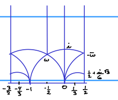
(30)

Modular Forms



M. Lalín

Modular forms for congruence subgroups

Let Γ be a congruence subgroup, $k\in\mathbb{Z}$, and f a meromorphic function on $\mathbb{H}$ that is weakly modular of weight k for the group Γ. Let c be a cusp of Γ, $t\in\mathbb{P}^1(\mathbb{Q})$ an element of the corresponding Γ-orbit in $\mathbb{P}^1(\mathbb{Q})$. Let $\gamma_t\in SL_2(\mathbb{Z})$ such that $\gamma_t\infty=t\in\mathbb{P}^1(\mathbb{Q})$. Then the meromorphic function $f|_k\gamma_t$ is invariant under the weight k action of H_c. Indeed, let $\gamma\in H_c$. Then $\gamma_t\gamma\gamma_t^{-1}=\tilde{\gamma}\in\Gamma$. And

$$(f|_k\gamma_t)|_k\gamma = f|_k\gamma_t\gamma = f|_k\tilde{\gamma}\gamma_t=(f|_k\tilde{\gamma})|_k\gamma_t=f|_k\gamma_t.$$

By definition of width and (ir)regularity of cusps, $f|_k\gamma_t$ is periodic with period

$$\tilde{h}_\Gamma(c)=\begin{cases} h_\Gamma(c) & \text{if } c \text{ regular,}\\ 2h_\Gamma(c) & \text{if } c \text{ irregular.}\end{cases}$$

Then, in $\mathbb{D}^*$, we can express $f|_k\gamma_t$ as a Laurent series in the variable $q_c=\exp(2\pi i z/\tilde{h}_\Gamma(c))$. So

$$(f|_k\gamma_t)(z)=\tilde{f}_c\left(\exp\left(2\pi i z/\tilde{h}_\Gamma(c)\right)\right),$$

where

$$\tilde{f}_c(q_c)=\sum_{n\in\mathbb{Z}} a_{c,n}q_c^n$$

Def: We say that f is **meromorphic at the cusp c** if $\tilde{f}_c$ can be continued to a meromorphic function on $\mathbb{D}$, that f is **holomorphic at c** if in addition $\tilde{f}_c$ is holomorphic at $q_c=0$, and that f **vanishes at c** if $\tilde{f}_c$ vanishes at $q_c=0$. If f is meromorphic at c, the **order** of f at c is the least integer n such that $a_{c,n}\neq 0$. The notation is $\mathrm{ord}_{\Gamma,c}(f)$.

Ex: Let Γ,Γ' be two congruence subgroups such that $\Gamma'\subseteq\Gamma$. Let f be a meromorphic function on $\mathbb{H}$ that is weakly modular of weight k for Γ (hence, also for Γ'). Let $c'\in \mathrm{Cusps}(\Gamma')$ and let c be its image under the natural map $\mathrm{Cusps}(\Gamma')\longrightarrow\mathrm{Cusps}(\Gamma)$.

(i) Prove that $h_\Gamma(c)\mid h_{\Gamma'}(c')$ and that $\tilde{h}_\Gamma(c)\mid\tilde{h}_{\Gamma'}(c')$

(ii) Prove $$\frac{\mathrm{ord}_{\Gamma',c'}(f)}{\tilde{h}_{\Gamma'}(c')}=\frac{\mathrm{ord}_{\Gamma,c}(f)}{\tilde{h}_\Gamma(c)}$$

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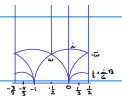

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Proof: (i) Since c is the image of c' we can choose t in both orbits. Then $\Gamma' \subseteq \Gamma \Rightarrow$

$$H_{c'} = \gamma_t^{-1}\Gamma'\gamma_t \cap SL_2(\mathbb{Z})_\infty \subseteq H_c = \gamma_t^{-1}\Gamma\gamma_t \cap SL_2(\mathbb{Z})_\infty$$

Notice that $\left\langle \begin{pmatrix}1 & h'\\ 0 & 1\end{pmatrix}\right\rangle \subseteq \left\langle \begin{pmatrix}-1 & h\\ 0 & -1\end{pmatrix}\right\rangle$ iff $\begin{pmatrix}-1 & h\\ 0 & -1\end{pmatrix}^2 \Big| \begin{pmatrix}1 & h'\\ 0 & 1\end{pmatrix}$ iff $2h \mid h'$. The other possibilities are obvious.

(ii) Let $\tilde{h}_{\Gamma'}(c') = d\,\tilde{h}_\Gamma(c)$. Then $q_c = q_{c'}^{d}$.

$$\tilde{f}_c(q_c) = \sum_{n\in\mathbb{Z}} a_{c,n}\, q_c^n = \sum_{n\in\mathbb{Z}} a_{c,n}\, q_{c'}^{dn}. \Rightarrow$$

$$\mathrm{ord}_{\Gamma',c'}(f) = d\,\mathrm{ord}_{\Gamma,c}(f). \quad \#.$$

Def: Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and let k be an integer. A **modular form** of weight k for the group Γ is a holomorphic function $f:\mathbb{H}\longrightarrow\mathbb{C}$ that is weakly modular of weight k for Γ and holomorphic at all cusps of Γ. Such f is called a **cusp form** (of weight k for Γ) if it vanishes at all cusps of Γ.

It is easy to check that the set of modular forms of weight k for Γ is a $\mathbb{C}$-vector space. We write $M_k(\Gamma)$ for this space, and $S_k(\Gamma)$ for the subspace of cusp forms.

Thm: Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and let k be an integer. Let $f:\mathbb{H}\longrightarrow\mathbb{C}$ be a holomorphic function which is weakly modular of weight k for Γ. Then the following are equivalent

(i) f is holomorphic at all cusps,

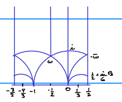
(ii) f is holomorphic at ∞ and there exist $c, d\in\mathbb{R}_{>0}$ such that the Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n q_\infty^n$ we have $|a_n| \le c n^d \ \forall n\in\mathbb{Z}_{>0}$.

Proof: (ii) $\Rightarrow$ (i) Let $z = x+iy$

$$|f(z)| \le \sum_{n=0}^{\infty} |a_n q_\infty^n| \le |a_0| + \sum_{n=1}^{\infty} c n^d \left|\exp(2\pi i n z/\tilde{h}_\infty(\Gamma))\right|$$

$$= a_0 + \sum_{n=1}^{\infty} c n^d e^{-2\pi n y/h}$$ Let $g:\mathbb{R}\longrightarrow\mathbb{R}$ $g(t) := t^d e^{-2\pi t y/h}$

We have $g'(t) = \left(d - \frac{2\pi y t}{h}\right) t^{d-1} e^{-2\pi t y/h}$. The function $g(t)$ increases in $\left[0, \frac{dh}{2\pi y}\right]$ and decreases in $\left[\frac{dh}{2\pi y}, \infty\right)$



We have $\displaystyle\sum_{n=1}^{\lfloor \frac{dh}{2\pi y}\rfloor -1} n^d e^{-2\pi n y/h} \le \int_1^{\lfloor \frac{dh}{2\pi y}\rfloor} g(t)\,dt$

$\displaystyle\sum_{n=\lceil \frac{dh}{2\pi y}\rceil}^{\infty} n^d e^{-2\pi n y/h} \le \int_{\lfloor \frac{dh}{2\pi y}\rfloor}^{\infty} g(t)\,dt$. Thus

$$|f(z)| \le |a_0| + C\left(\int_0^\infty g(t)\,dt + y^{-d}\right)$$

Take $s = 2\pi t y/h$

$$\int_0^\infty g(t)\,dt = \int_0^\infty \left(\frac{sh}{2\pi y}\right)^d e^{-s}\,\frac{h}{2\pi y}\,ds = \frac{C_2}{y^{d+1}}\int_0^\infty s^d e^{-s}\,ds$$

converges independently of y. Thus

$|f(z)| \le C_0 + C_1 y^{-d}$ as $y \to \infty$.

Now for $\gamma \in SL_2(\mathbb{Z})$, $z \longmapsto (f|_k\gamma)(z)$ is holomorphic and weight-k invariant under $\gamma^{-1}\Gamma\gamma$. Then it has a Laurent series $(f|_k\gamma)(z) = \sum_{n\in\mathbb{Z}} a_n' q_h^n$, $q_h = e^{2\pi i z/h}$

Now assume $0 \le x \le h$ $|cz+d|$ grows like y

$$|(f|_k\gamma)(z)| = |(cz+d)^{-k} f(\gamma(z))|$$

$$\le |cz+d|^{-k}\,|C_0 + C_1 \operatorname{Im}(\gamma(z))^{-d}| = |cz+d|^{-k}\left|C_0 + C_1\left(\frac{\operatorname{Im} z}{|cz+d|^2}\right)^{-d}\right|$$

$$\le C\, y^{-k}(C_0 + C_1 y^d) \le C\, y^{d-k}$$

Using $y = c\log(1/|q_h|)$,

$\displaystyle\lim_{q_h\to 0} |(f|_k\gamma)(z)\cdot q_h| \le c \lim_{q_h\to 0} y^{d-k}|q_h| = 0$, which implies that f is holomorphic at all cusps

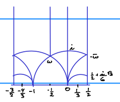
(i) $\Rightarrow$ (ii) will be discussed later. #.

Ex: Let $\Gamma' \subseteq \Gamma$ be two congruence subgroups, $k \in \mathbb{Z}$, f a meromorphic function on $\mathbb{H}$ that is weakly modular of weight k for Γ.

(i) Show that there is a canonical surjective map

$$\mathrm{Cusps}(\Gamma') \longrightarrow \mathrm{Cusps}(\Gamma)$$

(ii) Let $\mathfrak{c}' \in \mathrm{Cusps}(\Gamma')$ and let $\mathfrak{c} \in \mathrm{Cusps}(\Gamma)$ be its image. Show that f is holomorphic at $\mathfrak{c}$ iff f (viewed as a weakly modular function of weight k for Γ') is holomorphic at $\mathfrak{c}'$! Show that f vanishes at $\mathfrak{c}$ iff f (viewed as a weakly



modular function of weight k for Γ') vanishes at c!

(iii) Deduce that if f is a modular form (resp. a cusp form) of weight k for Γ, then it is a modular form (resp. a cusp form) of weight k for Γ' ($M_k(\Gamma)\subseteq M_k(\Gamma')$, $S_k(\Gamma)\subseteq S_k(\Gamma')$)

Proof: (i) $\Gamma'\subseteq\Gamma \Rightarrow \Gamma'\backslash\mathbb{P}^1(\mathbb{Q}) \longrightarrow \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$
$\Gamma' x \longrightarrow \Gamma x$

(ii) We previously proved

$$\frac{\mathrm{ord}_{\Gamma',c'}(f)}{\tilde{h}_{\Gamma'}(c')}=\frac{\mathrm{ord}_{\Gamma,c}(f)}{\tilde{h}_{\Gamma}(c)}$$

and it's easy to deduce (ii) from here.

(iii) trivial.

The Θ-function

Def: The Jacobi theta function is the holomorphic function $\theta: \mathbb{H}\longrightarrow\mathbb{C}$ defined by $\theta(z)=\sum_{n\in\mathbb{Z}} q^{n^2}=1+2\sum_{n=1}^{\infty} q^{n^2}$ ($q=\exp(2\pi i z)$).

This series converges uniformly by comparison with the geometric series, and this implies holomorphicity. It satisfies $\theta(z+1)=\theta(z)$ $\forall z\in\mathbb{H}$.

Thm: The function θ satisfies

$$\theta\left(\frac{-1}{4z}\right)=\sqrt{-2iz}\,\theta(z) \quad \forall z\in\mathbb{H},$$

where the branch of $\sqrt{-2iz}$ is taken to have positive real part.

Proof: Since both sides are holomorphic functions on $\mathbb{H}$, it suffices to prove this identity on the imaginary axis (this provides a set with a limit point where two holomorphic functions coincide). Let $z=\frac{ia}{2}$ with $a>0$.

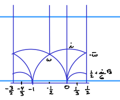
By Poisson Summation Formula,

$$\sum_{m\in\mathbb{Z}}\exp(-\pi a m^2)=\frac{1}{\sqrt{a}}\sum_{n\in\mathbb{Z}}\exp(-\pi n^2/a)$$

(The Fourier transform of $f_a(x)=\exp(-\pi a x^2)$ is $\hat{f}_a(t)=\frac{1}{\sqrt{a}}\exp(-\pi t^2/a)$.)

Substituting for $a=-2iz$,

$$\sum_{m\in\mathbb{Z}}\exp(2\pi i z m^2)=\frac{1}{\sqrt{-2iz}}\sum_{n\in\mathbb{Z}}\exp(-2\pi i n^2/(4z)) \quad \#$$



Coro: The function θ satisfies
$$\theta\left(\frac{z}{4z+1}\right) = \sqrt{4z+1}\,\theta(z) \quad \forall z \in \mathbb{H},$$
where the branch of $\sqrt{4z+1}$ is taken to have positive real part.

Proof: Let $z' = -1/(4z) - 1 \in \mathbb{H}$. Note that $\frac{z}{4z+1} = -\frac{1}{4z'}$.
We have
$$\theta\left(-\frac{1}{4z'}\right) = \sqrt{-2iz'}\,\theta(z') \Rightarrow \theta\left(\frac{z}{4z+1}\right) = \sqrt{-2iz'}\,\theta\left(-\frac{1}{4z} - 1\right)$$
$$= \sqrt{-2iz'}\,\theta\left(-\frac{1}{4z}\right) = \sqrt{-2iz'}\sqrt{-2iz}\,\theta(z) = \sqrt{4z+1}\,\theta(z). \quad \#$$

Lemma: Let $A = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $\langle A, T \rangle = \Gamma_1(4)$.

Proof: Denote $\Gamma = \langle A, T \rangle$. Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(4)$.
We have $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b' \\ c & nc+d \end{pmatrix}$. If $c \neq 0$, we can choose n such that $|nc+d| < |c|/2$. This gives $\gamma \in \Gamma$ such that $\alpha\gamma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ satisfies $|d'| < |c'|/2$. We also have
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 4n & 1 \end{pmatrix} = \begin{pmatrix} a' & b \\ c+4nd & d \end{pmatrix}$$
If $d \neq 0$, again we get $\gamma \in \Gamma$ such that $\alpha\gamma$ satisfies $|c'| < 2|d'|$. We apply this process until we reach either $c' = 0$ or $d' = 0$. If $c' = 0$, $\alpha\gamma = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \in \Gamma_1(4) \Rightarrow a' = d' = 1$ and $\alpha\gamma \in \Gamma$. If $d' = 0$, $\alpha\gamma = \begin{pmatrix} a' & b' \\ c' & 0 \end{pmatrix} \in \Gamma_1(4)$ contradiction.
Thus $\alpha\gamma \in \Gamma$ and $\gamma \in \Gamma \Rightarrow \alpha \in \Gamma$ #

Thm: Let k be an even positive integer. Then
$$\theta^k : z \longrightarrow \theta(z)^k$$
is a modular form of weight $k/2$ for the group $\Gamma_1(4)$.

Proof: It suffices to prove that $f := \theta^2 \in M_1(\Gamma_1(4))$. Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$. We have $f|_1 T = f$ and $f|_1 A = f$. By the Lemma, f is holomorphic and weakly modular of weight 1 for the group $\Gamma_1(4)$. By construction, it is also holomorphic at ∞. By a previous Theorem, it suffices to show that the absolute values of the Fourier coefficients are bounded by a polynomial in the index. But this is obvious since they are constant. #

Modular Forms

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EISENSTEIN SERIES OF WEIGHT 2. RECALL THAT $\dim M_2 = 0$ AND THAT THE "EISENSTEIN SERIES" E_2 IS NOT A MODULAR FORM. BUT WE CAN USE E_2 TO DEFINE MODULAR FORMS OF WEIGHT 2 FOR CONGRUENCE SUBGROUPS OF HIGHER LEVEL.

FOR EVERY POSITIVE INTEGER e DEFINE A HOLOMORPHIC FUNCTION $E_2^{(e)} : \mathbb{H} \longrightarrow \mathbb{C}$ BY

$$E_2^{(e)}(z) = E_2(z) - e\,E_2(ez)$$

LET $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(e)$. THEN

$$(cz+d)^{-2} E_2^{(e)}\left(\frac{az+b}{cz+d}\right)$$

$$= (cz+d)^{-2} E_2\left(\frac{az+b}{cz+d}\right) - e\,(cz+d)^{-2} E_2\left(e\,\frac{az+b}{cz+d}\right)$$

$$= (cz+d)^{-2} E_2\left(\frac{az+b}{cz+d}\right) - e\left(c/e\,(ez)+d\right)^{-2} E_2\left(\frac{a(ez)+be}{c/e\,(ez)+d}\right)$$

$$= E_2(z) - \frac{1}{4\pi i}\,\frac{c}{cz+d} - e\left(E_2(ez) - \frac{1}{4\pi i}\,\frac{c/e}{c/e\,(ez)+d}\right)$$

$$= E_2(z) - e\,E_2(ez) = E_2^{(e)}(z)$$

THUS $E_2^{(e)}(z)$ IS WEAKLY MODULAR OF WEIGHT 2 FOR $\Gamma_0(e)$. SINCE $E_2^{(e)}$ IS HOLOMORPHIC AT INFINITY AND $|a_n| = |\sigma_1(n)| < n^2$, THEN $E_2^{(e)}$ IS HOLOMORPHIC AT ALL THE CUSPS AND IT IS A MODULAR FORM FOR $\Gamma_0(e)$.