_0(f^*)) t^{k-s} \frac{dt}{N} - i^k \eta_f a_0(f^*) N^{-\frac{(s+k)}{2}} \frac{1}{s-k} - a_0(f) N^{-s/2} \frac{1}{s}$$

Because $|f(it) - a_0(f)|$ and $|f^*(it) - a_0(f^*)|$ are bounded by a constant times $\exp(-2\pi t)$ as $t \to \infty$, both integrals on the right-

HAND SIDE CONVERGE UNIFORMLY FOR s IN COMPACT SUBSETS OF $\mathbb{C}$.

COMBINING THE TWO EXPRESSIONS FOR THE MELLIN TRANSFORM OF $f(it)-a_0(f)$ COMPUTED ABOVE, WE GET,

$$\Lambda(f,s) = N^{s/2}\int_{1/\sqrt{N}}^{\infty}\big(f(it)-a_0(f)\big)t^s\frac{dt}{t} + i^k\eta_f N^{-s/2}\int_{1/\sqrt{N}}^{\infty}\big(f^*(it)-a_0(f^*)\big)t^{k-s}\frac{dt}{t}$$
$$-a_0(f)\frac{1}{s} + i^k\eta_f N^{-k/2}a_0(f^*)\frac{1}{s-k}.$$

THIS PROVES (i). COMPARING THE FORMULA ABOVE WITH THE CORRESPONDING FORMULA FOR $\Lambda(f^*,s)$,

$$\Lambda(f^*,s) = N^{s/2}\int_{1/\sqrt{N}}^{\infty}\big(f^*(it)-a_0(f^*)\big)t^s\frac{dt}{t} + i^k\eta_{f^*} N^{-s/2}\int_{1/\sqrt{N}}^{\infty}\big(f(it)-a_0(f)\big)t^{k-s}\frac{dt}{t}$$
$$-a_0(f^*)\frac{1}{s} + i^k\eta_{f^*}N^{-k/2}a_0(f)\frac{1}{s-k},$$ WE GET

$$\Lambda(f,k-s) = i^k\eta_f N^{-k/2}\Lambda(f^*,s) \qquad \#$$

Ex: LET f BE A NORMALIZED EIGENFORM OF WEIGHT k FOR $\Gamma_1(N)$. LET a_p AND $\chi(d)$ BE THE EIGENVALUES OF THE HECKE OPERATORS T_p AND THE DIAMOND OPERATORS $\langle d\rangle$ ON f, RESPECTIVELY. PROVE THAT $L(f,s)$ CAN BE EXPRESSED AS AN EULER PRODUCT,

$$L(f,s) = \prod_{p \text{ prime}}\left(1-a_p p^{-s}+\chi(p)p^{k-1-2s}\right)^{-1} \qquad \operatorname{Re}(s)>S$$

FOR $S\in\mathbb{R}$ SUFFICIENTLY LARGE, WHERE THE INFINITY PRODUCT CONVERGES ON COMPACT SUBSETS OF THE RIGHT HALF-PLANE $\{s\in\mathbb{C} \mid \operatorname{Re}(s)>S\}$ AND DEFINES AN HOLOMORPHIC FUNCTION ON THIS HALF-PLANE.

Proof (IDEA): ONE CAN SEE THAT NORMALIZED EIGENFORMS MUST SATISFY

(i) $a_1(f)=1$

(ii) $a_{p^r}(f) = a_p(f)a_{p^{r-1}}(f) - \chi(p)p^{k-1}a_{p^{r-2}}(f)$ $\forall p$ PRIME, $r\geq 2$.

(iii) $a_{mn}(f) = a_m(f)\,a_n(f)$ WHEN $(m,n)=1$.

MULTIPLYING (ii) BY p^{-rs} AND SUMMING OVER $r\geq 2$,

$$\sum_{r=2}^{\infty}a_{p^r}(f)p^{-rs} = \sum_{r=1}^{\infty}a_p(f)\,a_{p^r}(f)\,p^{-(r+1)s} + \sum_{r=0}^{\infty}\chi(p)p^{k-1}a_{p^r}(f)\,p^{-(r+2)s}$$

$$\Rightarrow \sum_{r=0}^{\infty}a_{p^r}p^{-rs}\left(1-a_p p^{-s}+\chi(p)p^{k-1-2s}\right) - \underset{=1}{a_1} - \underbrace{a_p p^{-s} + a_p a_1 p^{-s}}_{0\ \text{(i)}} = 0$$

THEN $$\left(\sum_{r=0}^{\infty}a_{p^r}p^{-rs}\right)\left(1-a_p p^{-s}+\chi(p)p^{k-1-2s}\right) = 1.$$

WE CAN COMPLETE BY THE FUNDAMENTAL THEOREM OF ARITHMETICS #.

THINGS THAT ARE PROVED IN HW 5 (OR BEFORE)

$f^*(z)=\overline{f(-\bar{z})}$, f PRIMITIVE $\Longleftrightarrow$ f^* PRIMITIVE $\quad W_N f=\eta_f f^*$

$a_n(f^*)=\overline{a_n(f)}$ IN $\Gamma_1(N)$.

$\eta_f\eta_{f^*}=(-N)^k$, $\eta_{f^*}=\overline{\eta_f}(-1)^k$, $|\eta_f|=N^{k/2}$

NOW, WE HAVE, $\Lambda(f,k-s)=\varepsilon_f\Lambda(f^*,s)$, $\varepsilon_f=i^k\eta_f N^{-k/2}$

SUPPOSE THAT ALL COEFFICIENTS $a_n(f)$ ARE REAL $\Rightarrow a_n(f^*)=a_n(f)$

$\Rightarrow f=f^* \Rightarrow \eta_f=\pm i^k N^{k/2} \Rightarrow \varepsilon_f=\pm 1.$

$$\Lambda(f,k-s)=\varepsilon_f\Lambda(f,s)$$

LET r BE THE ORDER OF VANISHING OF THE HOLOMORPHIC FUNCTION $L(f,s)$ AT $s=k/2$, I.E., $\Lambda(f,s)=c_r(s-k/2)^r+c_{r+1}(s-k/2)^{r+1}+\dots$

$\Rightarrow \Lambda(f,k-s)=c_r(k/2-s)^r+c_{r+1}(k/2-s)^{r+1}+\dots$

$=\varepsilon_f\left(c_r(s-k/2)^r+c_{r+1}(k/2-s)^{r+1}\dots\right)$

$\Rightarrow (-1)^r\varepsilon_f=1 \qquad \varepsilon_f=(-1)^r$