Sujets spéciaux en théorie des nombres - formes modulaires/ Special Topics in Number Theory - Modular Forms.
MAT 6684w
Homework 1. Due September 25, 2017
To get full credit solve **4** of the following problems (you are welcome to do them all). The answers may be submitted in English or French.

1. (a) Show that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ is transitive.

   (b) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ with $\gamma \neq \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Prove that γ has exactly one fixed point in $\mathbb{H}$ if $|a+d| < 2$, and no fixed points in $\mathbb{H}$ otherwise.

2. (a) Show that the stabiliser of i under the action of $\mathrm{SL}_2(\mathbb{R})$ is the group

$$\mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbb{R}, a^2 + b^2 = 1 \right\}.$$

   (b) Prove that there is a bijection

$$\begin{array}{rcl} \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) & \xrightarrow{\sim} & \mathbb{H} \\ \gamma \mathrm{SO}_2(\mathbb{R}) & \mapsto & \gamma i \end{array}$$

3. Express $\begin{pmatrix} 70 & 213 \\ 23 & 70 \end{pmatrix}$ in terms of the generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $\mathrm{SL}_2(\mathbb{Z})$.

4. (a) Show that $G_4\left(e^{\frac{2\pi i}{3}}\right) = 0$.

   (b) Show that $G_6(i) = 0$.

5. Recall the definition

$$\sigma_t(n) = \sum_{d|n} d^t \qquad t \geq 0 n \geq 1,$$

   where d runs over the set of positive divisors of n.

   (a) Let m, n and t be positive integers such that m and n are coprime. Show that

$$\sigma_t(mn) = \sigma_t(m)\sigma_t(n).$$

   (b) Let n and t be positive integers and let

$$n = \prod_{p \text{prime}} p^{e_p}$$

   be the prime factorization of n, where $e_p \geq 0$ and $e_p = 0$ for all but finitely many p. Show that

$$\sigma_t(n) = \prod_{p \text{ prime}} \frac{p^{(e_p+1)t} - 1}{p^t - 1}$$

6. Let $f : \mathbb{H} \to \mathbb{C}$ be a modular form of weight 0.

   (a) Show that there exists some $C \in \mathbb{R}_{>0}$ such that any element in $\mathbb{H}$ is $SL_2(\mathbb{Z})$-equivalent to some $z \in \mathbb{H}$ with $\text{Im}(z) \geq C$. (Take, for instance, $C = \sqrt{3}/2$)

   (b) Deduce that $|f|$ attains a maximum.

   (c) Conclude that the space of modular forms of weight zero consists exactly of the $\mathbb{C}$-constant functions. (Hint: maximum modulus principle.)

7. (a) Let f, g be modular forms of the same weight k. Show that $F(z) = f(z)\overline{g(z)}(\operatorname{Im} z)^k$ satisfies $F(\gamma z) = F(z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

   (b) Show that if f is a cusp form of weight k, then $|f(z)|(\operatorname{Im} z)^{k/2}$ is bounded on $\mathbb{H}$.

8. Define $f : \mathbb{H} \to \mathbb{C}$ by

$$f(z) = G_2(z) - \frac{\pi}{\operatorname{Im} z}.$$

   (a) Show that

$$f(\gamma z) = j(\gamma, z)^2 f(z) \qquad \forall \gamma \in \mathrm{SL}_2(\mathbb{Z}), z \in \mathbb{H}.$$

   (b) Is $f(z)$ a modular form?

9. Show that if f is any modular form of weight k, then $\partial f = \frac{1}{2\pi i} f' + 2kE_2 f$ is a modular form of weight $k + 2$. (This operator ∂ was introduced by Ramanujan.)

   Notice that $q\frac{df}{dq} = \frac{1}{2\pi i} f'$. (This may be useful for future applications.)