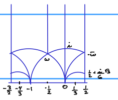
Modular Forms



M. Lalín

Hecke operators and eigenforms M_k has more structure than a complex vector space. It is a module over a commutative ring called the Hecke algebra

The operators T_α We extend the action of $SL_2(\mathbb{Z})$ on the set of meromorphic functions on $\mathbb{H}$ to an action of

$$GL_2^+(\mathbb{Q}) = \{\gamma \in GL_2(\mathbb{Q}) \mid \det\gamma > 0\}$$

$$(f|_k\gamma)(z) = \frac{(\det\gamma)^k}{(cz+d)^k} f\left(\frac{az+b}{cz+d}\right) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$$

Let Γ be a congruence subgroup and let $\alpha \in GL_2^+(\mathbb{Q})$.

Define $$\Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha$$

Let f be a modular form of weight k for Γ. Then $f|_k\alpha$ is invariant under the right action of $\alpha^{-1}\Gamma\alpha$, hence it is a modular form for Γ'. We define

$$T_\alpha f = \sum_{\gamma \in \Gamma'\backslash\Gamma} f|_k \alpha\gamma$$

Prop: Let Γ be a congruence subgroup, let k be an integer, and let $\alpha \in GL_2^+(\mathbb{Q})$. Then for any $f \in M_k(\Gamma)$, we also have $T_\alpha f \in M_k(\Gamma)$. Moreover, if $f \in S_k(\Gamma)$, then $T_\alpha f \in S_k(\Gamma)$

Proof: First notice that $f|_k \alpha\gamma = (f|_k\alpha)|_k\gamma$. Now, let $\delta \in \Gamma$.

$$T_\alpha f|_k \delta = \sum_{\gamma \in \Gamma'\backslash\Gamma} (f|_k\alpha\gamma)|_k\delta = \sum_{\gamma \in \Gamma'\backslash\Gamma} f|_k\alpha(\gamma\delta) =$$

$$= \sum_{\tilde\gamma \in \Gamma'\backslash\Gamma} f_k|\alpha\tilde\gamma = T_\alpha f$$

Clearly if f is holomorphic (zero) at the cusps, then $T_\alpha f$ is holomorphic (zero) at the cusps. #.

The map $f \longrightarrow T_\alpha f$ defines an endomorphism of the $\mathbb{C}$-vector space $M_k(\Gamma)$ which preserves $S_k(\Gamma)$.

Rem: There is an isomorphism
$$\Gamma'\backslash\Gamma \longrightarrow \Gamma\backslash\Gamma\alpha\Gamma$$
$$\Gamma'\gamma \longrightarrow \Gamma\alpha\gamma$$

Indeed, $\Gamma'\gamma_1 = \Gamma'\gamma_2 \Leftrightarrow \gamma_1\gamma_2^{-1} \in \Gamma' \Leftrightarrow \gamma_1\gamma_2^{-1} \in \Gamma \cap \alpha^{-1}\Gamma\alpha \Leftrightarrow$
$\gamma_1\gamma_2^{-1} \in \Gamma \wedge \alpha\gamma_1\gamma_2^{-1}\alpha^{-1} \in \Gamma \Leftrightarrow \Gamma\alpha\gamma_1 = \Gamma\alpha\gamma_2$ and we can write

$$T_\alpha f = \sum_{\gamma \in \Gamma\backslash\Gamma\alpha\Gamma} f|_k\gamma.$$

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A special case is when α normalises Γ. Then $\Gamma' = \Gamma$ and $T_\alpha f = f|_k \alpha$. If moreover $\alpha \in \Gamma$, then $T_\alpha f = f$.

Ex: Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and $k \in \mathbb{Z}$.

(i) Let $\Gamma' \subseteq \Gamma$ be a congruence subgroup and let $g \in M_k(\Gamma')$. Show that $g \in M_k(\Gamma)$ iff g is invariant under the weight k action of Γ.

(ii) Let $f \in M_k(\Gamma)$ and let $\alpha \in GL_2^+(\mathbb{Q})$. Show that there is a congruence subgroup $\Gamma' \subset \Gamma \cap \alpha^{-1}\Gamma\alpha$ such that $\forall \gamma \in \Gamma$, $f|_k \alpha\gamma \in M_k(\Gamma')$

(iii) Show that $T_\alpha f = \sum_{\gamma \in \Gamma' \backslash \Gamma} f|_k \alpha\gamma \in M_k(\Gamma)$

Proof: (i) The condition that g is invariant under the action of Γ is necessary. It is also sufficient because we proved that in that case g is holomorphic (vanishes) at c' iff it is holomorphic (vanishes) at c.

(ii) We have proved in Notes 4 that $\Gamma \cap \alpha^{-1}\Gamma\alpha$ is a congruence subgroup. Take $\Gamma' = \Gamma(N) \subseteq \Gamma \cap \alpha^{-1}\Gamma\alpha$. In particular $\Gamma' \trianglelefteq \Gamma \cap \alpha^{-1}\Gamma\alpha$.
Let $\gamma' \in \Gamma'$ then $\alpha\gamma\gamma'\gamma^{-1}\alpha^{-1} \in \Gamma'$ Write $\tilde{\gamma} = \alpha\gamma\gamma'\gamma^{-1}\alpha^{-1}$ We have

$$(f|_k \alpha\gamma)|_k \gamma' = f|_k(\alpha\gamma\gamma') = f|_k(\tilde{\gamma}\alpha\gamma) = (f|_k \tilde{\gamma})|_k \alpha\gamma = f|_k \alpha\gamma$$

(iii) Trivial from (i) and (ii)

<u>Hecke operators for $\Gamma_1(N)$</u> Choose $N>0$ and take $\Gamma = \Gamma_1(N)$

We have
$$\Gamma_1(N)\backslash\Gamma_0(N) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^\times$$
$$\Gamma_1(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow d \pmod N.$$

Furthermore, given $d \in \mathbb{Z}$, $(d,N)=1$, we can find $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$
For such α, let $\langle d \rangle f = T_\alpha f$ $\forall f \in M_k(\Gamma_1(N))$, in other words
$$(\langle d \rangle f)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

This definition only depends on the class of α in $\Gamma_1(N)\backslash\Gamma_0(N)$ which is given by $d \pmod N$. This gives an action of $(\mathbb{Z}/N\mathbb{Z})^\times$ on $M_k(\Gamma_1(N))$

Ex: Show that the space of $(\mathbb{Z}/N\mathbb{Z})^\times$-invariants in $M_k(\Gamma_1(N))$

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can be identified with $M_k(\Gamma_0(N))$

Proof: Notice that $\forall \alpha \in \Gamma_0(N)$, we can realize $f|_k\alpha$ as $\langle d\rangle f$. If $\langle d\rangle f = f$ $\forall d \in (\mathbb{Z}/N\mathbb{Z})^*$, then clearly $f|_k\alpha = f$ $\forall \alpha \in \Gamma_0(N)$ and $f \in M_k(\Gamma_0(N))$. #.

Now we take $\alpha = \begin{pmatrix}1&0\\0&p\end{pmatrix} \notin SL_2(\mathbb{Z})$.

Def: Let N be a positive integer and let p be a prime number. The Hecke operator T_p is the $\mathbb{C}$-linear endomorphism of $M_k(\Gamma_1(N))$ defined by

$$T_p f = \underbrace{\frac{1}{p}}_{\text{this factor gives nicer formulas.}} T_{\begin{pmatrix}1&0\\0&p\end{pmatrix}} f \qquad \forall f \in M_k(\Gamma_1(N)).$$

Lemma: Let N be a positive integer, let p be a prime number and let $\alpha = \begin{pmatrix}1&0\\0&p\end{pmatrix}$, $\Gamma = \Gamma_1(N)$, $\Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha$.
Then,

$$\Gamma' = \left\{ \gamma = \begin{pmatrix}a&b\\c&d\end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix}1&*\\0&1\end{pmatrix} \pmod N \wedge p \mid b \right\}$$

A system of coset representatives for the quotient $\Gamma' \backslash \Gamma$ is given by

$$\begin{cases} \left\{ \begin{pmatrix}1&b\\0&1\end{pmatrix} \;\middle|\; 0 \le b \le p-1 \right\} & \text{if } p \mid N, \\ \left\{ \begin{pmatrix}1&b\\0&1\end{pmatrix} \;\middle|\; 0 \le b \le p-1 \right\} \cup \left\{ \begin{pmatrix}ap&1\\cN&1\end{pmatrix} \right\} & \text{if } p \nmid N, \end{cases}$$

where $a, c \in \mathbb{Z}$ are such that $ap - cN = 1$

Proof: Let $\gamma = \begin{pmatrix}a&b\\c&d\end{pmatrix} \in \Gamma$. Then

$$\alpha^{-1}\gamma\alpha = \begin{pmatrix}1&0\\0&p^{-1}\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&0\\0&p\end{pmatrix} = \begin{pmatrix}1&0\\0&p^{-1}\end{pmatrix}\begin{pmatrix}a&bp\\c&dp\end{pmatrix} = \begin{pmatrix}a&bp\\c/p&d\end{pmatrix}$$

Hence Γ' is the set of matrices from Γ whose right coefficient is divisible by p. This gives the first claim.

To find systems of coset representatives, we consider the map $\Gamma = \Gamma_1(N) \longrightarrow SL_2(\mathbb{F}_p)$ in two cases: $p \mid N$ and $p \nmid N$.

If $p \mid N$, the image of the map above consists of all matrices of the form $\begin{pmatrix}1&*\\0&1\end{pmatrix}$ and the inverse image of $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ equals Γ'.
Then $\Gamma' \backslash \Gamma \doteq \left\{ \begin{pmatrix}1&0\\0&1\end{pmatrix} \right\} \backslash \left\{ \begin{pmatrix}1&b\\0&1\end{pmatrix} \;\middle|\; b \in \mathbb{F}_p \right\}$

This description shows $\left\{ \begin{pmatrix}1&b\\0&1\end{pmatrix} \;\middle|\; 0 \le b \le p-1 \right\}$ is a system of

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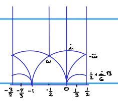

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Coset representatives of $\Gamma' \backslash \Gamma$

If $p \nmid N$ the reduction map $\Gamma \longmapsto SL_2(\mathbb{F}_p)$ is surjective. Indeed, we know $SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{F}_p)$ surjective. But $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ may be replaced by $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a & b \\ c+lpa & d+lpb \end{pmatrix}$ such that $d' \equiv 1 \pmod N$ (unless $N|b$, but then we can replace by $\begin{pmatrix} a+pc & b+pd \\ c & d \end{pmatrix}$ first if necessary). Then we do $\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a'+l'pb' & b' \\ c'+l'pd' & d' \end{pmatrix}$ such that $N|c''$. Since $a''d''-b''c''=1$, this implies $a'' \equiv 1 \pmod N$.

The inverse image of the group of lower triangular matrices in $SL_2(\mathbb{F}_p)$ under this map is Γ'. This implies

$$\Gamma' \backslash \Gamma = \left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \,\middle|\, a \in \mathbb{F}_p^\times, c \in \mathbb{F}_p \right\} \backslash SL_2(\mathbb{F}_p).$$

$\sqrt{256}=16$

Marco

A system of coset representatives for this quotient is given by

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{F}_p \right\} \cup \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \bmod p \right\},$$

(They are all non-equivalent $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b-b' \\ 0 & 1 \end{pmatrix} \in \Gamma' \Leftrightarrow b=b'$, $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & b+1 \end{pmatrix} \notin \Gamma'$. And they have the right cardinality since

$$|SL_2(\mathbb{F}_p)| = p(p^2-1) \qquad \left|\left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \right\}\right| = (p-1)p \Big).$$

Given c there is a unique $a \in \mathbb{Z}$ such that $ap - cN = 1$.

This implies that the set of matrices given in the statement is a system of coset representatives $\Gamma' \backslash \Gamma$ #.

We apply this lemma to the definition of $T_{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}}$. For $0 \le b \le p-1$,

$$\left(f|_k \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right)(z) = \left(f|_k \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \right)(z)$$
$$= \frac{p^k}{p^k} f\left(\frac{z+b}{p}\right) = f\left(\frac{z+b}{p}\right)$$

In the case $p \nmid N$, we also have $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} ap & 1 \\ cN & 1 \end{pmatrix} = \begin{pmatrix} ap & 1 \\ pcN & p \end{pmatrix} = \begin{pmatrix} a & 1 \\ cN & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$

Then

$$\left(f|_k \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} ap & 1 \\ cN & 1 \end{pmatrix} \right)(z) = \left(f|_k \begin{pmatrix} a & 1 \\ cN & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right)(z)$$
$$= \left((\langle p \rangle f)|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right)(z) = p^k (\langle p \rangle f)(pz)$$

Therefore

$$\left(T_{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}} f\right)(z) = \sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right) + \underbrace{p^k (\langle p \rangle f)(pz)}_{\text{only included if } p \nmid N.}$$

<u>Notation:</u> From now on, if f is a modular form (of some weight) for $\Gamma_1(N)$ we will write $a_n(f)$ for the n^{th} coefficient of the

q-EXPANSION OF f AT THE CUSP ∞ OF Γ.

THM: LET N BE A POSITIVE INTEGER, AND LET p BE A PRIME NUMBER. THE HECKE OPERATOR ON $M_k(\Gamma_1(N))$ IS GIVEN BY

$$(T_p f)(z) = \frac{1}{p}\sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right) + p^{k-1}(\langle p\rangle f)(pz)$$

AND ITS EFFECT ON q-EXPANSIONS AT THE CUSP ∞ OF Γ IS

$$a_n(T_p f) = a_{pn}(f) + p^{k-1} a_{n/p}(\langle p\rangle f) \quad \forall n \geq 0.$$

OR $\displaystyle T_p f = \sum_{n=0}^{\infty}\left(a_{pn}(f) + p^{k-1} a_{n/p}(\langle p\rangle f)\right) q^n \qquad q = \exp(2\pi i z)$

HERE $a_{n/p}(\langle p\rangle f)$ IS ONLY INCLUDED IF $p \nmid N$ AND $p \mid n$.

PROOF: THE FIRST FORMULA FOLLOWS FROM THE THE DEFINITION.

NOW $\displaystyle (T_p f)(z) = \frac{1}{p}\sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right) + p^{k-1}(\langle p\rangle f)(pz)$

$$= \frac{1}{p}\sum_{b=0}^{p-1}\sum_{n=0}^{\infty} a_n(f)\exp\left(2\pi i\frac{n(z+b)}{p}\right) + p^{k-1}\sum_{n=0}^{\infty} a_n(\langle p\rangle f)\exp(2\pi i p n z)$$

$$= \frac{1}{p}\sum_{n=0}^{\infty}\left(\sum_{b=0}^{p-1}\exp\left(\frac{2\pi i n b}{p}\right)\right) a_n(f)\exp\left(\frac{2\pi i n z}{p}\right)$$

$$+ p^{k-1}\sum_{n=0}^{\infty} a_n(\langle p\rangle f)\exp(2\pi i p n z) \qquad \text{NOTICE } \sum_{b=0}^{p-1}\exp\left(\frac{2\pi i n b}{p}\right) = \begin{cases} p & \text{IF } p\mid n \\ 0 & \text{IF } p\nmid n\end{cases}$$

$$= \sum_{\substack{n=0\\ p\mid n}}^{\infty} a_n(f)\exp\left(\frac{2\pi i n z}{p}\right) + p^{k-1}\sum_{n=0}^{\infty} a_n(\langle p\rangle f)\exp(2\pi i p n z)$$

$$= \sum_{n=0}^{\infty} a_{pn}(f)\exp(2\pi i n z) + p^{k-1}\sum_{n=0}^{\infty} a_n(\langle p\rangle f)\exp(2\pi i p n z) \quad \#$$

LATTICE INTERPRETATION OF HECKE OPERATORS. WE GIVE A MORE CONCEPTUAL EXPLANATION FOR $N=1$ (THIS CAN BE DONE FOR OTHER SUBGROUPS BUT IT'S MORE COMPLICATED).

LET $f \in M_k = M_k(SL_2(\mathbb{Z}))$. LET $\mathcal{L}$ BE THE SET OF ALL LATTICES IN $\mathbb{C}$ AND LET $\Lambda_z = \mathbb{Z}z + \mathbb{Z}$. THERE IS A UNIQUE FUNCTION $\mathcal{F}: \mathcal{L} \longrightarrow \mathbb{C}$ HOMOGENEOUS OF WEIGHT k AND GIVEN BY

$$\mathcal{F}(\Lambda_z) = f(z) \qquad \forall z \in \mathbb{H}$$

IF p IS A PRIME NUMBER, WE WRITE $T_p\mathcal{F}$ FOR THE HOMOGENEOUS FUNCTION CORRESPONDING TO $T_p f$.

PROP: LET $k \in \mathbb{Z}$, LET $f \in M_k(SL_2(\mathbb{Z}))$, AND LET $\mathcal{F}$ BE THE HOMOGENEOUS FUNCTION ASSOCIATED TO f. THEN FOR EVERY PRIME p

$$(T_p\mathcal{F})(\Lambda) = \frac{1}{p}\sum_{\substack{\Lambda' \supset \Lambda \\ [\Lambda':\Lambda]=p}} \mathcal{F}(\Lambda') \qquad \forall \Lambda \in \mathcal{L}$$

Proof: By homogeneity, it suffices to consider the case $\Lambda=\Lambda_z$ with $z\in\mathbb{H}$. Notice that for $N=1$, $p\equiv 1 \pmod{N}$ and $\langle p\rangle f = f$. Then

$$(T_p\mathcal{F})(\Lambda_z) = (T_p f)(z) = \frac{1}{p}\sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right) + p^{k-1} f(pz)$$

$$= \frac{1}{p}\sum_{b=0}^{p-1} \mathcal{F}\left(\mathbb{Z}\frac{z+b}{p}+\mathbb{Z}\right) + p^{k-1}\mathcal{F}(\mathbb{Z}pz+\mathbb{Z})$$

$$= \frac{1}{p}\left(\sum_{b=0}^{p-1} \mathcal{F}\left(\mathbb{Z}\frac{z+b}{p}+\mathbb{Z}\right) + \mathcal{F}\left(\mathbb{Z}z+\mathbb{Z}\frac{1}{p}\right)\right)$$

Let $\Lambda=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ be a lattice containing Λ_z with index p. Thus $1=a\omega_1+b\omega_2$; $z=c\omega_1+d\omega_2$, $\det\begin{pmatrix}a&b\\c&d\end{pmatrix}=p$. Turning to Hermitian normal form gives $\begin{pmatrix}a'&0\\c'&d'\end{pmatrix}$ with $a',d'>0$, $0\le c'<d'$. Also $a'd'=p \Rightarrow a'=p, d'=1$ or $a'=1, d'=p$. This gives $\begin{pmatrix}p&0\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\c&p\end{pmatrix}$ $0\le c<p-1$ as the only possibilities. Therefore, $\mathbb{Z}\frac{z+c}{p}+\mathbb{Z}$ $(0\le c\le p-1)$ and $\mathbb{Z}z+\mathbb{Z}\frac{1}{p}$ are precisely the lattices containing Λ_z with index p #.