Sujets spéciaux en théorie des nombres - formes modulaires/ Special Topics in Number Theory - Modular Forms.
MAT 6684w
Homework 2. Due October 11, 2017

To get full credit solve **4** of the following problems (you are welcome to do them all). The answers may be submitted in English or French.

1. (a) Prove that

$$\sigma_9(n) = \frac{21}{11}\sigma_5(n) - \frac{10}{11}\sigma_3(n) + \frac{5040}{11}\sum_{j=1}^{n-1}\sigma_3(j)\sigma_5(n-j) \qquad \forall n \in \mathbb{Z}_{>0}$$

   (b) Find similar expressions for σ_{13} in terms of σ_3 and σ_9 and in terms of σ_5 and σ_7.

2. (a) Find rational numbers λ and μ such that

$$\Delta = \lambda E_4^3 + \mu E_{12}.$$

   (b) Let $\tau(n)$ be the n-th coefficient in the q-expansion of Δ, in other words,

$$\Delta = \sum_{n=1}^{\infty} \tau(n) q^n.$$

   Prove Ramanujan's congruence:

$$\tau(n) \equiv \sigma_{11}(n) \bmod 691$$

3. (a) Prove that $\frac{7}{10}E_6 = \frac{1}{2\pi i}E_4' + 8E_2E_4$, $\frac{10}{21}E_8 = \frac{1}{2\pi i}E_6' + 12E_2E_6$. (Hint: Problem 9, Homework 1).

   (b) Prove that $\frac{5}{6}E_4 = \frac{1}{2\pi i}E_2' + 2E_2^2$. (Recall that E_2 is not a modular form.)

   (Points (a) and (b) are known are *Ramanujan's identities* for Eisenstein series.)

   (c) Conclude that the ring $\mathbb{C}[E_2, E_4, E_6]$ is closed under differentiation.

4. Consider the modular function $j : \mathbb{H} \to \mathbb{C}$.

   (a) Show that $j(i) = 1728$ and $j\left(e^{\frac{2\pi i}{3}}\right) = 0$.

   (b) Let $z \in \mathcal{D}$ (the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$). Prove that if z lies on the boundary of $\mathcal{D}$ or $\mathrm{Re}\, z = 0$, then $j(z) \in \mathbb{R}$.

   (c) Show that $\overline{j} : \mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H} \to \mathbb{C}$ given by $\overline{j}([z]) := j(z)$ is well-defined and prove that $\overline{j}$ is bijective. (Hint: Use the valence formula to show that $f_\lambda = (240E_4)^3 - \lambda\Delta$ has exactly one zero on the fundamental domain.)

   (Here $[z]$ denotes the orbit of z under the action of $\mathrm{SL}_2(\mathbb{Z})$.)

   (d) Prove the converse to part (b).

5. (a) Show that M_k is spanned by all $E_4^j E_6^\ell$ with $j, \ell \in \mathbb{Z}_{\geq 0}$ and $4j + 6\ell = k$.

   (b) Show that E_4 and E_6 are algebraically independent over $\mathbb{C}$.

   This question shows that the ring of modular forms (for $SL_2(\mathbb{Z})$) defined as $M := \bigoplus_{k \in \mathbb{Z}} M_k$ is isomorphic to the ring of polynomials over $\mathbb{C}$ in two variables $\mathbb{C}[x, y]$ with isomorphism $\mathbb{C}[x, y] \xrightarrow{\sim} M$ given by $(x, y) \mapsto (E_4, E_6)$.

   (If we grade the rings by assigning grade k to a modular form of weight k and grades 4 and 6 to x and y respectively, we get an isomorphism of graded rings.)

6. (a) Show that the modular functions for $\mathrm{SL}_2(\mathbb{Z})$ form a field F with addition and multiplication defined pointwise.

   (b) Prove that $F = \mathbb{C}(j)$ and that j is transcendental over $\mathbb{C}$.

7. Let $\mathcal{L}_1(N)$ be the set of pairs (Λ, P) where Λ is a lattice in $\mathbb{C}$ and P is a point of order N in the group $\mathbb{C}/\Lambda$.

   (a) Show that on $\mathcal{L}_1(N)$ there is an equivalence relation $\sim$ with the property that $(\Lambda, P) \sim (\Lambda', P')$ if and only if there exists $\alpha \in \mathbb{C}^\times$ such that for any $\omega \in \mathbb{C}$ with $\omega + \Lambda = P$ in $\mathbb{C}/\Lambda$ we have $\alpha\Lambda = \Lambda'$ and $\alpha\omega + \Lambda' = P'$ in $\mathbb{C}/\Lambda'$.

   (b) Recall that $\Gamma_1(N)$ is the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of matrices of the form $\begin{pmatrix} a & b \\ Nc & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$, $a \equiv d \equiv 1 \bmod N$ and $ad - Nbc = 1$. Prove that there is a bijection

   $$\mathcal{L}_1(N)/\sim \cong \Gamma_1(N) \setminus \mathbb{H}$$

   (Hint: consider lattices together with a suitable $\mathbb{Z}$-basis (ω_1, ω_2), and use a similar argument as for the bijection $\mathcal{L}_0(N)/\sim \cong \Gamma_0(N) \setminus \mathbb{H}$ constructed in the lecture.)

8. Let N be a positive integer, and let H be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$. Show that the set

   $$\Gamma_H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| a, d \bmod N \text{ are in } H \text{ and } c \equiv 0 \bmod N \right\}$$

   is a congruence subgroup, and determine its level.