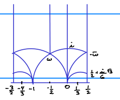
Modular Forms



M. Lalín

The theory of newforms

The Petersson inner product We have constructed commutative $\mathbb{C}$-algebras $\mathbb{T}(M_k(\Gamma_1(N)))$ and $\mathbb{T}(S_k(\Gamma_1(N)))$ acting on $M_k(\Gamma_1(N))$ and $S_k(\Gamma_1(N))$. We will add some additional structure, namely a Hermitian inner product on $S_k(\Gamma_1(N))$.

Lemma: Let U be a subset of $\mathbb{H}$ whose boundary consists of finitely many line segments and circle arcs. Let $f: U \longrightarrow \mathbb{C}$ be a continuous function and let $\gamma \in GL_2^+(\mathbb{R})$. Then

$$\int_{z\in U} f(z)\,\frac{dx\,dy}{y^2} = \int_{z\in\gamma^{-1}U} f(\gamma z)\,\frac{dx\,dy}{y^2} \qquad (z = x+iy)$$

Proof: We view $\mathbb{H}$ as an open set of $\mathbb{R}^2$ with coordinates (x,y) and $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ as a real differentiable map $\mathbb{H}\longrightarrow\mathbb{H}$.
We write $\gamma_1(x,y) = \operatorname{Re}\gamma(x+iy)$ $\gamma_2(x,y) = \operatorname{Im}\gamma(x+iy)$
The Jacobian matrix of this map is

$$J_\gamma(x,y) = \begin{pmatrix} \partial\gamma_1/\partial x & \partial\gamma_1/\partial y \\ \partial\gamma_2/\partial x & \partial\gamma_2/\partial y \end{pmatrix}$$

Since γ is holomorphic, it satisfies the Cauchy–Riemann equations

$$\frac{\partial\gamma_2}{\partial y} = \frac{\partial\gamma_1}{\partial x}, \quad \frac{\partial\gamma_2}{\partial x} = -\frac{\partial\gamma_1}{\partial y}$$

and γ' can be expressed as

$$\gamma'(z) = \frac{\partial\gamma_1}{\partial x} + i\frac{\partial\gamma_2}{\partial x}$$

Therefore, $\det J_\gamma(x,y) = \frac{\partial\gamma_1}{\partial x}\frac{\partial\gamma_2}{\partial y} - \frac{\partial\gamma_1}{\partial y}\frac{\partial\gamma_2}{\partial x} = \left(\frac{\partial\gamma_1}{\partial x}\right)^2 + \left(\frac{\partial\gamma_2}{\partial x}\right)^2 = |\gamma'(x+iy)|^2.$

On the other hand we have $\operatorname{Im}(\gamma z) = \det\gamma\,\frac{\operatorname{Im}(z)}{|cz+d|^2}$ and $\gamma'(z) = \frac{\det\gamma}{(cz+d)^2}$ Putting this together,

$$|\det J_\gamma(x,y)| = |\gamma'(z)|^2 = \frac{\operatorname{Im}(\gamma z)^2}{\operatorname{Im}(z)^2}$$

This implies

$$\int_{z\in U} f(z)\,\frac{dx\,dy}{y^2} = \int_{z\in\gamma^{-1}U} f(\gamma z)\,|\det J_\gamma(z)|\,\frac{dx\,dy}{\operatorname{Im}(\gamma z)^2}$$

$$= \int_{z\in\gamma^{-1}U} f(\gamma z)\,\frac{dx\,dy}{y^2} \qquad \#$$

Rem: We have proved that the differential 2-form $\frac{dx\wedge dy}{y^2}$ is $SL_2(\mathbb{R})$-invariant.

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. Recall that

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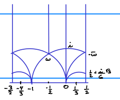

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$D_\Gamma = \bigcup_{\gamma \in R} \gamma D$, where R is a system of representatives for the quotient $\Gamma \backslash SL_2(\mathbb{Z})$ and D is the standard fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$. Notice that D_Γ depends on the choice of R.

Let $F: \mathbb{H} \longrightarrow \mathbb{C}$ be a continuous function that is Γ-invariant in the sense that $F(\gamma z) = F(z)$ $\forall \gamma \in \Gamma, z \in \mathbb{H}$.

By the Lemma, the value of $\int_{z \in D_\Gamma} F(z) \frac{dx\,dy}{y^2}$ (*) does not depend on the choice of R.

We consider subsets of D_Γ which are "neighborhoods of the cusps":

For $Y > 0$, let $U_Y = \{ z = x+iy \mid -1/2 \leq x \leq 1/2 \wedge y \geq Y \}$.

Then D_Γ is the union of some compact set $K \subset \mathbb{H}$ and the sets $\gamma U_Y = \{ \gamma z \mid z \in U_Y \}$ for $\gamma \in R$.

Lemma: Suppose that for all $\gamma \in SL_2(\mathbb{Z})$ there exist real numbers $c_\gamma > 0$ and $e_\gamma < 1$ such that

$|F(\gamma z)| \leq c_\gamma (\operatorname{Im} z)^{e_\gamma}$ for $\operatorname{Im} z$ sufficiently large.

Then the integral (*) converges.

Proof: The integral (*) restricted to K converges because K is compact. It remains to show that the integral converges on each of the sets γU_Y. For $\gamma \in R$

$$\left| \int_{z \in \gamma U_Y} F(z) \frac{dx\,dy}{y^2} \right| = \left| \int_{z \in U_Y} F(\gamma z) \frac{dx\,dy}{y^2} \right| \leq c_\gamma \int_{z \in U_Y} y^{e_\gamma} \frac{dx\,dy}{y^2}$$

$$= c_\gamma \int_{y=Y}^{\infty} y^{e_\gamma - 2}\, dy < \infty$$ since $e_\gamma - 2 < -1$. #.

Let $k \in \mathbb{Z}$. Let f, g be two modular forms of weight k for Γ. Consider the continuous (but generally non-holomorphic) function $F(z) = f(z)\overline{g(z)}(\operatorname{Im} z)^k$.

Lemma: The function $F(z)$ is Γ-invariant. If $k=0$ or if for each cusp c at least one of the forms vanishes at c, then F is bounded.

Proof: Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. By the modularity of f and g, we have, $f(\gamma z) = (cz+d)^k f(z)$, $\overline{g(\gamma z)} = \overline{(cz+d)}^k \overline{g(z)}$

AND $(\operatorname{Im}\gamma z)^k = |cz+d|^{-2k}(\operatorname{Im} z)^k$. MULTIPLYING ALL EQUATIONS YIELDS $F(\gamma z) = F(z)$.

SINCE F IS Γ-INVARIANT, IT SUFFICES TO PROVE THAT F IS BOUNDED ON D_Γ. AS BEFORE, WE CONSIDER THE REGIONS γU_Y.
LET $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ AND LET $\mathfrak{c}$ BE THE CORRESPONDING CUSP, I.E., THE Γ-ORBIT OF $\gamma\infty \in \mathbb{P}^1(\mathbb{Q})$ IN $\mathrm{Cusps}(\Gamma) = \Gamma \backslash \mathbb{P}^1(\mathbb{Q})$.
THEN BY THE DEFINITION OF THE q-EXPANSION OF f AT $\mathfrak{c}$, WE HAVE $(cz+d)^{-k} f(\gamma z) = (f|_k \gamma)(z) = \sum_{n=0}^{\infty} a_{n,\mathfrak{c}}(f) \exp(2\pi i n z/h_{\mathfrak{c}})$
WHERE $h_{\mathfrak{c}}$ IS THE WIDTH OF $\mathfrak{c}$, AND SIMILARLY FOR g.
THEREFORE $F(\gamma z) = f(\gamma z)\overline{g(\gamma z)}(\operatorname{Im}(\gamma z))^k$

$$= (f|_k\gamma)(z)\,(cz+d)^k\,\overline{(g|_k\gamma)(z)\,(cz+d)^k}\,|cz+d|^{-2k}(\operatorname{Im} z)^k$$

$$= \left(\sum_{n=0}^{\infty} a_{n,\mathfrak{c}}(f)\exp(2\pi i n z/h_{\mathfrak{c}})\right)\left(\overline{\sum_{n=0}^{\infty} a_{n,\mathfrak{c}}(f)\exp(2\pi i n z/h_{\mathfrak{c}})}\right)(\operatorname{Im} z)^k$$

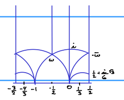
IF $a_{0,\mathfrak{c}}(f) = 0$ OR $a_{0,\mathfrak{c}}(g) = 0$, THEN THE ABSOLUTE VALUE OF THE EXPRESSION ABOVE IS BOUNDED BY A CONSTANT MULTIPLE OF $|\exp(2\pi i z/h_{\mathfrak{c}})|\, y^k$ FOR $y \geq Y$.
IN PARTICULAR F IS BOUNDED ON γU_Y FOR ANY $Y > 0$. SINCE THE COMPLEMENT OF $\bigcup_{\gamma \in R} \gamma U_Y$ IN D_Γ IS A COMPACT SUBSET $K \subseteq \mathbb{H}$ AND F IS CONTINUOUS, THEN IT IS BOUNDED ON D_Γ AND ON $\mathbb{H}$. #.

FOR $f, g \in M_k(\Gamma)$ DEFINE

$$\langle f, g\rangle_\Gamma = \int_{z \in D_\Gamma} f(z)\overline{g(z)}\, y^k \frac{dx\,dy}{y^2}, \quad (\star)$$

WHERE AS USUAL $z = x + iy$ IS WRITTEN AS (x,y). THIS IS INDEPENDENT OF THE CHOICE OF R. HOWEVER WE NEED EXTRA CONDITIONS TO ENSURE THAT THE INTEGRAL CONVERGES. BY THE LEMMA ABOVE, THIS IS WELL-DEFINED ON $M_k(\Gamma) \times S_k(\Gamma) \longrightarrow \mathbb{C}$

<u>DEF</u>: LET Γ BE A CONGRUENCE SUBGROUP AND LET k BE AN INTEGER. THE PETERSSON INNER PRODUCT ON THE $\mathbb{C}$-VECTOR SPACE $S_k(\Gamma)$ IS THE HERMITIAN INNER PRODUCT $\langle\,,\rangle_\Gamma$ DEFINED BY $(\star)$.
THIS PRODUCT DOES NOT EXTEND TO $M_k(\Gamma)$, SINCE $\langle f, f\rangle_\Gamma$



DIVERGES FOR EVERY $f \in M_k(\Gamma) \setminus S_k(\Gamma)$.

DEF: LET Γ BE A CONGRUENCE SUBGROUP, AND LET k BE AN INTEGER. THE EISENSTEIN SUBSPACE (OR THE SPACE OF EISENSTEIN SERIES) IN $M_k(\Gamma)$, DENOTED BY $\mathcal{E}_k(\Gamma)$ IS THE SPACE

$$\mathcal{E}_k(\Gamma) = \{ f \in M_k(\Gamma) \mid \langle f, g \rangle_\Gamma = 0 \text{ FOR ALL } g \in S_k(\Gamma) \}$$

THE EISENSTEIN SUBSPACE CAN BE SEEN AS AN "ORTHOGONAL COMPLEMENT" OF $S_k(\Gamma)$ WITH RESPECT TO $\langle , \rangle_\Gamma$, EVEN THOUGH $\langle , \rangle_\Gamma$ DOES NOT DEFINE AN INNER PRODUCT ON ALL OF $M_k(\Gamma)$.

THE ADJOINTS OF THE HECKE OPERATORS WE WOULD LIKE TO APPLY THE SPECTRAL THEOREM TO THE SPACES $S_k(\Gamma_1(N))$, EQUIPPED WITH THE PETERSSON INNER PRODUCT, TO OBTAIN DECOMPOSITIONS OF THESE SPACES INTO SMALLER ONES.

TO COMPUTE THE ADJOINTS OF THE VARIOUS OPERATORS IN THE HECKE ALGEBRA $\mathbb{T}(S_k(\Gamma_1(N)))$ WE START WITH A GENERAL CONGRUENCE SUBGROUP Γ. LET $\alpha \in GL_2^+(\mathbb{Q})$ AND LET T_α BE

$$T_\alpha f = \sum_{\gamma \in \Gamma_\alpha \backslash \Gamma} f|_k \alpha\gamma, \quad \text{WHERE } \Gamma_\alpha = \Gamma \cap \alpha^{-1}\Gamma\alpha$$

NOTATION: FOR $\alpha \in GL_2^+(\mathbb{Q})$, $\alpha^* = (\det\alpha)\alpha^{-1} \in GL_2^+(\mathbb{Q})$

MORE CONCRETELY, IF $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, THEN $\alpha^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

NOTATION: FOR $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$ AND $z \in \mathbb{H}$, $j(\gamma, z) = cz+d$,

$$(f|_k\gamma)(z) = \frac{(\det\gamma)^{k}}{j(\gamma,z)^k} f(\gamma z), \qquad \operatorname{Im}(\gamma z) = \frac{\det\gamma}{|j(\gamma,z)|^2} \operatorname{Im} z,$$

$$\frac{d}{dz}(\gamma z) = \frac{\det\gamma}{j(\gamma,z)^2}$$

LEMMA: FOR ALL $\gamma, \delta \in GL_2^+(\mathbb{Q})$ AND ALL $z \in \mathbb{H}$,

(i) $j(\gamma\delta, z) = j(\gamma, \delta z)\, j(\delta, z)$

(ii) $j(\gamma, z)\, j(\gamma^{-1}, \gamma z) = 1$

(iii) $j(id, z) = 1$

PROOF (iii) IS TRIVIAL, (ii) FOLLOWS FROM (i) AND (iii). FOR (i), LET $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\delta = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. THEN $\gamma\delta = \begin{pmatrix} * & * \\ ce+dg & cf+dh \end{pmatrix}$

THEN $j(\gamma\delta, z) = (ce+dg)z + (cf+dh) = c(ez+f) + d(gz+h)$

$$= \left(c\,\frac{ez+f}{gz+h} + d\right)(gz+h) = j(\gamma, \delta z)\, j(\delta, z) \qquad \#$$

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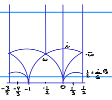

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Rem: The first equation is called the cocycle condition.

Prop: Let Γ be a congruence subgroup, and let k be an integer. Then for all $f, g \in M_k(\Gamma)$ such that at least one of f and g is a cusp form, and for all $\alpha \in GL_2^+(\mathbb{Q})$, we have

$$\langle T_\alpha f, g\rangle_\Gamma = \langle f, T_{\alpha^*} g\rangle_\Gamma$$

Coro: With the notation above, T_{α^*} is the adjoint of the operator T_α on $S_k(\Gamma)$ with respect to $\langle , \rangle_\Gamma$

Proof of Prop: Notice that

$$T_\alpha f = \sum_{\gamma \in \Gamma_\alpha \backslash \Gamma} \frac{(\det \alpha\gamma)^k}{j(\alpha\gamma, z)^k} f(\alpha\gamma z)$$

$$\langle T_\alpha f, g\rangle_\Gamma = \int_{z \in D_\Gamma} \sum_{\gamma \in \Gamma_\alpha \backslash \Gamma} \frac{(\det \alpha\gamma)^k}{j(\alpha\gamma, z)^k} f(\alpha\gamma z)\, \overline{g(z)}\, (\operatorname{Im} z)^k \frac{dx\,dy}{y^2}$$

We apply the following: $\det(\alpha\gamma) = \det(\alpha)$; $g(z) = j(\gamma,z)^{-k} g(\gamma z)$
$j(\alpha\gamma, z)^{-k} = j(\alpha, \gamma z)^{-k} j(\gamma, z)^{-k}$; $(\operatorname{Im} z)^k = |j(\gamma, z)|^{2k} (\operatorname{Im} \gamma z)^k$

Thus

$$\langle T_\alpha f, g\rangle_\Gamma = (\det \alpha)^k \int_{z \in D_\Gamma} \sum_{\gamma \in \Gamma_\alpha \backslash \Gamma} j(\alpha, \gamma z)^{-k} f(\alpha\gamma z)\, \overline{g(\gamma z)}\, (\operatorname{Im} \gamma z)^k \frac{dx\,dy}{y^2}$$

Instead of integrating over z and summing over γ, we can integrate directly over $z \in D_{\Gamma_\alpha}$. This gives.

$$\langle T_\alpha f, g\rangle_\Gamma = (\det \alpha)^k \int_{z \in D_{\Gamma_\alpha}} j(\alpha, z)^{-k} f(\alpha z)\, \overline{g(z)}\, (\operatorname{Im} z)^k \frac{dx\,dy}{y^2}.$$

Notice that $T_{\alpha^*} g = (\det \alpha)^k T_{\alpha^{-1}} g$

We apply the formula that we proved for $\langle T_\alpha f, g\rangle_\Gamma$ to the case $\langle T_{\alpha^{-1}} g, f\rangle_\Gamma$ as follows.

$$\langle f, T_{\alpha^*} g\rangle_\Gamma = (\det \alpha)^k \langle f, T_{\alpha^{-1}} g\rangle_\Gamma = (\det \alpha)^k \overline{\langle T_{\alpha^{-1}} g, f\rangle_\Gamma}$$

$$= (\det \alpha)^k \overline{(\det \alpha^{-1})^k \int_{z \in D_{\Gamma_{\alpha^{-1}}}} j(\alpha^{-1}, z)^{-k} g(\alpha^{-1} z)\, \overline{f(z)}\, (\operatorname{Im} z)^k \frac{dx\,dy}{y^2}}$$

$$= \int_{z \in D_{\Gamma_{\alpha^{-1}}}} \overline{j(\alpha^{-1}, z)}^{-k} f(z)\, \overline{g(\alpha^{-1} z)}\, (\operatorname{Im} z)^k \frac{dx\,dy}{y^2}$$

We make the change of variables $z = \alpha w$

Note that $\Gamma_{\alpha^{-1}} = \Gamma \cap \alpha \Gamma \alpha^{-1} = \alpha \Gamma_\alpha \alpha^{-1}$; $j(\alpha^{-1}, \alpha w) = j(\alpha, w)^{-1}$;

$\operatorname{Im}(\alpha w) = \dfrac{(\det \alpha)^k}{|j(\alpha, w)|^{2k}} \operatorname{Im}(w)^k$

Moreover, letting z range over $D_{\Gamma_{\alpha^{-1}}}$ has the effect of letting w range over a fundamental domain of Γ_α. We get

$$\langle f, T_{\alpha^*} g\rangle_\Gamma = (\det \alpha)^k \int_{w \in D_{\Gamma_\alpha}} j(\alpha, w)^{-k} f(\alpha w) \overline{g(w)} (\operatorname{Im} w)^k \frac{dx' dy'}{y'^2} \quad (w = x' + i y')$$

$= \langle T_\alpha f, g\rangle_\Gamma$ (same expression) #

Rem: The proof above can be described as

$$\langle T_\alpha f, g\rangle_\Gamma = \langle f|_k \alpha, g\rangle_{\Gamma_\alpha} = \langle f, g|_k \alpha^*\rangle_{\Gamma_{\alpha^*}} = \langle f, T_{\alpha^*} g\rangle_{\Gamma_{\alpha^*}}$$

Cor: Let Γ be a congruence subgroup, and let k be an integer. Then all operators T_α on $M_k(\Gamma)$, for $\alpha \in GL_2^+(\mathbb{Q})$, preserve the Eisenstein subspace $\mathcal{E}_k(\Gamma)$ of $M_k(\Gamma)$.

Proof: Let $f \in \mathcal{E}_k(\Gamma)$ and let $\alpha \in GL_2^+(\mathbb{Q})$. Then for all $g \in S_k(\Gamma)$, $\langle T_\alpha f, g\rangle_\Gamma = \langle f, T_{\alpha^*} g\rangle_\Gamma = 0$.

since $T_{\alpha^*} g \in S_k(\Gamma)$. Therefore $T_\alpha f$ is orthogonal to $S_k(\Gamma)$ and hence is in $\mathcal{E}_k(\Gamma)$ #.

Lemma: Let Γ be a congruence subgroup and let k be an integer. Let $\alpha, \beta \in GL_2^+(\mathbb{Q})$ be such that at least one of them normalizes Γ. Then $T_{\alpha\beta} = T_\beta T_\alpha$, $T_{\beta\alpha} = T_\alpha T_\beta$. as operators in $M_k(\Gamma)$.

Proof: By symmetry, assume that β normalizes Γ. Then

$\Gamma_\beta = \Gamma \cap \beta^{-1} \Gamma \beta = \Gamma$ and $\Gamma_{\beta\alpha} = \Gamma \cap \alpha^{-1}\beta^{-1}\Gamma\beta\alpha = \Gamma \cap \alpha^{-1}\Gamma\alpha = \Gamma_\alpha$

Moreover, conjugation by β gives the isomorphisms

$$\Gamma \xrightarrow{\sim} \Gamma, \quad \alpha^{-1}\Gamma\alpha \xrightarrow{\sim} \beta^{-1}\alpha^{-1}\Gamma\alpha\beta, \quad \Gamma_\alpha \xrightarrow{\sim} \Gamma_{\alpha\beta},$$

where all the maps are defined as $\gamma \to \beta^{-1}\gamma\beta$

Let $f \in M_k(\Gamma)$. Then $T_\alpha f = \sum_{\gamma \in \Gamma_\alpha \backslash \Gamma} f|_k \alpha\gamma$; $T_\beta f = f|_k \beta$

$$T_{\beta\alpha} f = \sum_{\gamma \in \Gamma_{\beta\alpha} \backslash \Gamma} f|_k \beta\alpha\gamma = \sum_{\gamma \in \Gamma_\alpha \backslash \Gamma} (f|_k \beta)|_k \alpha\gamma = T_\alpha(T_\beta f)$$

$$T_{\alpha\beta} f = \sum_{\gamma \in \Gamma_{\alpha\beta} \backslash \Gamma} f|_k \alpha\beta\gamma = \sum_{\substack{\delta \in \Gamma_\alpha \backslash \Gamma \\ \delta = \beta\gamma\beta^{-1}}} f|_k \alpha\beta(\beta^{-1}\delta\beta) = \sum_{\delta \in \Gamma_\alpha \backslash \Gamma} f|_k \alpha\delta\beta$$

$$= \sum_{\delta \in \Gamma_\alpha \backslash \Gamma} (f|_k \alpha\delta)|_k \beta = T_\beta(T_\alpha f) \quad \#.$$

Rem: The assumption that α or β normalizes Γ is necessary. For instance, take $\Gamma = \Gamma_1(N)$, let p prime $p \nmid N$ and $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, $\beta = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.
If the Lemma were true, we would have

$$T_{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}} T_{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} \overset{?}{=} T_{\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}}$$

and the next proposition would imply

$$p^2 \langle p \rangle^{-1} T_p T_p \overset{?}{=} p^k \mathrm{id},$$

which is in general false. For instance T_p is not invertible in general.

Prop: Let $N \geq 1$ and $k \in \mathbb{Z}$. In the Hecke algebra $\mathbb{T}(S_k(\Gamma_1(N)))$ consider the diamond operators $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ and the Hecke operators T_m for $m \geq 1$ with $(m,N)=1$. The adjoints of these operators with respect to the Petersson inner product are

$$\langle d \rangle^\dagger = \langle d \rangle^{-1} \ \forall d \in (\mathbb{Z}/N\mathbb{Z})^\times, \quad T_m^\dagger = \langle m \rangle^{-1} T_m \ \text{ if } (m,N)=1$$

Proof: Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ so that $\langle d \rangle = T_\alpha$. We have

$$\alpha^\iota = \alpha^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \Gamma_0(N),$$ which gives $\langle a \rangle$

Since $N | c$, we get $1 = \det \alpha = ad - bc \equiv ad \pmod N$

so $\langle a \rangle = \langle d \rangle^{-1}$ is the adjoint of $\langle d \rangle$.

We start with T_p with $p \nmid N$. We have

$$T_p = \frac{1}{p} T_{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}}, \quad T_p^\dagger = \frac{1}{p} T_{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}$$

We have to prove $T_{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} = \langle p \rangle^{-1} T_{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}}$

By the Chinese Remainder Theorem, since $(p,N)=1$, $\exists\, d$ $\begin{cases} d \equiv 1 \pmod N \\ d \equiv 0 \pmod p \end{cases}$ Since $(d,N)=1$, $\exists\, a, b$ such that $ad - bN = 1$

Then $\begin{pmatrix} a & b \\ N & d \end{pmatrix} \in \Gamma_1(N)$. This implies $T_{\begin{pmatrix} a & b \\ N & d \end{pmatrix}} = \mathrm{id}$. We have

$$\begin{pmatrix} a & b \\ N & d \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ap & b \\ Np & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} ap & b \\ N & d/p \end{pmatrix}, \quad \begin{pmatrix} ap & d \\ N & d/p \end{pmatrix} \in \Gamma_0(N) \text{ since } p | d.$$

We then get

$$T_{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} = T_{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} T_{\begin{pmatrix} a & b \\ N & d \end{pmatrix}} = T_{\begin{pmatrix} a & b \\ N & d \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} = T_{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} ap & b \\ N & d/p \end{pmatrix}} = T_{\begin{pmatrix} ap & b \\ N & d/p \end{pmatrix}} T_{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}}$$

Using $d \equiv 1 \pmod N$ $T\begin{pmatrix} ap & b \\ N & d/p \end{pmatrix} = \langle d/p \rangle = \langle p \rangle^{-1}$.

Thus, $T\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \langle p \rangle^{-1} T\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$

$\therefore$ as claimed

For the general case of T_m, first consider $m = p^r$ and do induction on r. The claim is trivial for $r=0$, true for $r=1$. Assume it is true for $m = 1, p, \ldots, p^{r-1}$. We have

$$T_{p^r} = T_p T_{p^{r-1}} - p^{k-1} \langle p \rangle T_{p^{r-2}}$$

This implies $T_{p^r}^+ = T_{p^{r-1}}^+ T_p^+ - p^{k-1} T_{p^{r-2}}^+ \langle p \rangle^+$

$$= \langle p^{r-1} \rangle^{-1} T_{p^{r-1}} \langle p \rangle^{-1} T_p - p^{k-1} \langle p^{r-2} \rangle^{-1} T_{p^{r-2}} \langle p \rangle^{-1}$$

By commutativity of the Hecke algebra,

$$T_{p^r}^+ = \langle p^r \rangle^{-1} T_p T_{p^{r-1}} - p^{k-1} \langle p^r \rangle^{-1} \langle p \rangle T_{p^{r-2}}$$

$= \langle p^r \rangle^{-1} T_{p^r}$ which proves the claim for $m = p^r$

Finally, for $(m,n)=1$,

$$T_{mn}^+ = T_n^+ T_m^+ = \langle n \rangle^{-1} T_n \langle m \rangle^{-1} T_m = \langle mn \rangle^{-1} T_{mn}.$$

The claim for general $(m,N)=1$ follows by induction on the number of prime factors #.

__Coro:__ The operators T_m for $(m,N)=1$ and $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ form a commuting system of normal operator

__Coro:__ The space $S_k(\Gamma_1(N))$ admits a basis consisting of simultaneous eigenvectors for the operators T_m for $(m,N)=1$ and $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$.