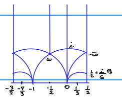


The valence formula for congruence subgroups

Notation: For any congruence subgroup Γ we write $\overline{\Gamma}$ for its image under the quotient $SL_2(\mathbb{Z}) \longrightarrow PSL_2(\mathbb{Z})$. We also write $\Gamma_z = \mathrm{Stab}_\Gamma z$ and $\overline{\Gamma}_z = \mathrm{Stab}_{\overline{\Gamma}} z$. $\forall z \in \mathbb{H}$.

Thm: (Valence formula for congruence subgroups). Let Γ be a congruence subgroup, and let k be an integer. Let f be a non-zero meromorphic function on $\mathbb{H}$ that is weakly modular of weight k for the group Γ and meromorphic at infinity. Let

$$\varepsilon_{\Gamma,c} = \begin{cases} 1 & \text{if } -1 \notin \Gamma \text{ and } c \text{ is regular,} \\ 2 & \text{if } -1 \in \Gamma \text{ or } c \text{ is irregular.} \end{cases}$$

and

$$\overline{\varepsilon}_{\Gamma,c} = \begin{cases} 1 & \text{if } c \text{ is regular,} \\ 2 & \text{if } c \text{ is irregular.} \end{cases}$$

Then we have

$$\sum_{z \in \Gamma \backslash \mathbb{H}} \frac{\mathrm{ord}_z(f)}{\# \Gamma_z} + \sum_{c \in \mathrm{Cusps}(\Gamma)} \frac{\mathrm{ord}_{\Gamma,c}(f)}{\varepsilon_{\Gamma,c}} = \frac{k}{24} [SL_2(\mathbb{Z}) : \Gamma]$$

and

$$\sum_{z \in \overline{\Gamma} \backslash \mathbb{H}} \frac{\mathrm{ord}_z(f)}{\# \overline{\Gamma}_z} + \sum_{c \in \mathrm{Cusps}(\Gamma)} \frac{\mathrm{ord}_{\Gamma,c}(f)}{\overline{\varepsilon}_{\Gamma,c}} = \frac{k}{12} [PSL_2(\mathbb{Z}) : \overline{\Gamma}]$$

Proof: Write $d = [SL_2(\mathbb{Z}) : \Gamma]$. Let R be a set of coset representatives for the quotient $\Gamma \backslash SL_2(\mathbb{Z})$. Then $\# R = d$.

Let $F(z) = \prod_{\gamma \in R} (f|_k \gamma)(z)$. This function is weakly modular of weight dk for the full modular group $SL_2(\mathbb{Z})$. By the Valence Formula,

$$\mathrm{ord}_\infty F + \frac{1}{2} \mathrm{ord}_i F + \frac{1}{3} \mathrm{ord}_\omega F + \sum_{\substack{z \in SL_2(\mathbb{Z}) \backslash \mathbb{H} \\ z \neq i, \omega, \infty}} \mathrm{ord}_z F = \frac{dk}{12}$$

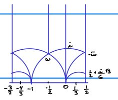
This can be rewritten as

$$\frac{1}{2} \mathrm{ord}_\infty F + \sum_{z \in SL_2(\mathbb{Z}) \backslash \mathbb{H}} \frac{\mathrm{ord}_z F}{\# SL_2(\mathbb{Z})_z} = \frac{dk}{24}$$

Let $z \in \mathbb{H}$. We apply the stabilizer lemma (ii) (Notes 4) to $G = SL_2(\mathbb{Z})$ and $H = \Gamma$ and $X = SL_2(\mathbb{Z})$-orbit of z.

$$\mathrm{ord}_z F = \sum_{\gamma \in \Gamma \backslash SL_2(\mathbb{Z})} \mathrm{ord}_z (f|_k \gamma) = \sum_{\gamma \in \Gamma \backslash SL_2(\mathbb{Z})} \mathrm{ord}_{\gamma z}(f)$$

Modular Forms



M. Lalín

$$= \sum_{v\in\Gamma\backslash SL_2(\mathbb{Z})z} [SL_2(\mathbb{Z})_v : \Gamma_v]\, \mathrm{ord}_v f, \quad (\star)$$

Where we have used that $\mathrm{ord}_{\gamma z} f$ depends only on γz and not on γ. $\Gamma\backslash SL_2(\mathbb{Z}) \longrightarrow \Gamma\backslash SL_2(\mathbb{Z})z$ fiber has $[SL_2(\mathbb{Z})_v : \Gamma_v]$ elements.

Since $SL_2(\mathbb{Z})_v$ is finite and independent of $v \in \Gamma\backslash SL_2(\mathbb{Z})z$, $[SL_2(\mathbb{Z})_v : \Gamma_v] = \dfrac{\#SL_2(\mathbb{Z})_z}{\#\Gamma_v}$. We divide $(\star)$ by $\#SL_2(\mathbb{Z})_z$

$$\frac{\mathrm{ord}_z F}{\#SL_2(\mathbb{Z})_z} = \sum_{v\in\Gamma\backslash SL_2(\mathbb{Z})z} \frac{\mathrm{ord}_v f}{\#\Gamma_v}$$

Summing over a system of orbit representatives for the quotient $SL_2(\mathbb{Z})\backslash\mathbb{H}$, we obtain

$$\sum_{z\in SL_2(\mathbb{Z})\backslash\mathbb{H}} \frac{\mathrm{ord}_z F}{\#SL_2(\mathbb{Z})_z} = \sum_{z\in SL_2(\mathbb{Z})\backslash\mathbb{H}} \sum_{v\in\Gamma\backslash SL_2(\mathbb{Z})z} \frac{\mathrm{ord}_v f}{\#\Gamma_v}$$

$$= \sum_{v\in\Gamma\backslash\mathbb{H}} \frac{\mathrm{ord}_v f}{\#\Gamma_v}$$

We will use $\dfrac{1}{2}\mathrm{ord}_\infty F = \displaystyle\sum_{c\in \mathrm{Cusp}(\Gamma)} \frac{\mathrm{ord}_{\Gamma,c}(f)}{\varepsilon_{\Gamma,c}}$ $\boxtimes$ (to be proved later)

Then

$$\sum_{v\in\Gamma\backslash\mathbb{H}} \frac{\mathrm{ord}_v f}{\#\Gamma_v} + \sum_{c\in\mathrm{Cusp}(\Gamma)} \frac{\mathrm{ord}_{\Gamma,c}(f)}{\varepsilon_{\Gamma,c}} =$$

$$= \sum_{z\in SL_2(\mathbb{Z})\backslash\mathbb{H}} \frac{\mathrm{ord}_z F}{\#SL_2(\mathbb{Z})_z} + \frac{1}{2}\mathrm{ord}_\infty(F) = \frac{k}{24}[SL_2(\mathbb{Z}):\Gamma],$$

which proves the first formula. The second formula follows by multiplying by $\#(\Gamma\cap\{\pm 1\})$ #

<u>Ex:</u> Consider $Z = SL_2(\mathbb{Z}) / \left\{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} \,\middle|\, b\in\mathbb{Z}\right\}$ equiped with the natural left action of $SL_2(\mathbb{Z})$. Let u be the class of the unit matrix in Z.

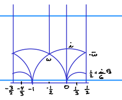
(i) Show that $\exists!$ map $Z \longrightarrow \mathbb{P}^1(\mathbb{Q})$ compatible with the $SL_2(\mathbb{Z})$-action and sending u to ∞.

(ii) Consider $\begin{array}{ccc}\Gamma\backslash Z & \longrightarrow & \mathrm{Cusps}(\Gamma)\\ x & \longmapsto & \bar{x}\end{array}$ obtained by quotienting by Γ the map of (i).

Show that for $c\in\mathrm{Cusps}(\Gamma)$, the fiber over c has cardinality $2/\varepsilon_{\Gamma,c}$

(38)

Modular Forms



M. Lalín

(iii) Show that for each $x \in \Gamma\backslash Z$ the fiber of the map $\Gamma\backslash SL_2(\mathbb{Z}) \longrightarrow \Gamma\backslash Z$ over x has cardinality $\tilde{h}_\Gamma(\bar{x})$.

Proof: (i) We send $\alpha \in Z$ to $\alpha\infty$. Notice that this is well defined because $\begin{pmatrix}1&b\\0&1\end{pmatrix}\infty=\infty$. It is clearly compatible with the action and sends μ to ∞. The map is unique because of the action of $SL_2(\mathbb{Z})$, because $\varphi(\alpha)=\alpha\varphi\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)$ $=\alpha\infty$

(ii) $\Gamma\backslash Z \longrightarrow \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$. Let $\Gamma\gamma_1, \Gamma\gamma_2 \in \Gamma\backslash Z$ in the fiber of c

Then: $\Gamma\gamma_1\infty = \Gamma\gamma_2\infty \Rightarrow \exists\gamma\in\Gamma$ $\gamma\gamma_1\infty=\gamma_2\infty \Rightarrow \gamma_2^{-1}\gamma\gamma_1 \in SL_2(\mathbb{Z})_\infty$

$\Rightarrow \gamma_2^{-1}\gamma\gamma_1 = \delta\begin{pmatrix}1&b\\0&1\end{pmatrix}$, $\delta=\pm1 \Rightarrow \gamma_2^{-1}\gamma\gamma_1=\delta$ in $Z \Rightarrow \gamma=\delta\gamma_2\gamma_1^{-1}$ in Z. $\Rightarrow \Gamma\delta\gamma_2\gamma_1^{-1} = \delta\Gamma\gamma_2\gamma_1^{-1}=\Gamma \Rightarrow$ either $\Gamma\gamma_1=\Gamma\gamma_2$ or $\Gamma\gamma_1=-\Gamma\gamma_2$

When is $\Gamma\gamma_t \neq -\Gamma\gamma_t$ in $\Gamma\backslash Z$? $\Leftrightarrow \gamma_t^{-1}\Gamma\gamma_t \neq -\gamma_t^{-1}\Gamma\gamma_t$

$\Leftrightarrow H_c \neq -H_c$ modulo $\left\{\begin{pmatrix}1&b\\0&1\end{pmatrix}\right\}$ $\Leftrightarrow -1\notin\Gamma$ and c regular

$\Leftrightarrow -1\notin\Gamma$ and c is regular.

(iii) $\Gamma\backslash SL_2(\mathbb{Z}) \longrightarrow \Gamma\backslash Z \longrightarrow \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ The fiber of c has cardinality $h_\Gamma(c)$ if $-1\in\Gamma$, and $2h_\Gamma(c)$ if $-1\notin\Gamma$. Using (ii) we get

$$\begin{cases} h_\Gamma(c)e_{\Gamma c}/2 & -1\in\Gamma \\ h_\Gamma(c)e_{\Gamma c} & -1\notin\Gamma \end{cases} = \begin{cases} h_\Gamma(c) & -1\in\Gamma \quad (c \text{ is always regular}) \\ 2h_\Gamma(c) & -1\notin\Gamma,\ c \text{ irregular} \\ h_\Gamma(c) & -1\notin\Gamma,\ c \text{ regular} \end{cases} = \tilde{h}_\Gamma(c)$$

Ex: Choose a congruence subgroup Γ' contained in Γ such that $\Gamma' \trianglelefteq SL_2(\mathbb{Z})$. Let $\tilde{h}_{\Gamma'}$ be the common value $\tilde{h}_{\Gamma'}(c)$ for all cusps c of Γ'.

(i) Show that all fibers of $\Gamma'\backslash SL_2(\mathbb{Z}) \longrightarrow \Gamma\backslash SL_2(\mathbb{Z})$ have cardinality $(\Gamma:\Gamma')$

(ii) Prove $\displaystyle\sum_{\gamma\in\Gamma'\backslash SL_2(\mathbb{Z})} \mathrm{ord}_{\Gamma',\overline{\gamma\infty}}(f) = [SL_2(\mathbb{Z}):\Gamma']\,\tilde{h}_{\Gamma'}\,\mathrm{ord}_\infty(f)$

(iii) Prove $\displaystyle\sum_{\gamma\in\Gamma'\backslash SL_2(\mathbb{Z})} \mathrm{ord}_{\Gamma',\overline{\gamma\infty}}(f) = [\Gamma:\Gamma']\,\tilde{h}_{\Gamma'} \sum_{x\in\Gamma\backslash Z} \mathrm{ord}_{\Gamma,\bar{x}}(f)$

(iv) Deduce $\boxtimes$

Modular Forms

M. Lalín

Proof: (i) By comparing indexes. $\Gamma'\backslash SL_2(\mathbb{Z}) \xrightarrow{\varphi} \Gamma\backslash SL_2(\mathbb{Z})$
the fiber has $[\Gamma:\Gamma'] = [SL_2(\mathbb{Z}):\Gamma'] / [SL_2(\mathbb{Z}):\Gamma]$ elements.

(ii) $\Gamma = SL_2(\mathbb{Z})$ in $\dfrac{\mathrm{ord}_{\Gamma',c'}(f)}{\tilde{h}_{\Gamma'}(c')} = \dfrac{\mathrm{ord}_{\Gamma,c}(f)}{\tilde{h}_{\Gamma}(c)}$ (the only cusp in $SL_2(\mathbb{Z})$ is ∞) gives

$$\frac{\mathrm{ord}_{\Gamma',\overline{\gamma\infty}}(f)}{\tilde{h}_{\Gamma'}} = \frac{\mathrm{ord}_{\Gamma',\overline{\infty}}(f|\gamma)}{\tilde{h}_{\Gamma'}} = \frac{\mathrm{ord}_{SL_2(\mathbb{Z}),\infty}(f|\gamma)}{\tilde{h}_{SL_2(\mathbb{Z})}=1}$$

$$\sum_{\gamma\in\Gamma'\backslash SL_2(\mathbb{Z})} \mathrm{ord}_{\Gamma',\overline{\gamma\infty}}(f) = \sum_{\gamma\in\Gamma'\backslash SL_2(\mathbb{Z})} \tilde{h}_{\Gamma'}\,\mathrm{ord}_{SL_2(\mathbb{Z}),\infty}(f|\gamma)$$

$$\overset{(i)}{=} \tilde{h}_{\Gamma'}[\Gamma:\Gamma'] \sum_{\gamma\in\Gamma\backslash SL_2(\mathbb{Z})} \mathrm{ord}_{SL_2(\mathbb{Z}),\infty}(f|\gamma) = \tilde{h}_{\Gamma'}[\Gamma:\Gamma']\,\mathrm{ord}_{\infty}(f)$$

(iii) $$\sum_{\gamma\in\Gamma'\backslash SL_2(\mathbb{Z})} \frac{\mathrm{ord}_{\Gamma',\overline{\gamma\infty}}(f)}{\tilde{h}_{\Gamma'}} = [\Gamma:\Gamma'] \sum_{\gamma\in\Gamma\backslash SL_2(\mathbb{Z})} \frac{\mathrm{ord}_{\Gamma,\overline{\gamma\infty}}(f)}{\tilde{h}_{\Gamma}(\overline{\gamma\infty})}$$

$$\overset{\text{(ii) previous ex}}{=} [\Gamma:\Gamma'] \sum_{x\in\Gamma\backslash\mathbb{Z}} \mathrm{ord}_{\Gamma,\bar{x}}(f)$$

(iv) $$[\Gamma:\Gamma']\,\tilde{h}_{\Gamma'}\,\mathrm{ord}_{\infty}(f) \overset{(ii)}{=} \sum_{\gamma\in\Gamma'\backslash SL_2(\mathbb{Z})} \mathrm{ord}_{\Gamma',\overline{\gamma\infty}}(f) \overset{(iii)}{=}$$

$$= [\Gamma:\Gamma'] \sum_{x\in\Gamma\backslash\mathbb{Z}} \mathrm{ord}_{\Gamma,\bar{x}}(f) \overset{\text{(ii) previous}}{=} [\Gamma:\Gamma'] \sum_{c\in \mathrm{Cusps}(\Gamma)} \frac{2\,\mathrm{ord}_{\Gamma,c}(f)}{e_{\Gamma,c}} \quad \#$$

Coro: Let $f\in M_k(\Gamma)$ be a modular form with q-expansion $\sum_{n=0}^{\infty} a_n q^n$ at some cusp c of Γ. Suppose $a_j = 0$ for $j = 0, 1, \ldots \left\lfloor \frac{k}{24}\, \varepsilon_{\Gamma,c}\,[SL_2(\mathbb{Z}):\Gamma] \right\rfloor$
Then $f=0$. Similarly, two forms in $M_k(\Gamma)$ are equal whenever their q-expansions at c agree to this precision.

Coro: The space of modular forms of a given weight for a given congruence subgroup Γ with at least one regular cusp has dimension at most $1+\left\lfloor \frac{k}{24}[PSL_2(\mathbb{Z}):\bar{\Gamma}] \right\rfloor$

Rem: There are complicated formulas for the dimensions of $M_k(\Gamma)$, $S_k(\Gamma)$. (Chapter 3, Diamond).

Dirichlet Characters

Def: Let N be a positive integer. A Dirichlet character modulo N is a function $\chi:\mathbb{Z}\longrightarrow\mathbb{C}$ such that $\exists\ \chi':(\mathbb{Z}/N\mathbb{Z})^{\times}\longrightarrow\mathbb{C}^{\times}$

with $\chi(d) = \begin{cases} \chi'(d \bmod N) & \text{if } (d,N)=1 \\ 0 & \text{if } (d,N)\neq 1. \end{cases}$

Sometimes we use this terminology for χ' itself.

Since $(\mathbb{Z}/N\mathbb{Z})^\times$ is finite, the image in $\mathbb{C}^\times$ is contained in the group of roots of unity.

For N fixed, the set of Dirichlet characters modulo N is a group under pointwise multiplication, which can be identified $\mathrm{Hom}((\mathbb{Z}/N\mathbb{Z})^\times, \mathbb{C}^\times)$. By decomposing $(\mathbb{Z}/N\mathbb{Z})^\times$ as a product of cyclic groups, $\mathrm{Hom}((\mathbb{Z}/N\mathbb{Z})^\times, \mathbb{C}^\times)$ is non-canonically isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times$. In particular, there are $\phi(N)$ Dirichlet characters modulo N.

Let M, N be positive integers with $M|N$ and let χ be a Dirichlet character modulo M. Then χ can be lifted to a Dirichlet character modulo N denoted $\chi^{(N)}$ as

$$\chi^{(N)}(m) = \begin{cases} \chi(m) & \text{if } (m,N)=1 \\ 0 & \text{if } (m,N)>1 \end{cases}$$

The conductor of a Dirichlet character χ modulo N is the smallest divisor M of N such that there exists a Dirichlet character χ_M modulo M satisfying $\chi = \chi_M^{(N)}$. A Dirichlet character modulo N is primitive if its conductor equals N.

Ex: Modulo 1 we have the trivial character $\mathbb{1}: (\mathbb{Z}/1\mathbb{Z})^\times = \{1\} \longrightarrow \mathbb{C}$

The corresponding Dirichlet character $\mathbb{1}: \mathbb{Z} \longrightarrow \mathbb{C}$ is the constant function 1. For any N, $\mathbb{1}$ lifts to a Dirichlet character modulo N

$$\mathbb{1}^{(N)}: \mathbb{Z} \longrightarrow \mathbb{C} \qquad m \longrightarrow \begin{cases} 1 & \text{if } (m,N)=1 \\ 0 & \text{if } (m,N)>1 \end{cases}$$

Ex: Let $N=4$. Then $(\mathbb{Z}/4\mathbb{Z})^\times$ has order 2. There exists a unique non-trivial Dirichlet character χ modulo 4 given by

$$\chi(m) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod 4 \\ -1 & \text{if } m \equiv -1 \pmod 4 \\ 0 & \text{if } m \equiv 0, 2 \pmod 4 \end{cases}$$

Ex: If $N=p$ prime number, put

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p | a \\ 1 & \text{if } p \nmid a \text{ and } a \equiv \square \pmod p \\ -1 & \text{if } p \nmid a \text{ and } a \not\equiv \square \pmod p \end{cases}$$

The map $a \longrightarrow \left(\frac{a}{p}\right)$ is a Dirichlet character modulo p. It is of conductor p if $p>2$ and of conductor 1 if $p=2$.

Given two Dirichlet characters χ_1, χ_2 modulo N_1, N_2, and let $N=[N_1; N_2]$ (least common multiple). Then one can form

$$\chi = \chi_1 \chi_2 = \chi_1^{(N)} \chi_2^{(N)},$$

a character modulo N.

<u>Applications of modular forms to sums of squares</u>

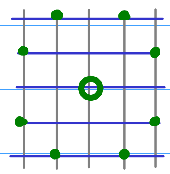
<u>Question:</u> Given positive integers n and k, in how many ways can n be written as a sum of k squares of integers?

Write $r_k(n) = \# \{ (x_1, \dots, x_k) \in \mathbb{Z}^k \mid x_1^2 + \dots + x_k^2 = n \}$

Then the question is how to find $r_k(n)$. Notice that changes in signs and order are counted as different solutions.

For example $r_2(5)=8$, as $5 = 1^2+2^2 = (-1)^2 + 2^2 = 1^2+(-2)^2 = (-1)^2+(-2)^2 = 2^2+1^2 = (-2)^2+1^2 = 2^2+(-1)^2 = (-2)^2+(-1)^2$.

This can be viewed geometrically as saying that in the square lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ there are 8 points whose distance from the origin is $\sqrt{5}$.



<u>Thm</u> (Fermat) Let n be an odd positive integer. Then n is the sum of two squares iff every prime $p \mid n$ with $p \equiv 3 \pmod 4$ occurs an even number of times in the prime factorization of n.

<u>Coro:</u> Let p be a prime number. Then p is a sum of two squares iff $p=2$ or $p \equiv 1 \pmod 4$

<u>Thm</u> (Lagrange): Every non-negative integer is a sum of four squares.

(The Thms above are proved without modular forms)

In the next results χ is the Dirichlet character modulo 4 given before.

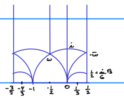
<u>Thm:</u> (Jacobi) The functions $r_2(n)$ and $r_4(n)$ are given by

$$r_2(n) = 4 \sum_{d \mid n} \chi(d), \qquad r_4(n) = 8 \sum_{d \mid n,\, 4 \nmid n} d \qquad \forall n \geq 1$$

In the first sum d runs over all the positive divisors of n.

In the second sum d runs over those that are not divisible by 4.

Modular Forms



M. Lalín

Thm (Jacobi, Eisenstein, Smith): The functions $r_6(n)$ and $r_8(n)$ are given by $r_6(n)=\sum_{d|n}\left(16\,\chi(n/d)-4\,\chi(d)\right)d^2$,
$r_8(n)=16\sum_{d|n}(-1)^{n-d}d^3$ $\quad\forall n\geq 1$

Thm (Glaisher, conjectured by Liouville) The functions $r_{10}(n)$ and $r_{12}(n)$ are (partially) given by

$$r_{10}(n)=\frac{4}{5}\sum_{d|n}\left(\chi(d)+16\,\chi(n/d)\right)d^2+\frac{8}{5}\sum_{\substack{z\in\mathbb{Z}[i]\\|z|^2=n}} z^4 \qquad \forall n\geq 1$$

$$r_{12}(n)=8\sum_{d|n}d^5-512\sum_{d|n/4}d^5 \quad \forall\, n\geq 2 \text{ even.}$$

Recall that $\theta(z)=\sum_{n\in\mathbb{Z}}q^{n^2}$. Thus, $\theta^k(z)=\sum_{n=0}^{\infty}r_k(n)q^n$ $\forall z\in\mathbb{H}$.
We proved that $\theta^k\in M_{k/2}(\Gamma_1(4))$. We have $[SL_2(\mathbb{Z}):\Gamma_1(4)]=12$.
By the valence formula, $\dim M_{k/2}(\Gamma_1(4))\leq 1+\lfloor k/4\rfloor$
The group $\Gamma_1(4)$ has three cusps $(0,1/2,\infty)$, two of which are regular.

Let $k=4$. Then $\dim M_2(\Gamma_1(4))\leq 2$. We know two linearly independent elements, namely $E_2^{(2)}(z)=E_2(z)-2E_2(2z)$ and $E_2^{(4)}(z)=E_2(z)-4\,E_2(4z)$. They form a basis for $M_2(\Gamma_1(4))$.
We have $E_2(z)=-\frac{1}{24}+\sum_{n=1}^{\infty}\sigma_1(n)q^n$ and find
$E_2(z)=-\frac{1}{24}+q+3q^2+O(q^3)$, $\quad E_2(2z)=-\frac{1}{24}+q^2+O(q^3)$
$E_2(4z)=-\frac{1}{24}+O(q^3)$.
On the other hand, $\quad\theta(z)^4=1+8q+24q^2+O(q^3)$
If $\theta(z)^4=c_1E_2(z)+c_2E_2(2z)+c_4E_2(4z)$, the coefficients must satisfy

$$\begin{pmatrix}-1/24 & -1/24 & -1/24\\ 1 & 0 & 0\\ 3 & 1 & 0\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}1\\8\\24\end{pmatrix} \quad \text{we get} \quad \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}8\\0\\-32\end{pmatrix}$$

Thus $\theta(z)^4=8E_2(z)-32E_2(4z)=8E_2^{(4)}(z)$.
Taking coefficients,

$$r_4(n)=8\Big(\sum_{d|n}d-4\underbrace{\sum_{d|(n/4)}d}_{\text{only included if } 4|n}\Big)=8\sum_{d|n,\,4\nmid d}d$$

#

For $k=8$, one can see that $M_4(\Gamma_1(4))$ has dimension 3, with basis given by $E_4(z)$, $E_4(2z)$, $E_4(4z)$. We have

$$E_4(z)=\frac{1}{240}+\sum_{n=1}^{\infty}\sigma_3(n)q^n$$

43

Modular Forms

M. Lalín

As before, one can obtain $\theta(z)^8 = 16E_4(z) - 32E_4(2z) + 256E_4(4z)$

This implies

$$r_8(n) = 16\left(\sum_{d|n} d^3 - 2\sum_{d|(n/2)} d^3 + 16\sum_{d|(n/4)} d^3\right)$$

Let $2^j \| n$. $n = 2^j n_1$

$$r_8(n) = 2^4\sum_{d|n} d^3 - 2^5 \sum_{d|n/2} d^3 + 2^8 \sum_{d|n/4} d^3$$

$$= 2^4 \sum_{\ell=0}^{j} \sum_{\substack{d|n\\ 2^\ell\|d}} d^3 - 2^5 \sum_{\ell=0}^{j-1}\sum_{\substack{d|n\\ 2^\ell\|d}} d^3 + 2^8\sum_{\ell=0}^{j-2}\sum_{\substack{d|n\\ 2^\ell\|d}} d^3$$

$$= 2^4 \sum_{\substack{d|n\\ 2^j\|d}} d^3 - 2^4\sum_{\substack{d|n\\ 2^{j-1}\|d}} d^3 + 2^4\cdot 15 \sum_{\ell=0}^{j-2}\sum_{\substack{d|n\\ 2^\ell\|d}} d^3$$

$$= 2^4\left(2^{3j} - 2^{3(j-1)} + 15\sum_{\ell=0}^{j-2} 2^{3\ell}\right)\sum_{d|n_1} d^3$$

$$= 2^4\left(2^{3(j-1)}\cdot 7 + 15\left(\frac{2^{3(j-1)}-1}{7}\right)\right)\sum_{d|n_1} d^3 = 2^4\left(\frac{2^{3(j+1)}-1}{7} - 2\right)\sum_{d|n_1} d^3$$

$$= 2^4\left(2^{3j}+2^{3(j-1)}+\dots+2^3-1\right)\sum_{d|n_1} d^3 = 2^4\sum_{d|n}(-1)^{n-d} d^3 \quad \#$$