Sujets spéciaux en théorie des nombres - formes modulaires/ Special Topics in Number Theory - Modular Forms.
MAT 6684w
Homework 1. Due September 25, 2017
To get full credit solve **4** of the following problems (you are welcome to do them all). The answers may be submitted in English or French.

1. (a) Show that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ is transitive.

   (b) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ with $\gamma \neq \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Prove that γ has exactly one fixed point in $\mathbb{H}$ if $|a+d| < 2$, and no fixed points in $\mathbb{H}$ otherwise.

Solution: (a) It suffices to show that for any $z \in \mathbb{H}$, there is some $\gamma \in \mathrm{SL}_2(\mathbb{R})$ such that $\gamma(i) = z$. Therefore, all the points are in the same orbit at i and the action is transitive.

Let $z = x + yi$ with $y > 0$. Write $y = a^2$. Then

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} i = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} yi = x + yi.$$

Therefore, the action is transitive.

(b) We solve the equation

$$\frac{az+b}{cz+d} = z \Leftrightarrow cz^2 + (d-a)z - b = 0.$$

If $c = 0$, then $\gamma = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$, $|a+d| = 2$, and this matrix has no fixed points in $\mathbb{H}$ if $b \neq 0$ (which is given by hypothesis).

If $c \neq 0$, the discriminant of the quadratic equation is $\Delta = (d-a)^2 + 4bc = (a+d)^2 - 4$ (since $ad - bc = 1$). Then, we get a (unique) solution in $\mathbb{H}$ iff $\Delta < 0$ iff $|a+d| < 2$. (The solution is unique because we get one solution with imaginary part positive and the other with imaginary part negative).

2. (a) Show that the stabiliser of i under the action of $\mathrm{SL}_2(\mathbb{R})$ is the group

$$\mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| \, a, b \in \mathbb{R}, a^2 + b^2 = 1 \right\}.$$

   (b) Prove that there is a bijection

$$\begin{array}{rcl} \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) & \xrightarrow{\sim} & \mathbb{H} \\ \gamma\mathrm{SO}_2(\mathbb{R}) & \mapsto & \gamma i \end{array}$$

Solution: (a) Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} i = \frac{ai+b}{ci+d} = i$, then we have $ai + b = -c + di$, giving $a = d$ and $b = -c$, and still $ad - bc = a^2 + b^2 = 1$.

(b) The map is well-defined by (a). Let $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{R})$ such that $\gamma_1 i = \gamma_2 i$. But this is iff $\gamma_1^{-1}\gamma_2 \in \mathrm{SO}_2(\mathbb{R})$ and this is iff $\gamma_1 \mathrm{SO}_2(\mathbb{R}) = \gamma_2 \mathrm{SO}_2(\mathbb{R})$. Therefore, this map is injective. Surjectivity is consequence of the fact that the action is transitive, because of what we proved in question 1(a), in other words, for any $z \in \mathbb{H}$, there is a $\gamma \in \mathrm{SL}_2(\mathbb{R})$ such that $\gamma i = z$.

3. Express $\begin{pmatrix} 70 & 213 \\ 23 & 70 \end{pmatrix}$ in terms of the generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $\mathrm{SL}_2(\mathbb{Z})$.

Solution: Use an iterative process: begin with the given matrix and multiply it on the right by T^{-1}. This substracts a copy of the first column from the second.

If we repeat the process of doing this until the bottom right entry is smaller than the bottom left one, then then multiply (again on the right) by S to swap the columns, then we are essentially performing Euclid's algorithm on the elements of the bottom row.

Since the original matrix was in $\mathrm{SL}_2(\mathbb{Z})$, the entries of the bottom row are coprime, so we must terminate with the bottom row being $(0, \pm 1)$. Since we never leave $\mathrm{SL}_2(\mathbb{Z})$, the top row must now be $(\pm 1, n)$ for some $n \in \mathbb{Z}$, and this is clearly in the subgroup generated by T and $S^2 = -1$. We have thus expressed the given matrix as a product of elements of $\mathrm{SL}_2(\mathbb{Z})$. In this case, if m is the given matrix, we find that $m = S^2 T^3 S^{-1} T^{-23} S^{-1} T^3$.

4. (a) Show that $G_4\left(e^{\frac{2\pi i}{3}}\right) = 0$.

   (b) Show that $G_6(i) = 0$.

Solution: (a) Let $\omega = e^{\frac{2\pi i}{3}}$. Notice that $-1/\omega = \omega + 1$. Since $G_4(z+1) = G_4(z)$ and $G_4(-1/z) = z^4 G_4(z)$,

$$G_4(\omega) = G_4(\omega + 1) = G_4(-1/\omega) = \omega^4 G_4(\omega).$$

Since $\omega^4 = \omega \neq 0$, we get $G_4(\omega) = 0$.

(b) Since $G_6(z+1) = G_6(z)$ and $G_6(-1/z) = z^6 G_6(z)$,

$$G_6(i) = G_6(-1/i) = i^6 G_6(i) = -G_6(i).$$

Therefore, $G_6(i) = 0$.

5. Recall the definition

$$\sigma_t(n) = \sum_{d|n} d^t \qquad t \geq 0 n \geq 1,$$

where d runs over the set of positive divisors of n.

(a) Let m, n and t be positive integers such that m and n are coprime. Show that

$$\sigma_t(mn) = \sigma_t(m)\sigma_t(n).$$

(b) Let n and t be positive integers and let

$$n = \prod_{p \text{prime}} p^{e_p}$$

be the prime factorization of n, where $e_p \geq 0$ and $e_p = 0$ for all but finitely many p. Show that

$$\sigma_t(n) = \prod_{p \text{ prime}} \frac{p^{(e_p+1)t} - 1}{p^t - 1}$$

Solution: (a) Since m and n are coprime, every divisor d of mn can be written uniquely in the form $d_1 d_2$ with $d_1 \mid m$ and $d_2 \mid n$. Then

$$\sum_{d|mn} d^t = \sum_{\substack{d_1|m \\ d_2|n}} d_1^t d_2^t = \left(\sum_{d_1|m} d_1^t\right)\left(\sum_{d_2|m} d_2^t\right)$$

(b) First suppose that $n = p^e$. Then the sum of t-powers of divisors is $1 + p^t + p^{2t} + \cdots + p^{et} = \frac{p^{(e+1)t}-1}{p^t-1}$. Then apply part (a).

6. Let $f : \mathbb{H} \to \mathbb{C}$ be a modular form of weight 0.

(a) Show that there exists some $C \in \mathbb{R}_{>0}$ such that any element in $\mathbb{H}$ is $SL_2(\mathbb{Z})$-equivalent to some $z \in \mathbb{H}$ with $\mathrm{Im}(z) \geq C$. (Take, for instance, $C = \sqrt{3}/2$)

(b) Deduce that $|f|$ attains a maximum.

(c) Conclude that the space of modular forms of weight zero consists exactly of the $\mathbb{C}$-constant functions. (Hint: maximum modulus principle.)

Solution: (a) Using the fundamental domain $\mathcal{D}$, we see that z is equivalent to a point in the fundamental domain, and those satisfy $\mathrm{Im}(z) \geq \sqrt{3}/2$.

(b) Since f is a modular form of weight 0, it suffices to consider $z \in \mathcal{D}$ and therefore it suffices to consider $\tilde{f}$ in the closed disk of radius $e^{-\pi\sqrt{3}}$ ($\tilde{f}$ extends to $q = 0$ because f is a modular form). Since $|\tilde{f}|$ is continuous and the disk is compact, then it must attain its maximum in the disk.

(c) We have an holomorphic function $\tilde{f}$ defined in the open unit disk that attains a maximum in the closed disk of radius $e^{-\pi\sqrt{3}}$, which is in the interior of the unit disk. Therefore $\tilde{f}$ is constant, and so is f.

7. (a) Let f, g be modular forms of the same weight k. Show that $F(z) = f(z)\overline{g(z)}(\mathrm{Im}\, z)^k$ satisfies $F(\gamma z) = F(z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

   (b) Show that if f is a cusp form of weight k, then $|f(z)|(\mathrm{Im}\, z)^{k/2}$ is bounded on $\mathbb{H}$.

Solution: (a) We have

$$\begin{aligned} F(\gamma z) =& f(\gamma z)\overline{g(\gamma z)}(\mathrm{Im}\,\gamma z)^k \\ =& j(\gamma, z)^k f(z)\overline{j(\gamma, z)^k g(z)}\left(\frac{\mathrm{Im}\, z}{|j(\gamma, z)|^2}\right)^k \\ =& F(z). \end{aligned}$$

(b) If f is a cusp form, then $F(z) = |f(z)|(\mathrm{Im}\, z)^{k/2}$ is a continuous function on $\mathcal{D}$. Notice that

$$|q(\mathrm{Im}\, z)^{k/2}| = e^{-2\pi\,\mathrm{Im}(z)}(\mathrm{Im}\, z)^{k/2}.$$

Then, the absolute convergence of the q-series implies

$$F(z) = \left|\sum_{n=1}^{\infty} a_n q^n (\mathrm{Im}\, z)^{k/2} \to 0 \text{ as } \mathrm{Im}\, z \to \infty\right|.$$

Therefore $F(z)$ is bounded on $\mathcal{D}$. Since $F(\gamma z) = F(z)$ by part (a), and every $\mathrm{SL}_2(\mathbb{Z})$-orbit in $\mathbb{H}$ contains a point o $\mathcal{D}$, it follows that any upper bound for F on $\mathcal{D}$ is also an upper bound for F on $\mathbb{H}$.

8. Define $f : \mathbb{H} \to \mathbb{C}$ by

$$f(z) = G_2(z) - \frac{\pi}{\mathrm{Im}\, z}.$$

   (a) Show that

$$f(\gamma z) = j(\gamma, z)^2 f(z) \qquad \forall \gamma \in \mathrm{SL}_2(\mathbb{Z}), z \in \mathbb{H}.$$

(b) Is $f(z)$ a modular form?

Solution: (a) We have seen that

$$G_2(\gamma z) = j(\gamma, z)^2 G_2(z) - 2\pi i c j(\gamma, z).$$

Thus,

$$\begin{aligned}
f(\gamma z) =& G_2(\gamma z) - \frac{\pi}{\operatorname{Im}(\gamma z)} = j(\gamma, z)^2 G_2(z) - 2\pi i c j(\gamma, z) - \frac{\pi |j(\gamma, z)|^2}{\operatorname{Im} z} \\
=& j(\gamma, z)^2 G_2(z) - j(\gamma, z) \frac{\pi}{\operatorname{Im} z} \left(2ic \operatorname{Im} z + (c\overline{z} + d)\right) \\
=& j(\gamma, z)^2 G_2(z) - j(\gamma, z)^2 \frac{\pi}{\operatorname{Im} z} \\
=& j(\gamma, z)^2 f(z).
\end{aligned}$$

(b) $f(z)$ is not a modular form because it is not holomorphic because $G_2(z)$ is holomorphic but $\frac{\pi}{\operatorname{Im} z}$ is not (it does not satisfy the Cauch–Riemann equations).

9. Show that if f is any modular form of weight k, then $\partial f = \frac{1}{2\pi i} f' + 2k E_2 f$ is a modular form of weight $k+2$. (This operator ∂ was introduced by Ramanujan.)

   Notice that $q \frac{df}{dq} = \frac{1}{2\pi i} f'$. (This may be useful for future applications.)

Solution: We have that $f'(z+1) = f'(z)$ and $f(-1/z) = z^k f(z)$ implies that $z^{-2} f'(-1/z) = k z^{k-1} f(z) + z^k f'(z)$.

On the other hand, we have $E_2(z+1) = E_2(z)$ and $z^{-2} E_2(-1/z) = E_2(z) - \frac{1}{4\pi i z}$.

It is then clear that $\partial f(z+1) = \partial f(z)$.

$$\begin{aligned}
\partial f(-1/z) =& \frac{1}{2\pi i} f'(-1/z) + 2k E_2(-1/z) f(-1/z) \\
=& \frac{1}{2\pi i} \left(k z^{k+1} f(z) + z^{k+2} f'(z)\right) + 2k \left(z^2 E_2(z) - \frac{z}{4\pi i}\right) z^k f(z) \\
=& z^{k+2} \left(\frac{1}{2\pi i} f'(z) + 2k E_2(z) f(z)\right).
\end{aligned}$$

∂f is holomorphic (since f, f', E_2 are holomorphic) on $\mathbb{H}$ and has a convergent q-expansion on some neighbourhood of ∞ with no negative powers of q (since this is true for f, f', E_2), so it is a modular form.