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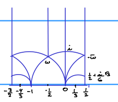
Modular Forms



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Modular Curves Recall that $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. We define $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ the extended upper complex plane.

Def: The topology of $\mathbb{H}^*$ is defined as follows. Let $\tau \in \mathbb{H}$, then we take the usual open neighborhoods of τ contained in $\mathbb{H}$. For the cusp ∞, we take as a basis of open neighborhoods the sets $\{\tau \in \mathbb{H} : \operatorname{Im}(\tau) > k\} \cup \{\infty\}$ for every $k > 0$.
For a cusp $c \neq \infty$ we take as a basis of open neighborhoods $\{$the interior of a circle in $\mathbb{H}$ tangent to the real axis at $c\} \cup \{c\}$

Remark We know $\exists \gamma \in SL_2(\mathbb{Z})$ such that $\gamma\infty = c$. Then γ sends $\{\tau \in \mathbb{H} : \operatorname{Im}(\tau) > k\}$ to the interior of a circle in $\mathbb{H}$ tangent to the real axis at c.

Remark $\mathbb{H}^*$ is Hausdorff (distinct points of $\mathbb{H}^*$ have disjoint neighborhoods)

Def: Let $\Gamma < SL_2(\mathbb{Z})$ be a congruence subgroup. We define
$Y(\Gamma) = \Gamma \backslash \mathbb{H}$ and $X(\Gamma) = \Gamma \backslash \mathbb{H}^*$ they are called modular curves.

$Y(\Gamma)$ and $X(\Gamma)$ are given the quotient topologies, namely $U \subset Y(\Gamma)$ is open iff $\pi^{-1}(U) \subset \mathbb{H}$ is open and similarly for $X(\Gamma)$

Prop: For any $\tau_1, \tau_2 \in \mathbb{H}^*$ $\exists$ neighborhoods $\tau_1 \in U_1$, $\tau_2 \in U_2$ such that if $\gamma \in SL_2(\mathbb{Z})$ satisfies $\gamma(U_1) \cap U_2 \neq \emptyset$, then $\gamma(\tau_1) = \tau_2$.

Proof: Prop 2.1.1 Diamond & Shurman

We then say that $SL_2(\mathbb{Z})$ acts properly discontinuously on $\mathbb{H}^*$

Coro: $X(\Gamma)$ is Hausdorff.

Proof: Let $P_1 \neq P_2 \in X(\Gamma)$. Let $\tau_1, \tau_2 \in \mathbb{H}^*$ that lift P_1, P_2. Let U_1, U_2 as in the proposition and $V_i = \pi(U_i)$. We claim that $V_1 \cap V_2 = \emptyset$. Suppose $V_1 \cap V_2 \neq \emptyset$. Then $\pi^{-1}(V_1) \cap \pi^{-1}(V_2) \neq \emptyset$. Hence
$\left(\bigcup_{\gamma \in \Gamma} \gamma U_1\right) \cap \left(\bigcup_{\gamma' \in \Gamma} \gamma' U_2\right) \neq \emptyset$ So there are $\gamma, \gamma' \in \Gamma$, $\gamma U_1 \cap \gamma' U_2 \neq \emptyset$
This gives $(\gamma')^{-1}(\gamma U_1) \cap U_2 \neq \emptyset$. By proposition $(\gamma')^{-1}\gamma\tau_1 = \tau_2$, a contradiction since $P_1 \neq P_2 \Rightarrow \tau_1, \tau_2$ are in different orbits. #

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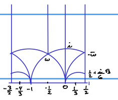

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Prop: $X(\Gamma)$ is compact

Proof: It suffices to find a compact subset of $\mathbb{H}^*$ that maps surjectively to $X(\Gamma)$. Let $D^* = \{\infty\} \cup \{z \in \mathbb{H}, -\frac{1}{2} \le \operatorname{Re} z \le \frac{1}{2}, |z| \ge 1\}$.
We saw that D^* contains a point for each $SL_2(\mathbb{Z})$-orbit on $\mathbb{H}^*$.
If $\gamma_1, \dots, \gamma_h$ are coset representatives for $\Gamma \backslash SL_2(\mathbb{Z})$ then $\bigcup_{i=1}^{h} \gamma_i D^*$ surjects onto $X(\Gamma)$. Since D^* is compact, we are done #

Def: A Riemann surface is a connected Hausdorff topological space X together with a collection $(U_i, V_i, \phi_i)_{i \in I}$ where $V_i \subset X$ is open, $U_i \subset \mathbb{C}$ is open, and $\phi_i : U_i \longrightarrow V_i$ is an homeomorphism, such that if $V_i \cap V_j \neq \emptyset$, the map $\phi_j^{-1} \circ \phi_i : U_i \cap \phi_i^{-1}(V_i \cap V_j) \longrightarrow U_j \cap \phi_j^{-1}(V_i \cap V_j)$ is holomorphic.

We will show that $X(\Gamma)$, $Y(\Gamma)$ have natural Riemann surface structures.

Def: We say that $P \in Y(\Gamma)$ is an elliptic point if for some $\tau \in \mathbb{H}$ lifting of P, $\operatorname{Stab}_{\bar{\Gamma}}(\tau) \neq \{1\}$, where $\bar{\Gamma}$ is the image of Γ in $PSL_2(\mathbb{Z})$.

If P is elliptic for Γ, then it maps to an elliptic point of $Y(SL_2(\mathbb{Z}))$.
There are only two of these points in $Y(SL_2(\mathbb{Z}))$, namely i and $\omega = e^{2\pi i/3}$.

If P is neither elliptic nor a cusp, one can easily find a chart around P. Let τ be a lifting of P and apply the Prop with $\tau_1 = \tau_2 = \tau$. Let $U = U_1 \cap U_2$. Then U is a neighborhood of τ such that $\gamma U \cap U = \emptyset$ for any $\gamma \neq 1 \in \bar{\Gamma}$. Let V be the image of U in $Y(\Gamma)$. Then $\phi = \pi|_U$ is a homeomorphism $U \xrightarrow{\sim} V$.

If P is elliptic, Prop gives a $\tau \in U$ such that $U \cap \gamma U \neq \emptyset$ iff $\gamma \in \operatorname{Stab}_{\bar{\Gamma}}(\tau)$ (which has order 2 or 3). By replacing U by $\bigcap_{\gamma \in \operatorname{Stab}_{\bar{\Gamma}}(\tau)} \gamma U$, we may assume that U is fixed by $\operatorname{Stab}_{\bar{\Gamma}}(\tau)$. Let $V = \pi(U)$. Then $\pi|_U$ gives a homeomorphism
$\operatorname{Stab}_{\bar{\Gamma}}(\tau) \backslash U \longrightarrow V$.

Choose $\delta \in SL_2(\mathbb{C})$ such that $\delta \tau = 0$ and $\delta \bar{\tau} = \infty$. Let $U' = \delta U$.
Then $\delta \operatorname{Stab}_{\bar{\Gamma}}(\tau) \delta^{-1}$ is a finite cyclic subgroup of Möbius transformations fixing 0 and ∞, hence it is a cyclic group C of

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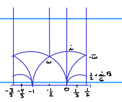

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Rotation by $e^{2\pi i/n}$, $n=2$ or 3. We have a diagram of homeomorphisms

$$\begin{tikzcd} \mathrm{Stab}_\Gamma(\tau)\backslash U \arrow[r] \arrow[dr] & C\backslash U' \arrow[d] \\ & V \end{tikzcd}$$

Two points in U' are in the same C-orbit iff they map to the same point under $z \mapsto z^n$. We can extend this diagram

$$\begin{tikzcd} \mathrm{Stab}_\Gamma(\tau)\backslash U \arrow[r] \arrow[dr] & C\backslash U' \arrow[r, "z\mapsto z^n"] \arrow[d] & U'' \arrow[dl] \\ & V & \end{tikzcd}$$

and the right arrow $U'' \longrightarrow V$ gives a coordinate chart around P.

If P is a cusp, we argue similarly. Let δ mapping P to ∞. Then $\delta\,\mathrm{Stab}_\Gamma(\hat{\infty})\,\delta^{-1}$ is the group of translations by $h\mathbb{Z}$ for some h fixed positive integer and $z \longrightarrow e^{2\pi i z/h}$ gives the local coordinate.

Then one checks that the coordinate charts are compatible and this gives a Riemann surface structure on $X(\Gamma)$ (and $Y(\Gamma)$)

Def: A modular function of level N is a meromorphic function $f: X(\Gamma_0(N)) \longrightarrow \mathbb{P}^1(\mathbb{C})$

Genus, ramification, Riemann-Hurwitz. Riemann surfaces are (compact) connected oriented smooth 2-manifolds, so they are tori with g holes

($g=0$ for the sphere)

Def: The genus of a compact connected Riemann surface M is the unique integer g such that $H^1(M,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$

The genus is related to the Euler characteristic

$$\chi(M) = \sum_{j\geq 0} (-1)^j \,\mathrm{rk}\, H^j(M,\mathbb{Z})$$

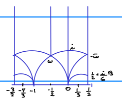
For our surfaces, $H^0(M,\mathbb{Z}) \simeq H^2(M,\mathbb{Z}) \simeq \mathbb{Z}$, $H^j(M,\mathbb{Z}) = 0\ \forall j \geq 3$

$\Rightarrow \chi(M) = 2-2g$

Prop: For $\Gamma = SL_2(\mathbb{Z})$, the space $X(\Gamma)$ is isomorphic as a Riemann surface to $\mathbb{P}^1(\mathbb{C}) \simeq S^2$

Proof: The j-invariant is $SL_2(\mathbb{Z})$-invariant and gives a holomorphic map $j: X(SL_2(\mathbb{C})) \longrightarrow \mathbb{P}^1(\mathbb{C})$, which is bijective, and has holomorphic

Modular Forms



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inverse by the open map theorem from complex analysis. #

We want to find $g(X(\Gamma))$. We know that $g(X(SL_2(\mathbb{Z})))=0$ and $X(\Gamma)\longrightarrow X(SL_2(\mathbb{Z}))$

Def: For $f: X\longrightarrow Y$, $P\in X$, the ramification degree $e_p(f)$ is the unique integer $e\geq 1$ such that f looks like $z\longrightarrow z^e$ locally.

If X, Y compact then $\sum_{P\in f^{-1}(Q)} e_p(f)$ is independent of $Q\in Y$ and is called the degree of f.

Remark: $f: X(\Gamma)\longrightarrow X(SL_2(\mathbb{Z}))$, $\deg f=[PSL_2(\mathbb{Z}):\bar{\Gamma}]$

Thm: (Riemann-Hurwitz) For $f: X\longrightarrow Y$ non-constant of degree N, X, Y compact connected Riemann surfaces, we have

$$2g(X)-2=N(2g(Y)-2)+\sum_{P\in X}(e_p(f)-1)$$

Coro: For any Γ, we have

$$g(X(\Gamma))=1+\frac{[PSL_2(\mathbb{Z}):\bar{\Gamma}]}{12}-\frac{\varepsilon_2}{4}-\frac{\varepsilon_3}{3}-\frac{\varepsilon_\infty}{2},$$

where ε_2 (resp. ε_3) is the number of elliptic points of order 2 (resp. 3) and ε_∞ is the number of cusps.

Proof: We need to analyze the ramification of $f: X(\Gamma)\longrightarrow SL_2(\mathbb{Z})$ for each $P\in X(\Gamma)$.

If $P\in Y(\Gamma)$ is not in the $SL_2(\mathbb{Z})$-orbit of i or ω,

$$\begin{tikzcd} \mathbb{H} \arrow[d] \arrow[dd, bend left, "\alpha"] \\ Y(\Gamma) \arrow[d] \\ Y(SL_2(\mathbb{Z})) \end{tikzcd}$$

The map α is unramified at any τ lifting a non-elliptic point of $Y(SL_2(\mathbb{Z}))$ so $e_p(f)=1$. If P maps to i, it is either non-elliptic or elliptic of order 2. If P is elliptic of order 2, then $Y(\Gamma)\longrightarrow Y(SL_2(\mathbb{Z}))$ is locally an isomorphism at P so $e_p(f)=1$. If P is non-elliptic, then the local coordinate for $SL_2(\mathbb{Z})$ is a square of that for Γ, so $e_p(f)=2$. So we have

$$\#\{\text{points over } i\}=N=\varepsilon_2+2\#\{\text{non-elliptic points over } i\}$$

Hence, there are $\frac{N-\varepsilon_2}{2}$ non-elliptic points over i and

$$\sum_{P\in f^{-1}(i)} e_p-1=\frac{N-\varepsilon_2}{2}$$

If P maps to ω, then $e_p(f)=1$ if P is elliptic and $e_p(f)=3$

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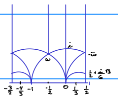

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If P is non-elliptic, and $\displaystyle\sum_{P\in f^{-1}(\infty)} e_P - 1 = \frac{2(N-\mathcal{E}_3)}{3}$.

If P is a cusp, let h be its width. A local coordinate for $X(SL_2(\mathbb{Z}))$ at ∞ is $(e^{2\pi i/h})^h$, so $e_P(f)=h$ and

$$\sum_{P\in f^{-1}(\infty)} e_P - 1 = \left(\sum_{P\in f^{-1}(\infty)} e_P\right) - \mathcal{E}_\infty = N-\mathcal{E}_\infty$$

Putting all together,

$$2g(X(\Gamma))-2 = (-2)N + \frac{N-\mathcal{E}_2}{2} + \frac{2(N-\mathcal{E}_3)}{3} + N-\mathcal{E}_\infty$$

$$g(X(\Gamma)) = 1+\frac{N}{12} - \frac{\mathcal{E}_2}{4} - \frac{\mathcal{E}_3}{3} - \frac{\mathcal{E}_\infty}{2} \qquad \#$$

Ex Take $\Gamma = \Gamma_0(p)$. We saw that there are two cusps. A system of coset representatives of $\Gamma_0(p)\backslash SL_2(\mathbb{Z})$ is $\left\{\begin{pmatrix}1&0\\j&1\end{pmatrix}\right\}_{j=0\dots p-1} \cup \left\{\begin{pmatrix}1&-1\\1&0\end{pmatrix}\right\}$. One can prove that (see DS, 3.1.4) $\mathcal{E}_2=\begin{cases}2 & p\equiv 1 \pmod 4\\ 0 & p\equiv 3 \pmod 4\\ 1 & p=2\end{cases}$

$\mathcal{E}_3=\begin{cases}2 & p\equiv 1 \pmod 3\\ 0 & p\equiv 2 \pmod 3\\ 1 & p=3\end{cases}$ We saw $\mathcal{E}_\infty=2$. This yields $g=\begin{cases}\left[\frac{p+1}{12}\right]-1 & p\equiv 1 \pmod{12}\\ \left[\frac{p+1}{12}\right] & \text{otherwise.}\end{cases}$

In particular, $g(X(\Gamma_0(11))) = g(X(\Gamma_0(17))) = g(X(\Gamma_0(19))) = 1$

The only primes p with $g(\Gamma_0(p))=0$ are $\{2,3,5,7,13\}$

Remark For any g there exists finitely many congruence subgroups Γ of $PSL_2(\mathbb{Z})$ of genus g (J.G. Thompson)

Remark The genus formula may be combined with Riemann-Roch to get the valence formula and dimension formulas.