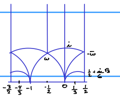
Modular Forms



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The modular form Δ and the modular function j.

Def Let $\Delta = \dfrac{(240 E_4)^3 - (-504 E_6)^2}{1728}$

Since E_4 and E_6 are modular forms of weight 4 and 6, Δ is a modular form of weight 12. The constants are chosen so that the constant term in the q-expansion vanishes. Indeed, $E_4 = \frac{1}{240} + \sum_{n=1}^{\infty} \sigma_3(n) q^n$, $E_6 = -\frac{1}{504} + \sum_{n=1}^{\infty} \sigma_5(n) q^n$

The q-expansion is

$$\Delta = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 + \dots$$

Prop: The coefficients of Δ are all integers.

Proof: $1728 = 2^6 \cdot 3^3$, $240 = 2^4 \cdot 3 \cdot 5$, $504 = 2^3 \cdot 3^2 \cdot 7$. $\Rightarrow$

$1728 \mid 240^3$, $1728 \mid 504^2$.

Def: Let $j = \dfrac{(240 E_4)^3}{\Delta}$

This is not a modular form because it has a pole at infinity (Δ vanishes but E_4 does not.) It is a modular function, in the sense that it satisfies

$f(\gamma z) = f(z)$ $\forall \gamma \in SL_2(\mathbb{Z})$ $\forall z \in \mathbb{H}$ and is meromorphic on $\mathbb{H}$ and at infinity.

The j-function appears in the theory of lattices and elliptic curves. As usual $j(\Lambda)$ is defined by $j(\omega_1/\omega_2)$ where $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\omega_1/\omega_2 \in \mathbb{H}$. Then one can show that there is a bijection

$$\mathcal{L}/\text{homothety} \xrightarrow{\sim} \mathbb{C}$$
$$[\Lambda] \longrightarrow j(\Lambda)$$

The set of modular functions is the field $\mathbb{C}(j)$

The q-expansion is given by

$$j(z) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots$$

The coefficients are linked to the representation theory of the monster group.

The η-function

Def: The Dedekind eta function is

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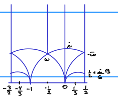

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$$\eta : \mathbb{H} \longrightarrow \mathbb{C} \qquad z \longmapsto q_{24} \prod_{n=1}^{\infty} (1-q^n),$$

where $q_{24} = \exp(2\pi i z/24)$

Prop: The Dedekind eta function $\eta : \mathbb{H} \longrightarrow \mathbb{C}$ is holomorphic and nonvanishing.

Proof: Since $\sum_{n=1}^{\infty} -q^n$ converges absolutely and uniformly on compact subsets of $\mathbb{H}$ because $|q|<1$, then the infinite product associated to η converges to a holomorphic function on $\mathbb{H}$ whose zeros coincide with those of the infinite product. But these factors have no zeros. #

Prop: We have $\eta(-1/z) = \sqrt{-iz}\,\eta(z)$, where the branch of $\sqrt{-iz}$ is taken to have positive real part.

Proof: Let $z \in \mathbb{H}$. Calculating the logarithmic derivative,

$$\frac{d}{dz}\log(\eta(z)) = \frac{2\pi i}{24} + \sum_{n=1}^{\infty} \frac{-2\pi i n q^n}{1-q^n} = \frac{\pi i}{12} - 2\pi i \sum_{n=1}^{\infty} n \sum_{m=1}^{\infty} q^{nm}$$

$$= \frac{\pi i}{12} - 2\pi i \sum_{m,n=1} n q^{nm} = \frac{\pi i}{12} - 2\pi i \sum_{l=1}^{\infty} \sigma(l) q^l = -2\pi i E_2(z)$$

By the transformation property for $E_2(z)$, we have

$$\frac{d}{dz}\log(\eta(-1/z)) = -\frac{2\pi i}{z^2} E_2(-1/z) = -2\pi i E_2(z) + \frac{1}{2z} = \frac{d}{dz}\log\left(\sqrt{-iz}\,\eta(z)\right)$$

Thus, there is a constant $c \in \mathbb{C}$ such that $\eta(-1/z) = c\sqrt{-iz}\,\eta(z)$. Setting $z=i$ shows $c=1$. #

Obs: Since $\eta(z+1) = \exp(2\pi i/24)\eta(z)$, we obtain a complete description of the action of $SL_2(\mathbb{Z})$ on $\eta(z)$.

The η-function can be used to find an infinite product expansion for Δ. Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be $f(z) = (\eta(z))^{24}$. Since η is holomorphic, the action of $SL_2(\mathbb{Z})$ on η implies that f is a weakly modular form of weight 12. Moreover, $f = q + O(q^2)$ so f is a cusp form of weight 12. We will later see that the cusp forms of weight 12 form a one-dimensional $\mathbb{C}$-vector space. Since the Fourier coefficients of q in both Δ and η^{24} equal 1, we get

$$\Delta = \frac{(240E_4)^3 - (-504E_6)^2}{1728} = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

The Fourier coefficients of Δ are denoted by $\tau(n)$, $\Delta = \sum_{n=1}^{\infty} \tau(n) q^n$. The function $n \to \tau(n)$ is called Ramanujan's τ-function.

Rem Ramanujan conjectured in 1916 the following properties

(i) τ is multiplicative, i.e., $\tau(mn) = \tau(m)\tau(n)$ $\forall m,n \in \mathbb{Z}_{>0}$, $(m,n)=1$.

(ii) $\tau(p^r) = \tau(p)\tau(p^{r-1}) - p^{11}\tau(p^{r-2})$ $\forall p$ prime and $r \in \mathbb{Z}_{\geq 2}$.

(iii) $|\tau(p)| \leq 2\, p^{11/2}$ $\forall p$ prime.

(i) and (ii) were proved by Mordell in 1917 and the last by Deligne in 1974 as a consequence of Weil conjectures.

The valence formula

Lemma: Let f be meromorphic on $\mathbb{H}$ and weakly modular of weight k, let $z \in \mathbb{H}$, let $\gamma \in SL_2(\mathbb{Z})$. Then

$$\mathrm{ord}_z f = \mathrm{ord}_{\gamma z} f$$

Proof: Since $f|_k \gamma = f$, we have $\mathrm{ord}_{\gamma z} f = \mathrm{ord}_z\, j(\gamma,z)^k f = \mathrm{ord}_z f$ since $j(\gamma,z) \neq 0, \infty$ on $\mathbb{H}$. #

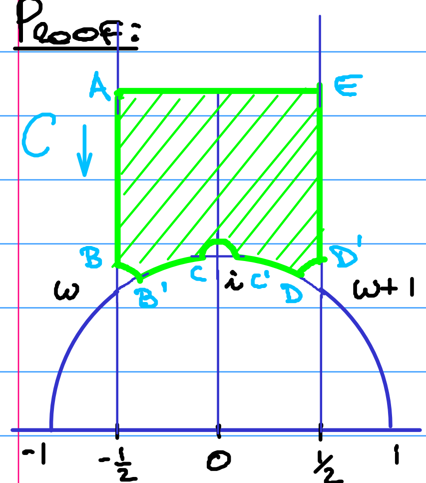
For "$z=\infty$," we define $\mathrm{ord}_{z=\infty} f = \mathrm{ord}_{q=0} \tilde{f}$, where $\tilde{f}$ is defined on $\mathbb{D} = \{ q \in \mathbb{C} \mid |q|<1 \}$ by $\tilde{f}(q) = f\left(\frac{\log q}{2\pi i}\right)$.

Thus, f is holomorphic at ∞ (resp. zero at ∞) iff $\mathrm{ord}_\infty f \geq 0$ (resp. $\mathrm{ord}_\infty > 0$).

Thm: (Valence formula) Let f be a non-zero meromorphic function on $\mathbb{H}$ that is weakly modular of weight k (for the group $SL_2(\mathbb{Z})$) and meromorphic at ∞. Then we have

$$\mathrm{ord}_\infty f + \frac{1}{2}\mathrm{ord}_i f + \frac{1}{3}\mathrm{ord}_\omega f + \sum_{\substack{z \in SL_2(\mathbb{Z})\backslash\mathbb{H} \\ z \neq i, \omega, \infty}} \mathrm{ord}_z(f) = \frac{k}{12}$$

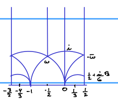
Proof:



By the Lemma, we may take all representatives $z \in SL_2(\mathbb{Z})\backslash\mathbb{H}$ as elements of the fundamental domain $\mathcal{D}$. Assume for simplicity of exposition that $\partial\mathcal{D}$ contains no zeroes or poles for f except possibly for i, ω, and $\omega+1$.

Let C be the contour as in the picture.

The small arcs around $i, \omega, \omega+1$ have radius r, and we will let $r \to 0$. The segment AE has imaginary part



R, and we will let $R \to \infty$. If ∂D contains zeros or poles of f, it has to be modified with additional small arcs going around the zeros and poles in the counterclockwise direction. When R is sufficiently large and r sufficiently small, the region surrounded by C contains all the zeros and poles of f in D except possibly those in $i, \omega, \omega+1, \infty$.

By the argument principle, $\displaystyle\oint_C \frac{f'(z)}{f(z)}\,dz = 2\pi i \sum_{\substack{z \in SL_2(\mathbb{Z})\backslash\mathbb{H} \\ z \neq i, \omega, \infty}} \mathrm{ord}_z f$ (*)

On the other hand we may consider the integral in each part of the contour.

$$\int_{D'}^{E} \frac{f'}{f}(z)\,dz = \int_{B}^{A} \frac{f'}{f}(z+1)\,dz = -\int_{A}^{B} \frac{f'}{f}(z)\,dz$$

From $f(-1/z) = z^k f(z)$, we differentiate,

$z^{-2} f'(-1/z) = k z^{k-1} f(z) + z^k f'(z)$ and divide by the previous equation,

$z^{-2}\frac{f'}{f}(-1/z) = \frac{k}{z} + \frac{f'}{f}(z)$ Thus

$$\int_{C'}^{D} \frac{f'}{f}(z)\,dz = \int_{C}^{B'} \frac{f'}{f}(-1/z')\,(z')^{-2}\,dz',$$

where we set $z' = -\frac{1}{z}$, $dz = (z')^{-2}dz'$.

$$\int_{C'}^{D} \frac{f'}{f}(z)\,dz = \int_{C}^{B'} \left(\frac{k}{z} + \frac{f'}{f}(z)\right) dz' = k\int_{C}^{B'} \frac{dz}{z} - \int_{B'}^{C} \frac{f'}{f}(z)\,dz.$$ Thus

$\displaystyle\int_{C'}^{D} \frac{f'}{f}(z)\,dz + \int_{B'}^{C} \frac{f'}{f}(z)\,dz \longrightarrow k\frac{\pi i}{6}$ as $r \to 0$, since the angle $B'OC$ tends to $\pi/6$ as $r \to 0$.

We also have, as $r \to 0$,

$$\int_{B}^{B'} \frac{f'}{f}(z)\,dz \longrightarrow -\frac{\pi i}{3}\,\mathrm{ord}_\omega(f), \quad \int_{C'}^{C} \frac{f'}{f}(z)\,dz \longrightarrow -\pi i\,\mathrm{ord}_i(f)$$

$$\int_{D}^{D'} \frac{f'}{f}(z)\,dz \longrightarrow -\frac{\pi i}{3}\,\mathrm{ord}_{\omega+1}(f) = -\frac{\pi i}{3}\,\mathrm{ord}_\omega(f).$$

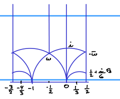
Finally, we evaluate the integral on EA. Consider $q = \exp(2\pi i z)$

By definition $f(z) = \tilde{f}(\exp(2\pi i z))$ and it follows that

$f'(z) = 2\pi i \exp(2\pi i z)\,\tilde{f}'(\exp(2\pi i z)) \Rightarrow$

$\frac{f'}{f}(z) = 2\pi i \exp(2\pi i z)\,\frac{\tilde{f}'}{\tilde{f}}(\exp(2\pi i z))$.

Also, $\frac{d}{dz}\exp(2\pi i z) = 2\pi i \exp(2\pi i z)$. Thus $dq = 2\pi i \exp(2\pi i z)\,dz$



$$\int_E^A \frac{f'}{f}(z)\,dz = -\oint_{|q|=\exp(-2\pi R)} \frac{\tilde{f}'}{\tilde{f}}(q)\,dq = -2\pi i \,\mathrm{ord}_{q=0}\tilde{f} = -2\pi i\,\mathrm{ord}_{z=\infty} f$$

FINALLY, $\displaystyle\oint_C \frac{f'}{f}(z)\,dz = k\frac{\pi i}{6} - \pi i\,\mathrm{ord}_i(f) - \frac{2\pi i}{3}\mathrm{ord}_\omega(f) - 2\pi i\,\mathrm{ord}_\infty f$

COMBINING WITH ($*$), WE GET THE RESULT. #

APPLICATIONS OF THE VALENCE FORMULA

DEF: LET M_k BE THE $\mathbb{C}$-VECTOR SPACE OF MODULAR FORMS OF WEIGHT k. LET $S_k \subseteq M_k$ BE THE SUBSPACE OF CUSP FORMS.

THM: (i) THE EISENSTEIN SERIES E_4 HAS A SIMPLE ZERO AT $z=\omega$ AND NO OTHER ZEROS.

(ii) THE EISENSTEIN SERIES E_6 HAS A SIMPLE ZERO AT $z=i$ AND NO OTHER ZEROS.

(iii) THE MODULAR FORM Δ OF WEIGHT 12 HAS A SIMPLE ZERO AT $z=\infty$ AND NO OTHER ZEROS.

PROOF: IF f IS A MODULAR FORM, THE NUMBERS $\mathrm{ord}_z f$ IN THE VALENCE FORMULA ARE NONNEGATIVE BECAUSE f IS HOLOMORPHIC AT $\mathbb{H}$ AND AT ∞. FOR Δ, THE q-EXPANSION SHOWS THAT $\mathrm{ord}_\infty \Delta = 1$. THE LOCATION OF THE ZEROES IS GIVEN BY THE ONLY POSSIBLE SOLUTIONS TO THE VALENCE FORMULA #

CORO: MULTIPLICATION BY Δ IS AN ISOMORPHISM

$$M_k \xrightarrow{\sim} S_{k+12}$$
$$f \longrightarrow \Delta \cdot f$$

IN PARTICULAR, $\dim S_{k+12} = \dim M_k$

THM: THE SPACES M_k AND S_k ARE FINITE DIMENSIONAL FOR EVERY k. MOREOVER, WE HAVE $M_k = \{0\}$ IF $k<0$ OR k ODD, AND FOR $k \geq 0$,

$$\dim M_k = \begin{cases} [k/12] & k \equiv 2 \pmod{12} \\ [k/12]+1 & k \not\equiv 2 \pmod{12} \end{cases}$$

PROOF: THE FACT THAT $M_k = \{0\}$ FOR $k<0$ FOLLOWS FROM THE VALENCE FORMULA. THE VALENCE FORMULA ALSO IMPLIES THAT $M_0 = \mathbb{C}$ (SEE PG 6 HWK 1) AND $M_2 = \{0\}$. IF k ODD AND $f \in M_k$, APPLYING $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ TO THE MATRIX $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ IMPLIES $f \equiv 0$

SUPPOSE $k \geq 4$ EVEN. SINCE E_k DOES NOT VANISH AT ∞, WE

Can write any $f \in M_k$ as a linear combination of E_k and a cusp. In other words, $M_k = S_k \oplus \mathbb{C} \cdot E_k$ for all even $k \geq 4$. In particular, $\dim M_k = \dim S_k + 1 = \dim M_{k-12} + 1$. For all even $k \geq 4$. The theorem follows by induction starting from the known values for $k \leq 2$. #

Thm: Let f be a modular form of weight k with q-expansion $\sum_{n=0}^{\infty} a_n q^n$. Suppose that $a_j = 0$ for $j = 0, 1, \dots, [k/12]$. Then $f = 0$.

Proof: Suppose that f is nonzero. The hypothesis implies $\mathrm{ord}_\infty f \geq [k/12] + 1 > k/12$. Then the left hand side of the valence formula is strictly larger than $k/12$, leading to a contradiction. $\Rightarrow f = 0$. #.

Coro: Let f, g be modular forms of the same weight k, with q-expansions $\sum_{n=0}^{\infty} a_n q^n$ and $\sum_{n=0}^{\infty} b_n q^n$ respectively. Suppose that $a_j = b_j$ for $j = 0, 1, \dots, \lfloor k/12 \rfloor$. Then $f = g$.

Ex: Using the fact that the space M_8 is one-dimensional, prove that $\sigma_7(n) = \sigma_3(n) + 120 \sum_{j=1}^{n-1} \sigma_3(j)\sigma_3(n-j)$ $\forall\, n \geq 1$.

Proof: $\dim M_8 = \lfloor 8/12 \rfloor + 1 = 1$.

We have, $E_8, E_4^2 \in M_8$. Thus, $\exists\, c \in \mathbb{C}$ such that $E_8 = c E_4^2$.

By looking at constant coefficients, we get $-\frac{B_8}{16} = c\,\frac{B_4^2}{64}$. $B_4 = B_8 = -\frac{1}{30}$. Thus, $c = 120$. And $E_8 = 120 E_4^2$.

$$-\frac{B_8}{16} + \sum_{n=1}^{\infty} \sigma_7(n) q^n = 120 \left(-\frac{B_4}{8} + \sum_{n=1}^{\infty} \sigma_3(n) q^n \right)^2.$$

Equating for n,

$$\sigma_7(n) = \underbrace{2 \cdot 120 \left(-\frac{B_4}{8} \right) \sigma_3(n)}_{\sigma_3(n).} + 120 \sum_{j=1}^{n-1} \sigma_3(j)\sigma_3(n-j)$$