Modular Forms

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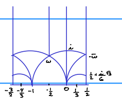


Modular Forms for $SL_2(\mathbb{Z})$

Def: Let f be a function on $\mathbb{H}$. We say that f is **weakly modular** of weight $k \in \mathbb{Z}$ if it is meromorphic and it satisfies

$$f(\gamma z) = (cz+d)^k f(z) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \quad \forall z \in \mathbb{H}.$$

We will reformulate this definition.

Def: For $\gamma \in SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{R} \;\; ad-bc=1 \right\}$, the **slash operator** of weight k on (f,γ) is defined by

$$(f|_k \gamma)(z) := j(\gamma,z)^{-k} f(\gamma z) \quad \forall z \in \mathbb{H}$$

Obs: The formula above defines a right action of $SL_2(\mathbb{R})$ on the set of meromorphic functions on $\mathbb{H}$.

Proof: Indeed, if $\gamma_1, \gamma_2 \in SL_2(\mathbb{R})$ we have, on one hand,

$$((f|_k \gamma_1)|_k \gamma_2)(z) := (c_2 z + d_2)^{-k} (f_k|\gamma_1)(\gamma_2 z)$$

$$= (c_2 z+d_2)^{-k} (c_1 \gamma_2 z + d_1)^{-k} f(\gamma_1\gamma_2 z) = (c_1(a_2 z + b_2) + d_1(c_2 z + d_2))^{-k} f(\gamma_1\gamma_2 z)$$

On the other hand, $\gamma_1 \cdot \gamma_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2 \end{pmatrix}$

and $(f|_k \gamma_1\gamma_2)(z) = ((c_1a_2+d_1c_2)z + c_1b_2+d_1d_2)^{-k} f(\gamma_1\gamma_2 z)$

Thus, $(f|_k \gamma_1)|_k \gamma_2 = f|_k \gamma_1\gamma_2$. #

We can say that f is weakly modular if and only if it is meromorphic and $f|_k \gamma = f$ (f is invariant under the action of $SL_2(\mathbb{Z})$ defined by the slash operator). Since $SL_2(\mathbb{Z}) = \langle S, T \rangle$, it suffices to check invariance under S and T. We need

$f(z+1) = f(z) \quad \forall z \in \mathbb{H}$ (invariance by T)

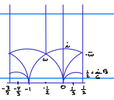
$f(-1/z) = z^k f(z) \quad \forall z \in \mathbb{H}$ (invariance by S)

Obs: Weak modularity with $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ gives $f(z) = (-1)^k f(z) \;\; \forall z \in \mathbb{H}$

$\Rightarrow$ if k is odd, the only meromorphic function on $\mathbb{H}$ that is weakly modular of weight k is the zero function.

Notation: We write
$$g: \mathbb{H} \longrightarrow \mathbb{C}$$
$$z \longrightarrow \exp(2\pi i z)$$

Let f be weakly periodic of weight k. Thus $f(z+1) = f(z)$. The f can be written as $f(z) = \tilde{f}(\exp(2\pi i z))$, where $\tilde{f}$ is a meromorphic function on the punctured unit disc



$\mathbb{D}^* = \{q \in \mathbb{C} \mid 0 < |q| < 1\}$. In other words, f is defined as $\tilde{f}(q) = f\left(\frac{\log q}{2\pi i}\right)$. The logarithm is multivalued but this is still well defined because choosing a different value for the logarithm amounts to adding an integer multiple of $2\pi i$ and $f(z+1) = f(z)$.

Def: Let f be a meromorphic function on $\mathbb{H}$ that is weakly modular of weight k. We say that f is **meromorphic at infinity** (or at the **cusp**) if $\tilde{f}$ can be continued to a meromorphic function on the open unit disc $\mathbb{D} = \{q \in \mathbb{C} \mid |q| < 1\}$.
We say that f is **holomorphic at infinity** (or at the **cusp**) if the meromorphic continuation of $\tilde{f}$ is holomorphic at $q = 0$.

The condition that $\tilde{f}$ can be continued to a meromorphic function on $\mathbb{D}$ is equivalent to $\tilde{f}$ having a Laurent series

$$\tilde{f}(q) = \sum_{n=N}^{\infty} a_n q^n \quad (a_n \in \mathbb{C},\ N \in \mathbb{Z})$$

convergent on $\{q \in \mathbb{C} \mid 0 < |q| < \varepsilon\}$ for some $\varepsilon > 0$.
Then f is holomorphic at infinity iff $a_n = 0 \ \forall n < 0$.
In this case, the **value of f at infinity** is defined by

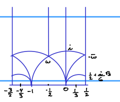
$$f(\infty) = \tilde{f}(0) = a_0.$$

Def: Let k be an integer. A **modular function** of weight k (for the group $SL_2(\mathbb{Z})$) is a holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ that is weakly modular of weight k and meromorphic at infinity. A **modular form** of weight k (for the group $SL_2(\mathbb{Z})$) is a modular function that is holomorphic at infinity. A **cusp form** is a modular form f of weight k satisfying $f(\infty) = 0$.

Eisenstein Series: Let k be an even integer with $k \geq 4$. We define the **Eisenstein Series** of weight k (for $SL_2(\mathbb{Z})$) by

$$G_k : \mathbb{H} \longrightarrow \mathbb{C}$$

$$z \longrightarrow G_k(\Lambda_z) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(mz+n)^k}$$



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Prop: The series above converges absolutely and uniformly on subsets of $\mathbb{H}$ of the form $R_{r,s} = \{x+iy \mid |x| \le r,\ y \ge s\}$

Proof: Let $z = x+iy \in R_{r,s}$. We have

$$|mz+n|^2 = (mx+n)^2 + m^2y^2 \ge (mx+n)^2 + m^2 s^2$$

Suppose that $|n| \le 3r|m|$. Then

$$|mz+n|^2 \ge m^2 s^2 \ge \frac{m^2 s^2}{2} + \frac{s^2 n^2}{2(3r)^2} \ge \min\{s^2/2,\ s^2/18r^2\}(m^2+n^2)$$

Suppose that $|n| \ge 3r|m|$

$$|mz+n|^2 \ge (|mx|-|n|)^2 + m^2 s^2 \ge n^2\, 2/3 + m^2 s^2 \ge \min\{2/3, s^2\}(m^2+n^2)$$

$$|nmx| \le \left|n \cdot \frac{n}{3 \cdot r} \cdot r\right| = \frac{n^2}{3}$$

Let $c^2 = \min\{s^2/2,\ s^2/18r^2,\ 2/3,\ s^2\}$, we get

$$|mz+n| \ge c\,(m^2+n^2)^{1/2} \quad \forall m,n \in \mathbb{Z},\ z \in R_{r,s}$$

This implies that for $z \in R_{r,s}$, we have,

$$|G_k(z)| \le \frac{1}{c^k} \sum_{(m,n)\ne(0,0)} \frac{1}{(m^2+n^2)^{k/2}}$$

Consider $L(j) = \{(m,n) \in \mathbb{Z} \mid \max\{|m|,|n|\} = j\}$ $j = 1,2,3,\ldots$

Then $\# L(j) = 8j$ and $j^2 \le m^2+n^2 \le 2j^2$ $\forall (m,n) \in L(j)$.

Thus

$$|G_k(z)| \le \frac{1}{c^k} \sum_{j=1}^{\infty} \frac{8j}{j^k} = \frac{8}{c^k} \sum_{j=1}^{\infty} \frac{1}{j^{k-1}} = \frac{8}{c^k}\zeta(k-1) < \infty$$

which is finite and independent of $z \in R_{r,s}$. #

Thus G_k is an holomorphic function on $\mathbb{H}$.

Thm: For every $k \in \mathbb{Z}_{\ge 4}$, $G_k : \mathbb{H} \longrightarrow \mathbb{C}$ is a modular form of weight k.

Proof: G_k is holomorphic on $\mathbb{H}$. And we saw earlier that any homogeneous $F : \mathcal{L} \longrightarrow \mathbb{C}$ of weight k leads to $f : \mathbb{H} \longrightarrow \mathbb{C}$ such that $f(\gamma z) = j(\gamma,z)^k f(z)$. It remains to see that f is holomorphic at infinity. #

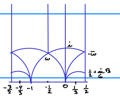
The q-expansions of Eisenstein series

Def: The **Riemann zeta function** is a complex analytic function given by $\zeta(s) = \sum_{n=1} \frac{1}{n^s}$, where $s \in \mathbb{C}$, $Re(s) > 1$.

Def: The **sum of t-th powers of positive divisors** of $n \in \mathbb{Z}$ is

$$\sigma_t(n) = \sum_{d|n,\, d>0} d^t$$

(11)



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We can write $G_k(z) = \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}$

$= \sum_{n\neq 0} \frac{1}{n^k} + \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k}$

Since $k\in\mathbb{Z}$ is even, we have,

$G_k(z) = 2\sum_{n=1}^{\infty} \frac{1}{n^k} + 2\sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k} = 2\zeta(k) + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k}$

Prop: Let $k\in\mathbb{Z}_{\geq 2}$. Then

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{\ell=1}^{\infty} \ell^{k-1} \exp(2\pi i \ell z) \quad \forall z\in\mathcal{H}.$$

Proof: We start by the classical formula.

$$\pi \frac{\cos(\pi z)}{\sin(\pi z)} = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right) \quad \forall z\in\mathbb{C}\setminus\mathbb{Z}.$$

Using $\exp(\pm iz) = \cos z \pm i \sin z$,

$$\pi\frac{\cos(\pi z)}{\sin(\pi z)} = \pi i \frac{\exp(\pi i z) + \exp(-\pi i z)}{\exp(\pi i z) - \exp(-\pi i z)} = -\pi i - 2\pi i \frac{\exp(2\pi i z)}{1-\exp(2\pi i z)}$$

$= -\pi i - 2\pi i \sum_{h=1}^{\infty} \exp(2\pi i h z)$. Thus

$$\frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right) = -\pi i - 2\pi i \sum_{\ell=1}^{\infty} \exp(2\pi i \ell z) \quad \forall z\in\mathcal{H}$$

Taking derivatives, $\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^2} = (2\pi i)^2 \sum_{\ell=1}^{\infty} \ell \exp(2\pi i \ell z)$, which gives the equality for $k=2$.

By induction, suppose that we have the equality for k.

We differentiate

$$\sum_{n\in\mathbb{Z}} \frac{-k}{(z+n)^{k+1}} = \frac{-(-2\pi i)^{k+1}}{(k-1)!} \sum_{\ell=1}^{\infty} \ell^{k} \exp(2\pi i \ell z). \qquad \#.$$

Applying the proposition to the sum for $G_k(z)$,

$G_k(z) = 2\zeta(k) + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k}$

$= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty} \ell^{k-1} \exp(2\pi i \ell m z)$ Take $n=\ell m$.

$= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sum_{\substack{d\mid n\\ d>0}} d^{k-1} \exp(2\pi i n z)$

$$= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)\, q^n.$$

THE **BERNOULLI NUMBERS** B_k, $k \in \mathbb{Z}_{\geq 0}$ ARE GIVEN BY

$$\frac{t}{\exp(t)-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k$$

PROP: (i) $\pi z \dfrac{\cos(\pi z)}{\sin(\pi z)} = \displaystyle\sum_{\substack{k\geq 0\\ \text{EVEN}}} \frac{(2\pi i)^k B_k}{k!} z^k \quad \forall\, |z|<1.$

(ii) $\pi z \dfrac{\cos(\pi z)}{\sin(\pi z)} = 1 - 2\displaystyle\sum_{\substack{k\geq 2\\ \text{EVEN}}} \zeta(k) z^k \quad \forall\, |z|<1.$

(iii) $\zeta(k) = -\dfrac{(2\pi i)^k B_k}{2\cdot k!}$

(iv) $B_k \neq 0 \iff k=1$ OR k EVEN.

PROOF: (i) AS WE SAW EARLIER,

$$\pi z \frac{\cos(\pi z)}{\sin(\pi z)} = -\pi i z - \frac{2\pi i z}{\exp(-2\pi i z)-1} = -\pi i z + \sum_{k=0}^{\infty} \frac{B_k}{k!}(-2\pi i z)^k$$

REPLACING z BY $-z$ AND ADDING, WE GET THE RESULT.

(ii) $\pi z \dfrac{\cos(\pi z)}{\sin(\pi z)} = 1 + \displaystyle\sum_{n=1}^{\infty}\left(\frac{z}{z-n} + \frac{z}{z+n}\right) = 1 + 2\sum_{n=1}^{\infty}\frac{z^2}{z^2-n^2}$

$$= 1 - 2\sum_{n=1}^{\infty}\frac{z^2}{n^2}\sum_{j=0}^{\infty}\frac{z^{2j}}{n^{2j}} = 1 - 2\sum_{k=1}^{\infty} z^{2k}\zeta(2k)$$

(iii) = (i) + (ii). (iv) IN PROOF OF (i) SUBSTRACT SERIES FOR z AND $-z$. WE GET $B_1 = -\frac{1}{2}$ AND $B_k = 0$ FOR k ODD. ALSO (iii) IMPLIES THAT $B_k \neq 0$ FOR k EVEN. #

EX: THE FIRST NONZERO BERNOULLI NUMBERS ARE

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, ...

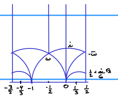
FINALLY, WE GET

$$G_k(z) = -\frac{(2\pi i) B_k}{k!} + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)\, q^n.$$

IT IS USEFUL TO SCALE $G_k(z)$ AS

$E_k(z) = \dfrac{(k-1)!}{2(2\pi i)^k} G_k(z)$. SO THAT $E_k(z) = -\dfrac{B_k}{2k} + \displaystyle\sum_{n=1}^{\infty}\sigma_{k-1}(n)\, q^n.$

(ALL THE COEFFICIENTS IN THE EXPANSION OF $E_k(z)$ ARE RATIONAL)



The Eisenstein series of weight 2

The construction of $G_k(z)$ $k \geq 4$ does not generalize completely to $k=2$ because $\displaystyle\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{1}{(mz+n)^2}$ does not converge.

One can define

$$G_2(z) = 2\zeta(2) + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2}$$

and set

$$E_2(z) = -\frac{1}{8\pi^2} G_2(z)$$

Then we still have

$$G_2(z) = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

and $E_2(z) = \dfrac{-B_2}{4} + \displaystyle\sum_{n=1}^{\infty} \sigma_1(n) q^n$

However, $G_2(z)$ and $E_2(z)$ are NOT modular forms

Prop: The functions $G_2(z)$ and $E_2(z)$ satisfy the following transformations

$$z^{-2} G_2(-1/z) = G_2(z) - \frac{2\pi i}{z} \quad \text{and} \quad z^{-2} E_2(-1/z) = E_2(z) - \frac{1}{4\pi i z}$$

Lemma: For all $z \in \mathbb{H}$,

(i) $\displaystyle\sum_{m\neq 0}\sum_{n\in\mathbb{Z}} \left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right) = 0$ and

(ii) $\displaystyle\sum_{n\in\mathbb{Z}}\sum_{m\neq 0} \left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right) = -\frac{2\pi i}{z}$

Proof: (i) Consider the telescoping sum

$$\sum_{-N\leq n<N} \left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right) = \frac{1}{mz-N} - \frac{1}{mz+N}$$

We have

$$\sum_{n\in\mathbb{Z}} \left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right) = \lim_{N\to\infty} \sum_{-N\leq n<N} \left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right)$$

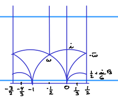
$$= \lim_{N\to\infty} \left(\frac{1}{mz-N} - \frac{1}{mz+N}\right) = 0.$$

(ii) On the other hand,

$$\sum_{n\in\mathbb{Z}}\sum_{m\neq 0} \left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right) = \lim_{N\to\infty} \sum_{-N\leq n<N}\sum_{m\neq 0} \left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right)$$

$$= \lim_{N\to\infty} \sum_{m\neq 0}\sum_{-N\leq n<N} \left(\frac{1}{mz+n} - \frac{1}{mz+n+1}\right) = \lim_{N\to\infty} \sum_{m\neq 0} \left(\frac{1}{mz-N} - \frac{1}{mz+N}\right)$$

The series does not converge uniformly, so we cannot exchange the limit with the sum. We can rewrite the sum as



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$$\sum_{m\neq 0}\left(\frac{1}{mz-N}-\frac{1}{mz+N}\right)=\sum_{m=1}^{\infty}\left(\frac{1}{mz-N}+\frac{1}{-mz-N}-\frac{1}{mz+N}-\frac{1}{-mz+N}\right)$$

$$=\frac{2}{z}\sum_{m=1}^{\infty}\left(\frac{1}{-N/z-m}+\frac{1}{-N/z+m}\right)=\frac{2}{z}\left(\frac{z}{N}-\pi i-2\pi i\sum_{l=1}^{\infty}\exp(-2\pi i lN/z)\right)$$

THE SERIES ON THE RIGHT-HAND SIDE CONVERGES UNIFORMLY FOR N IN THE INTERVAL $[1,\infty)$ BECAUSE

$$\sum_{l=L}^{\infty}\left|\exp(-2\pi i lN/z)\right|\leq\underbrace{\sum_{l=L}^{\infty}|q|^{l}}_{\to 0 \text{ AS } L\to\infty},\quad q=\exp(-2\pi i/z).$$

WE CAN EXCHANGE THE SUM AND THE LIMIT.

$$\sum_{n\in\mathbb{Z}}\sum_{m\neq 0}\left(\frac{1}{mz+n}-\frac{1}{mz+n+1}\right)=\lim_{N\to\infty}\frac{2}{z}\left(\frac{z}{N}-\pi i-2\pi i\sum_{l=1}^{\infty}\exp(-2\pi i lN/z)\right)$$

$$=-\frac{2\pi i}{z}\quad\#$$

<u>PROOF OF PROP:</u> WE HAVE $G_2(z)=2\zeta(2)+\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2}$

SUBSTRACTING FROM LEMMA (i),

$$G_2(z)=2\zeta(2)+\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2}-\left(\frac{1}{mz+n}-\frac{1}{mz+n+1}\right)$$

$$=2\zeta(2)+\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2(mz+n+1)}$$

ON THE OTHER HAND, $z^{-2}G_2(-1/z)=2\zeta(2)z^{-2}+\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(nz-m)^2}$

$$=2\zeta(2)+\sum_{m\in\mathbb{Z}}\sum_{n\neq 0}\frac{1}{(nz-m)^2}=2\zeta(2)+\sum_{n\in\mathbb{Z}}\sum_{m\neq 0}\frac{1}{(mz+n)^2}$$

SUBSTRACTING THE IDENTITY FROM LEMMA (ii)

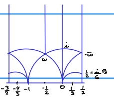
$$z^{-2}G_2(-1/z)+\frac{2\pi i}{z}=2\zeta(2)+\sum_{n\in\mathbb{Z}}\sum_{m\neq 0}\frac{1}{(mz+n)^2(mz+n+1)}$$

THE SERIES ON THE RIGHT-HAND SIDE IS ABSOLUTELY CONVERGENT (ARGUMENT SIMILAR TO THE CONVERGENCE OF EISENSTEIN SERIES) WE CAN EXCHANGE THE ORDER OF SUMMATION AND PROVE THAT.

$z^{-2}G_2(-1/z)=G_2(z)-\frac{2\pi i}{z}$. THE OTHER FORMULA IS OBVIOUS. #

<u>PROP:</u> LET $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$. THEN

$$(cz+d)^{-2}E_2\left(\frac{az+b}{cz+d}\right)=E_2(z)-\frac{1}{4\pi i}\,\frac{c}{cz+d}$$



Proof: Let $\gamma_1, \gamma_2 \in SL_2(\mathbb{Z})$ and suppose that

$$E_2|_2\gamma_j(z) = E_2(z) - \frac{1}{4\pi i}\frac{c_j}{c_j z + d_j}.$$

Then

$$\big((E_2|_2\gamma_1)|_2\gamma_2\big)(z) = (c_2z+d_2)^{-2}(E_2|_2\gamma_1)(\gamma_2 z)$$

$$= (c_2z+d_2)^{-2}E_2(\gamma_2 z) - \frac{1}{4\pi i}\frac{c_1}{(c_1\gamma_2 z+d_1)(c_2z+d_2)^2}$$

$$= E_2|_2\gamma_2(z) - \frac{1}{4\pi i}\frac{c_1}{\big(c_1(a_2z+b_2)+d_1(c_2z+d_2)\big)(c_2z+d_2)}$$

$$= E_2(z) - \frac{1}{4\pi i}\frac{c_2}{c_2z+d_2} - \frac{1}{4\pi i}\frac{c_1}{j(\gamma_1\gamma_2, z)(c_2z+d_2)} = \circledast$$

Notice that

$$c_2 j(\gamma_1\gamma_2, z) + c_1 = c_2\big((c_1a_2+d_1c_2)z + c_1b_2+d_1d_2\big)+c_1$$

$$= (c_1a_2+d_1c_2)c_2z + c_1\underbrace{c_2b_2}_{a_2d_2-1} + d_1c_2d_2 + c_1 = (c_1a_2+d_1c_2)(c_2z+d_2)$$

$$\Rightarrow \circledast = E_2(z) - \frac{1}{4\pi i}\frac{(c_1a_2+d_1c_2)}{j(\gamma_1\gamma_2, z)} = (E_2|_2\gamma_1\gamma_2)(z).$$

Now we know the result is true for S and for $S^{-1} = -S$.
By construction $(E_2|_2T)(z) = E_2(z)$ and similarly for T^{-1}.
Since $SL_2(\mathbb{Z})$ is generated by S and T, the result follows. #