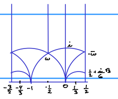

The Hecke algebra Let $N \geq 1$ and $k \in \mathbb{Z}$. The operators $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ and T_p for p prime preserve the $\mathbb{C}$-vector spaces $M_k(\Gamma_1(N))$ and $S_k(\Gamma_1(N))$

Def: Let $N, k \in \mathbb{Z}$ $N \geq 1$. The **Hecke algebra** acting on $M_k(\Gamma_1(N))$ is the $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}\, M_k(\Gamma_1(N))$ generated by

- the $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$
- the T_p for p prime.

This algebra is denoted by $\mathbb{T}(M_k(\Gamma_1(N)))$.

One can similarly define $\mathbb{T}(S_k(\Gamma_1(N)))$ as a $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}\, S_k(\Gamma_1(N))$ generated by the same operators. Then there is a surjective ring homomorphism

$$\mathbb{T}(M_k(\Gamma_1(N))) \twoheadrightarrow \mathbb{T}(S_k(\Gamma_1(N)))$$

defined by sending each operator to its restriction.

Prop: For every $N \geq 1$ the Hecke algebra $\mathbb{T}(M_k(\Gamma_1(N)))$ is commutative.

Proof: We first note that the diamond operators commute because $(\mathbb{Z}/N\mathbb{Z})^\times$ is commutative. More precisely,

$$\langle d \rangle \langle e \rangle = \langle de \rangle = \langle ed \rangle = \langle e \rangle \langle d \rangle.$$

Next we show that T_p and $\langle d \rangle$ commute $\forall p$ prime $\forall d \in (\mathbb{Z}/N\mathbb{Z})^\times$

Consider $\gamma_d = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. By multiplying by a power of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we may assume that $p \mid b$. We put $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, $\Gamma = \Gamma_1(N)$ $\Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha$. We compute

$$T_p(\langle d \rangle f) = T_p(f|_k \gamma_d) = \frac{1}{p} \sum_{\gamma \in \Gamma \backslash \Gamma\alpha\Gamma} (f|_k \gamma_d)|_k \gamma$$
$$= \frac{1}{p} \sum_{\gamma \in \Gamma' \backslash \Gamma} f|_k (\gamma_d \gamma)$$

Notice that γ_d normalizes both Γ and Γ'. Indeed, we have that $\Gamma_1(N) \triangleleft \Gamma_0(N)$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ pc & pd \end{pmatrix} = \begin{pmatrix} a & b/p \\ pc & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \quad \text{and} \quad \gamma_d^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ then}$$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/p \end{pmatrix} = \begin{pmatrix} d & -b/p \\ -c & a/p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/p \end{pmatrix} \begin{pmatrix} d & -b/p \\ -pc & a \end{pmatrix} \quad \text{Now } \Gamma_1(N) \triangleleft \Gamma_0(N)$$

implies $\begin{pmatrix} d & -b/p \\ -pc & a \end{pmatrix} \Gamma_1(N) \begin{pmatrix} a & b/p \\ pc & d \end{pmatrix} = \Gamma_1(N)$ and then

$$\gamma_d^{-1} \alpha^{-1} \Gamma \alpha \gamma_d = \alpha^{-1} \begin{pmatrix} d & -b/p \\ -pc & a \end{pmatrix} \Gamma \begin{pmatrix} a & b/p \\ pc & d \end{pmatrix} \alpha = \alpha^{-1} \Gamma \alpha$$

Conjugation by γ_d induces a permutation of the set $\Gamma \backslash \Gamma\alpha\Gamma$

We can replace the sum over $\gamma \in \Gamma \backslash \Gamma\alpha\Gamma$ by a sum over $\delta = \gamma_d \gamma \gamma_d^{-1}$

$T_p(\langle d\rangle f) = \frac{1}{p}\sum_{\delta\in\Gamma_1\backslash\Gamma_{\Delta}} f|_k(\delta\gamma_d) = \frac{1}{p}\left(\sum_{\delta\in\Gamma_1\backslash\Gamma_{\Delta}} f|_k\delta\right)\Big|_k\gamma_d$

$=\langle d\rangle(T_p f)$

Next we prove that T_p and $T_{p'}$ commute $\forall p, p'$ prime.

Let $f\in M_k(\Gamma_1(N))$. For all $n\geq 0$,

$a_n(T_{p'}f) = a_{p'n}(f) + (p')^{k-1}a_{n/p'}(\langle p\rangle f)$

and $a_n(T_pT_{p'}f) = a_{pn}(T_{p'}f) + p^{k-1}a_{n/p}(\langle p\rangle T_{p'}f)$

Since $\langle p\rangle$ and $T_{p'}$ commute,

$a_n(T_pT_{p'}f) = a_{pn}(T_{p'}f) + p^{k-1}a_{n/p}(T_{p'}(\langle p\rangle f))$

$= a_{p'pn}(f) + (p')^{k-1}a_{pn/p'}(\langle p'\rangle f)$

$\quad + p^{k-1}a_{p'n/p}(\langle p\rangle f) + p^{k-1}(p')^{k-1}a_{n/(pp')}(\langle p\rangle\langle p'\rangle f)$

(Recall that $a_m(f)=0$ if $m\notin\mathbb{Z}$ and $\langle p\rangle=0$ if $p|N$)

The right-hand side is symmetric in p and p' $\forall n$ showing that $T_pT_{p'}f = T_{p'}T_pf$. Since f is arbitrary, we get the result. #

<u>Def</u>: Let $N\geq 1$ and $k\in\mathbb{Z}$. The Hecke operators $T_n\in\mathbb{T}(M_k(\Gamma_1(N)))$ for $n\geq 1$ are defined starting from T_p:

$T_1 = \mathrm{Id}$, T_p as before for p prime

$T_{p^r} = T_pT_{p^{r-1}} - p^{k-1}\langle p\rangle T_{p^{r-2}}$ for p prime and $r=2,3,\dots$

$T_n = \prod_{p \text{ prime}} T_{p^{e_p}}$ for $n=\prod_{p\text{ prime}} p^{e_p}$

The ordering of the factors does not matter since the algebra is commutative. Moreover,

$T_{mn} = T_mT_n$ if $(m,n)=1$.

<u>The effect of Hecke operators on q-expansions</u>

<u>Prop</u>: Let $f\in M_k(\Gamma_1(N))$ and let $m\geq 1$. The q-expansion coefficients of T_mf at the cusp ∞ of $\Gamma_1(N)$ are given by

$$a_n(T_mf) = \sum_{d|(m,n)} d^{k-1}a_{mn/d^2}(\langle d\rangle f) \qquad \forall n\geq 0$$

In particular, $a_0(T_mf) = \sum_{d|m} d^{k-1}a_0(\langle d\rangle f)$, $a_1(T_mf) = a_m(f)$

<u>Proof</u>: Since $T_1 = \mathrm{Id}$ the claim is true for $m=1$. It is also true for $m=p$. Let p be a prime number. We prove by induction the case $m=p^r$. Suppose that the claim is true for $m=p^j$

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$0 \le j \le r-1$. We have to prove

$$a_n(T_{p^r} f) = \sum_{\substack{0\le j\le r\\ p^j \mid n}} p^{j(k-1)} a_{p^{r-2j}n}(\langle p^j\rangle f) \qquad \forall n\ge 0.$$

By definition, $T_{p^r} f = T_p T_{p^{r-1}} f - p^{k-1}\langle p\rangle T_{p^{r-2}} f$, so

$$a_n(T_{p^r}f) = a_n(T_pT_{p^{r-1}}f) - p^{k-1}a_n(\langle p\rangle T_{p^{r-2}}f)$$

$$= a_{pn}(T_{p^{r-1}}f) + p^{k-1}a_{n/p}(\langle p\rangle T_{p^{r-1}}f) - p^{k-1}a_n(\langle p\rangle T_{p^{r-2}}f)$$

$$= a_{pn}(T_{p^{r-1}}f) + p^{k-1}a_{n/p}(T_{p^{r-1}}\langle p\rangle f) - p^{k-1}a_n(T_{p^{r-2}}\langle p\rangle f)$$

$$= \sum_{\substack{0\le j\le r-1\\ p^j \mid pn}} p^{j(k-1)} a_{p^{r-1-2j}pn}(\langle p\rangle^j f) + p^{k-1}\sum_{\substack{0\le j\le r-1\\ p^{j+1}\mid n}} p^{j(k-1)} a_{p^{r-1-2j}n/p}(\langle p\rangle^{j+1} f)$$

$$- p^{k-1}\sum_{\substack{0\le j\le r-2\\ p^j\mid n}} p^{j(k-1)} a_{p^{r-2-2j}n}(\langle p\rangle^{j+1}f)$$

$$= \sum_{\substack{0\le j\le r-1\\ p^j\mid pn}} p^{j(k-1)}a_{p^{r-2j}n}(\langle p\rangle^j f) + \sum_{\substack{0\le j\le r-1\\ p^{j+1}\mid n}} p^{(j+1)(k-1)} a_{p^{r-2-2j}n}(\langle p\rangle^{j+1}f)$$

$$- \sum_{\substack{0\le j\le r-2\\ p^j\mid n}} p^{(j+1)(k-1)} a_{p^{r-2-2j}n}(\langle p\rangle^{j+1}f)$$

$$= \sum_{\substack{0\le j\le r-1\\ p^j\mid pn}} p^{j(k-1)} a_{p^{r-2j}n}(\langle p\rangle^j f) + \sum_{\substack{1\le j\le r\\ p^j\mid n}} p^{j(k-1)} a_{p^{r-2j}n}(\langle p\rangle^j f)$$

$$- \sum_{\substack{1\le j\le r-1\\ p^j\mid pn}} p^{j(k-1)} a_{p^{r-2j}n}(\langle p\rangle^j f)$$

The first and third sums cancel except for the term $j=0$.

$$a_n(T_{p^r}f) = a_{p^rn}(f) + \sum_{\substack{1\le j\le r\\ p^j\mid n}} p^{j(k-1)} a_{p^{r-2j}n}(\langle p\rangle^j f)$$

This proves the case $m=p^r$. Now we prove that if the claim holds for m, m' coprime, it holds for $m\cdot m'$!

$$a_n(T_{mm'}f) = a_n(T_m(T_{m'}f)) = \sum_{d\mid(m,n)} d^{k-1} a_{mn/d^2}(\langle d\rangle T_{m'}f)$$

$$= \sum_{d\mid(m,n)} d^{k-1} a_{mn/d^2}(T_{m'}\langle d\rangle f)$$

$$= \sum_{d\mid(m,n)} d^{k-1} \sum_{d'\mid(m', mn/d^2)} (d')^{k-1} a_{m'(mn/d^2)/(d')^2}(\langle d'\rangle(\langle d\rangle f))$$

$$= \sum_{d \mid (m,n)} \sum_{d' \mid (m', mn/d^2)} (dd')^{k-1} a_{mm'n/(dd')^2} (\langle dd' \rangle f)$$

Since $(m,m')=1$, $(m', mn/d^2) = (m', n)$ $\forall d \mid m$

Moreover, if d and d' range over the divisors of (m,n) and (m',n) respectively $e = dd'$ ranges over the divisors of (mm',n)

Then $$a_n(T_{mm'} f) = \sum_{e \mid (mm',n)} e^{k-1} a_{mm'n/e^2} (\langle e \rangle f)$$

which proves the claim for mm'. For general m, the claim is proven by induction on the number of prime factors of m #

Hecke eigenforms

Def: A (Hecke) eigenform is a non-zero modular form $f \in M_k(\Gamma_1(N))$ that is an eigenvector for all Hecke operators T_p for p prime and all diamond operators $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. A normalized eigenform is an eigenform f satisfying $a_1(f) = 1$.

Let $f \in M_k(\Gamma_1(N))$ be an eigenform. If λ_n is the eigenvalue of T_n on f, then taking a_1 in the equation $T_n f = \lambda_n f$ yields

$$a_n(f) = \lambda_n a_1(f) \quad \forall n \geq 1. \qquad (*)$$

If f is an eigenform with $a_1(f) = 0$, then the above shows $a_n(f) = 0$ $\forall n \geq 1$. Since we assume $f \neq 0$, this is only possible if $k=0$ and $f =$ constant. Then, for $k \geq 1$, any eigenform can be scaled to a normalized eigenform.

Thm: Let $f \in M_k(\Gamma_1(N))$ be a normalized eigenform. Then the eigenvalues of the Hecke operators on f are equal to the q-expansion coefficients of f at the cusp ∞ of $\Gamma_1(N)$:

$$T_n f = a_n(f) f \quad \forall n \geq 1.$$

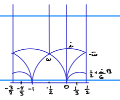
Proof: Follows from $(*)$ and the fact that $a_1(f) = 1$. #

Now consider the eigenvalues of the diamond operators

We write $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}$, for the map that sends every $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ to the eigenvalue of $\langle d \rangle$ on f. Then $\forall d, e \in (\mathbb{Z}/N\mathbb{Z})^\times$,

$$\chi(de) f = \langle de \rangle f = \langle d \rangle (\langle e \rangle f) = \langle d \rangle \chi(e) f = \chi(d) \chi(e) f$$

So χ is a Dirichlet character modulo N.



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Def: Let $N \geq 1$, $k \in \mathbb{Z}$ and $\chi \in \mathrm{Hom}((\mathbb{Z}/N\mathbb{Z})^*, \mathbb{C}^\times)$. The **space of modular forms of weight k and character χ for $\Gamma_1(N)$** is the $\mathbb{C}$-linear subspace of $M_k(\Gamma_1(N))$ consisting of the forms f satisfying $\langle d \rangle f = \chi(d) f \quad \forall d \in (\mathbb{Z}/N\mathbb{Z})^*$.

It is denoted by $M_k(\Gamma_1(N), \chi)$

Remark: Some people write $M_k(N, \chi)$ or $M_k(\Gamma_0(N), \chi)$.

Notice $M_k(\Gamma_0(N), \chi)$ is not generally a subspace of $M_k(\Gamma_0(N))$

Similarly, the **space of cusp forms of weight k and character χ for $\Gamma_1(N)$** is

$$S_k(\Gamma_1(N), \chi) = M_k(\Gamma_1(N), \chi) \cap S_k(\Gamma_1(N))$$

in $M_k(\Gamma_1(N))$

Prop: The $\mathbb{C}$-vector space $M_k(\Gamma_1(N))$ has a decomposition

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_k(\Gamma_1(N), \chi),$$

where χ runs through all Dirichlet characters modulo N.

The analogous statement holds for $S_k(\Gamma_1(N))$.

Proof: Any f satisfying $\langle d \rangle f = f|_k \gamma_d = \chi(d) f$ for two different characters must be zero. It suffices to show that any $f \in M_k(\Gamma_1(N))$ can be written as a sum of $f_\chi \in M_k(\Gamma_1(N), \chi)$

Let
$$f_\chi = \frac{1}{\phi(N)} \sum_{d \in (\mathbb{Z}/N\mathbb{Z})^\times} \bar{\chi}(d) f|_k \gamma_d, \quad \gamma_d = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

For any $\gamma_{d'} = \begin{pmatrix} a & b \\ c & d' \end{pmatrix} \in \Gamma_0(N)$

$$f_\chi|_k \gamma_{d'} = \frac{1}{\phi(N)} \sum_{d \in (\mathbb{Z}/N\mathbb{Z})^\times} \bar{\chi}(d) f|_k(\gamma_d \gamma_{d'}) =$$

$$= \frac{1}{\phi(N)} \sum_{e \in (\mathbb{Z}/N\mathbb{Z})^\times} \bar{\chi}(e) \chi(d') f|_k(\gamma_e) = \chi(d') f_\chi.$$

Finally, we sum f_χ over all the characters χ and reverse the order

$$\sum_\chi f_\chi = \sum_{d \in (\mathbb{Z}/N\mathbb{Z})^\times} \overbrace{\frac{1}{\phi(N)} \left(\sum_\chi \bar{\chi}(d) \right)}^{\circledast} f|_k \gamma_d = f,$$

Since
$$\circledast = \begin{cases} 1 & \text{if } d = 1, \\ 0 & \text{otherwise.} \end{cases} \quad \#.$$

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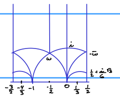

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Thm: Let $f \in M_k(\Gamma_1(N), \chi)$ be a normalized eigenform, and let $\sum_{n=0}^{\infty} a_n q^n$ be the q-expansion of f at ∞.
Then the a_n for $n \geq 1$ can be expressed recursively as follows

- $a_1 = 1$
- $a_{p^r} = a_p a_{p^{r-1}} - p^{k-1} \chi(p) a_{p^{r-2}}$ for p prime and $r = 2, 3, \ldots$
- $a_n = \prod_{p \text{ prime}} a_{p^{e_p}}$ for $n = \prod_{p \text{ prime}} p^{e_p}$.

Proof: Follows from the definition of Hecke characters and ⊛ #.

Coro: With the previous notation,

$$a_{mn} = a_m a_n \quad \text{if } (m,n) = 1.$$

Ex: We take $N = 1$ and $k = 12$. Recall that there is a cusp form $\Delta \in S_{12}(SL_2(\mathbb{Z}))$, unique up to scaling. Since the Hecke algebra preserves the space of cusps forms, Δ must be an eigenform. It has trivial character because $N = 1$.

Recall the Ramanujan τ-function defined by the q-expansion of Δ:

$$\Delta = \sum_{n=0}^{\infty} \tau(n) q^n.$$

Applying the results above, we get

$$\tau(p^r) = \tau(p)\tau(p^{r-1}) - p^{11} \tau(p^{r-2}) \quad \text{for } p \text{ prime and } r = 2, 3, \ldots$$

and $\tau(mn) = \tau(m)\tau(n)$ if $(m,n) = 1$,

which were conjectured by Ramanujan.