Sujets spéciaux en théorie des nombres - formes modulaires/ Special Topics in Number Theory - Modular Forms.

MAT 6684w

Homework 3. Due October 30, 2017

To get full credit solve **3** of the following problems (you are welcome to do them all). The answers may be submitted in English or French.

1. Show that the cusps of $\Gamma_1(4)$, viewed as $\Gamma_1(4)$-orbits in $\mathbb{P}^1(\mathbb{Q})$, are represented by the elements 0, 1/2 and ∞ of $\mathbb{P}^1(\mathbb{Q})$. For each of these cusps $\mathfrak{c}$, determine whether $\mathfrak{c}$ is regular or irregular, and compute its width $h_\Gamma(\mathfrak{c})$.

Solution: The orbit of $0 \in \mathbb{P}^1(\mathbb{Q})$ is

$$\begin{aligned}\Gamma_1(4)\cdot 0 &= \left\{ \begin{pmatrix} a & b \\ 4c & d \end{pmatrix} 0 \,\middle|\, a,b,c,d\in\mathbb{Z}, a\equiv d\equiv 1 \bmod 4, ad-4bc=1 \right\} \\ &= \left\{ \frac{b}{d} \,\middle|\, b,d\in\mathbb{Z}, d\equiv 1 \bmod 4, (b,d)=1 \right\}.\end{aligned}$$

The orbit of $\frac{1}{2} \in \mathbb{P}^1(\mathbb{Q})$ is

$$\begin{aligned}\Gamma_1(4)\cdot \frac{1}{2} &= \left\{ \begin{pmatrix} a & b \\ 4c & d \end{pmatrix} \frac{1}{2} \,\middle|\, a,b,c,d\in\mathbb{Z}, a\equiv d\equiv 1 \bmod 4, ad-4bc=1 \right\} \\ &= \left\{ \frac{a+2b}{4c+2d} \,\middle|\, a,b,c,d\in\mathbb{Z}, a\equiv d\equiv 1 \bmod 4, (a+2b)d-b(4c+2d)=1 \right\} \\ &= \left\{ \frac{a_1}{c_1} \,\middle|\, a_1,c_1\in\mathbb{Z}, a_1 \text{ odd }, c_1\equiv 2 \bmod 4, (a_1,c_1)=1 \right\}.\end{aligned}$$

The orbit of $\infty \in \mathbb{P}^1(\mathbb{Q})$ is

$$\begin{aligned}\Gamma_1(4)\cdot \infty &= \left\{ \begin{pmatrix} a & b \\ 4c & d \end{pmatrix} \infty \,\middle|\, a,b,c,d\in\mathbb{Z}, a\equiv d\equiv 1 \bmod 4, ad-4bc=1 \right\} \\ &= \left\{ \frac{a}{4c} \,\middle|\, a,c\in\mathbb{Z}, a\equiv 1 \bmod 4, (a,c)=1 \right\} \\ &= \left\{ \frac{a}{c_1} \,\middle|\, a,c_1\in\mathbb{Z}, a\equiv 1 \bmod 4, 4\mid c_1, (a,c_1)=1 \right\}.\end{aligned}$$

It is clear that the orbits are disjoint (by comparing denominators of the elements). Let us check that each element of $\mathbb{P}^1(\mathbb{Q})$ is in exactly one orbit. Let $r=\frac{a}{c}\in\mathbb{Q}$ with

$(a,c) = 1$. If c is odd, then either c or $-c$ is $\equiv 1 \bmod 4$. Then replacing by $\frac{-a}{-c}$ if necessary, we get $r \in \Gamma_1(4) \cdot 0$. Now assume c is even but not divisible by 4. Then clearly $r \in \Gamma_1(4) \cdot \frac{1}{2}$. Finally, if $4 \mid c$, a is odd, and by possibly considering $\frac{-a}{-c}$, we get $r \in \Gamma_1(4) \cdot \infty$.

For $\mathfrak{c} = 0$, take $\gamma_0 = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then

$$\begin{aligned} H_0 =& S^{-1}\Gamma_1(4)S \cap \mathrm{SL}_2(\mathbb{Z})_\infty \\ =& \left\{ \begin{pmatrix} d & -4c \\ -b & a \end{pmatrix} \right\} \cap \mathrm{SL}_2(\mathbb{Z})_\infty \\ =& \left\{ \begin{pmatrix} 1 & -4c \\ 0 & 1 \end{pmatrix} \right\} \end{aligned}$$

since $a \equiv d \equiv 1 \bmod 4$.

Thus, the cusp is regular, and $h_{\Gamma_1(4)}(0) = 4$.

For $\mathfrak{c} = \frac{1}{2}$, take $\gamma_{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Then

$$\begin{aligned} H_{\frac{1}{2}} =& \gamma_{\frac{1}{2}}^{-1}\Gamma_1(4)\gamma_{\frac{1}{2}} \cap \mathrm{SL}_2(\mathbb{Z})_\infty \\ =& \left\{ \begin{pmatrix} a+2b & b \\ -2a-4b+4c+2d & -2b+d \end{pmatrix} \right\} \cap \mathrm{SL}_2(\mathbb{Z})_\infty \\ =& \left\{ \begin{pmatrix} 1 & 2b_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2b_1+1 \\ 0 & -1 \end{pmatrix} \right\} \end{aligned}$$

Above we have solved for $a + 2b = d - 2b = 1$ or $a + 2b = d - 2b = -1$ and $-2a - 4b + 4c + 2d = 0$, keeping in mind that we also have $a \equiv d \equiv 1 \pmod 4$.

Thus, the cusp is irregular, and $h_{\Gamma_1(4)}(1/2) = 1$.

For $\mathfrak{c} = \infty$, take $\gamma_\infty = 1$. Then

$$\begin{aligned} H_\infty =& \Gamma_1(4) \cap \mathrm{SL}_2(\mathbb{Z})_\infty \\ =& \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\} \end{aligned}$$

Thus, the cusp is regular, and $h_{\Gamma_1(4)}(\infty) = 1$.

2. Let p be an odd prime number. Determine a set of representatives for the $\Gamma_1(p)$-orbits in $\mathbb{P}^1(\mathbb{Q})$. For each of the corresponding cusps $\mathfrak{c}$ of $\Gamma_1(p)$, compute its width $h_\Gamma(\mathfrak{c})$.

Solution: The orbit of $0 \in \mathbb{P}^1(\mathbb{Q})$ is

$$\begin{aligned}\Gamma_1(p)\cdot 0 &= \left\{ \left(\begin{array}{cc} a & b \\ pc & d \end{array} \right) 0 \,\middle|\, a,b,c,d\in\mathbb{Z}, a\equiv d\equiv 1 \bmod p, ad-pbc=1 \right\} \\ &= \left\{ \frac{b}{d} \,\middle|\, b,d\in\mathbb{Z}, d\equiv 1 \bmod p, (b,d)=1 \right\}.\end{aligned}$$

The orbit of $\infty \in \mathbb{P}^1(\mathbb{Q})$ is

$$\begin{aligned}\Gamma_1(p)\cdot \infty &= \left\{ \left(\begin{array}{cc} a & b \\ pc & d \end{array} \right) \infty \,\middle|\, a,b,c,d\in\mathbb{Z}, a\equiv d\equiv 1 \bmod p, ad-pbc=1 \right\} \\ &= \left\{ \frac{a}{pc} \,\middle|\, a,c\in\mathbb{Z}, a\equiv 1 \bmod p, (a,c)=1 \right\} \\ &= \left\{ \frac{a}{c_1} \,\middle|\, a,c_1\in\mathbb{Z}, a\equiv 1 \bmod p, p\mid c_1, (a,c_1)=1 \right\}.\end{aligned}$$

The orbit of $\frac{1}{n} \in \mathbb{P}^1(\mathbb{Q})$ for $2\le n<p/2$ is

$$\begin{aligned}\Gamma_1(p)\cdot \frac{1}{n} &= \left\{ \left(\begin{array}{cc} a & b \\ pc & d \end{array} \right) \frac{1}{n} \,\middle|\, a,b,c,d\in\mathbb{Z}, a\equiv d\equiv 1 \bmod p, ad-pbc=1 \right\} \\ &= \left\{ \frac{a+bn}{pc+dn} \,\middle|\, a,b,c,d\in\mathbb{Z}, a\equiv d\equiv 1 \bmod p, (a+bn)d-(pc+dn)b=1 \right\} \\ &= \left\{ \frac{a_1}{c_1} \,\middle|\, a_1,c_1\in\mathbb{Z}, c_1\equiv n \bmod p, (a_1,c_1)=1 \right\}.\end{aligned}$$

The orbit of $\frac{n}{p} \in \mathbb{P}^1(\mathbb{Q})$ for $2\le n<p/2$ is

$$\begin{aligned}\Gamma_1(p)\cdot \frac{n}{p} &= \left\{ \left(\begin{array}{cc} a & b \\ pc & d \end{array} \right) \frac{n}{p} \,\middle|\, a,b,c,d\in\mathbb{Z}, a\equiv d\equiv 1 \bmod p, ad-pbc=1 \right\} \\ &= \left\{ \frac{an+bp}{pcn+dp} \,\middle|\, a,b,c,d\in\mathbb{Z}, a\equiv d\equiv 1 \bmod p, (an+bp)d-(pcn+dp)b=n \right\} \\ &= \left\{ \frac{a_1}{c_1} \,\middle|\, a_1,c_1\in\mathbb{Z}, a_1\equiv n \bmod p, p\mid c_1, (a_1,c_1)=1 \right\}.\end{aligned}$$

It is clear that each element is in exactly one orbit. Indeed, one has to consider all the possible congruences modulo p for the denominator and do an argument similar to what we did in problem 1.

Notice that 0 corresponds to 1 and ∞ corresponds to $1/p$.

For $\mathfrak{c} = \frac{1}{n}$, take $\gamma_{\frac{1}{n}} = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$. Then

$$\begin{aligned} H_{\frac{1}{n}} =& \gamma_{\frac{1}{n}}^{-1}\Gamma_1(p)\gamma_{\frac{1}{n}} \cap \mathrm{SL}_2(\mathbb{Z})_\infty \\ =& \left\{ \begin{pmatrix} a+nb & b \\ -na-n^2b+pc+nd & -nb+d \end{pmatrix} \right\} \cap \mathrm{SL}_2(\mathbb{Z})_\infty \\ =& \left\{ \begin{pmatrix} 1 & pb_1 \\ 0 & 1 \end{pmatrix} \right\} \end{aligned}$$

where, again, we have considered that $a+nb = -nb+d = \pm 1$ and $-na-n^2b+pc+nd = 0$ as well as $a \equiv d \equiv 1 \bmod p$.

Thus, the cusp is regular, and $h_{\Gamma_1(p)}(1/n) = p$.

For $\mathfrak{c} = \frac{n}{p}$, take $\gamma_{\frac{n}{p}} = \begin{pmatrix} n & s \\ p & r \end{pmatrix}$ with $rn - sp = 1$. Then

$$\begin{aligned} H_{\frac{n}{p}} =& \gamma_{\frac{n}{p}}^{-1}\Gamma_1(p)\gamma_{\frac{n}{p}} \cap \mathrm{SL}_2(\mathbb{Z})_\infty \\ =& \left\{ \begin{pmatrix} a+k_1p & r^2b+k_2p \\ k_3p & d+k_4p \end{pmatrix} \right\} \cap \mathrm{SL}_2(\mathbb{Z})_\infty \\ =& \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\} \end{aligned}$$

(the computation is long).

Thus, the cusp is regular, and $h_{\Gamma_1(p)}(n/p) = 1$.

Solution: Another way (combining Anayo, Joëlle, Simone, and Subham's solutions, sketch): We have that $\Gamma_1(p) \subset \Gamma_0(p)$. Then $\mathrm{Cusps}(\Gamma_1(p))$ maps to $\mathrm{Cusps}(\Gamma_0(p)) = \{[\infty], [0]\}$. Since $\Gamma_0(p)$ acts transitively in $[\infty]$, we have a surjective map

$$\Gamma_1(p)\backslash\Gamma_0(p) \to \Gamma_1(p)\backslash\Gamma_0(p) \cdot \infty$$

A set of coset representatives of $\Gamma_1(p)\backslash\Gamma_0(p)$ is $\left\{ \begin{pmatrix} n & s \\ p & r \end{pmatrix} : 1 \leq n \leq (p-1)/2 \right\}$. This yields cusps of the form $[n/p]$ sent to $[\infty]$.

Similarly

$$\Gamma_1(p)\backslash\Gamma_0(p) \to \Gamma_1(p)\backslash\Gamma_0(p) \cdot 0$$

A set of coset representatives of $\Gamma_1(p)$ $\Gamma_0(p)$ is $\left\{\begin{pmatrix} r & 1 \\ sp & n \end{pmatrix} : 1 \leq n \leq (p-1)/2\right\}$.
This yields cusps of the form $[1/n]$ sent to $[0]$.

We have $h_{[n/p]} \mid h_{[\infty]} = 1$ and $h_{[1/n]} \mid h_{[0]} = p$. Finally

$$\sum_{\mathrm{Cusps}(\Gamma_1(p))} h_{\mathfrak{c}} = [\mathrm{SL}_2(\mathbb{Z}) : \{\pm 1\}\Gamma_1(p)] = [\mathrm{SL}_2(\mathbb{Z}) : \{\pm 1\}\Gamma_0(p)][\{\pm 1\}\Gamma_0(p) : \{\pm 1\}\Gamma_1(p)]$$

$$= (p+1)\frac{(p-1)}{2}$$

This implies that $h_{[1/n]} = p$.

3. (a) Let χ be a Dirichlet character modulo N. Prove that

$$\sum_{j=0}^{N-1} \chi(j) = \begin{cases} \phi(N) & \chi = \mathbf{1}_N, \\ 0 & \text{otherwise,} \end{cases}$$

where ϕ denotes Euler's totient function.

(b) Let j be an integer. Prove that

$$\sum_{\chi \mod N} \chi(j) = \begin{cases} \phi(N) & j \equiv 1 \pmod N \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is over all Dirichlet characters modulo N.

Solution: (a) The case $\chi = \mathbf{1}_N$ comes from the fact that $\mathbf{1}_N(j) = 1$ unless $(j, N) > 1$, and in that case $\mathbf{1}_N(j) = 0$.

If $\chi \neq \mathbf{1}_N$, we have j_0 such that $(j_0, N) = 1$ and $\chi(j_0) \neq 1$. Then the numbers $j_0 j$ with $j = 0, \dots, N-1$ are a system of residues modulo N.

$$\chi(j_0) \sum_{j=0}^{N-1} \chi(j) = \sum_{j=0}^{N-1} \chi(j_0 j) = \sum_{j=0}^{N-1} \chi(j)$$

and since $\chi(j_0) \neq 1$, we get that the sum is zero.

(b)

If $j \equiv 1 \pmod N$, this is trivial since $\chi(j) = 1$ for all χ, and we saw in class that $|\mathrm{Hom}((\mathbb{Z}/N\mathbb{Z})^\times, \mathbb{C}^\times)| = \phi(N)$. If $(j, N) > 1$, then $\chi(j) = 0$ and the sum is trivially 0.

Suppose that $(j, N) = 1$. Then there is χ_1 such that $\chi_1(j) \neq 0, 1$ (write $(\mathbb{Z}/N\mathbb{Z})^\times \simeq \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_n\mathbb{Z}$, let $(\ell_1, \ldots, \ell_n)$ be the image of j. Assume wlog that $\ell_1 \neq 0$ and consider $\chi_1(r_1, \ldots, r_n) = \exp(2\pi i r_1/a_1)$).

Then

$$\chi_1(j) \sum_{\chi \mod N} \chi(j) = \sum_{\chi \mod N} \chi_1\chi(j) = \sum_{\chi \mod N} \chi(j)$$

and since $\chi_1(j) \neq 0, 1$, we get that the sum is zero.

4. For integers $k > 0$ and $n \geq 0$, write

$$r_k(n) = \#\{(x_1, \ldots, x_k) \in \mathbb{Z}^k \mid x_1^2 + \cdots + x_k^2 = n\}.$$

Let χ be the unique non-trivial Dirichlet character modulo 4. Assume without proof that there exist modular forms $E_1^{\mathbf{1},\chi} \in M_1(\Gamma_1(4))$, and $E_3^{\mathbf{1},\chi}, E_3^{\chi,\mathbf{1}} \in M_3(\Gamma_1(4))$ with q-expansions

$$E_1^{\mathbf{1},\chi} = \frac{1}{4} + \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(d) \right) q^n,$$

$$E_3^{\mathbf{1},\chi} = -\frac{1}{4} + \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(d) d^2 \right) q^n,$$

$$E_3^{\chi,\mathbf{1}} = \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(n/d) d^2 \right) q^n.$$

(a) Prove that

$$r_2(n) = 4 \sum_{d|n} \chi(d) \qquad \forall n \geq 1.$$

(b) Prove that

$$r_6(n) = \sum_{d|n} (16\chi(n/d) - 4\chi(d)) d^2 \qquad \forall n \geq 1.$$

Solution: Recall that

$$\theta^k = \sum_{n=0}^{\infty} r_k(n) q^n$$

and that $\theta^{2k} \in M_{k/2}(\Gamma_1(4))$. In addition, the valence formula implies that

$$\dim M_k(\Gamma_1(4)) \leq 1 + \lfloor k/2 \rfloor.$$

(a) We have $\theta^2 \in M_1(\Gamma_1(4))$, which has dimension at most 1. Therefore, $\theta^2 = cE_1^{\mathbf{1},\chi}$. We compare the q-expansions.

$$\theta^2 = r_2(0) + r_2(1)q + \cdots = 1 + 4q + \cdots$$

$$E_1^{\mathbf{1},\chi} = \frac{1}{4} + q + \cdots$$

Therefore, $c = 4$ and

$$r_2(n) = 4\sum_{d|n} \chi(d).$$

(b) We have $\theta^6 \in M_3(\Gamma_1(4))$, which has dimension at most 2. Therefore, $\theta = c_1 E_3^{\mathbf{1},\chi} + c_2 E_3^{\chi,\mathbf{1}}$. Again we check first coefficients.

$$\theta^6 = r_6(0) + r_6(1)q + r_6(2)q^2 + \cdots = 1 + 12q + 60q^2 + \cdots$$

$$E_3^{\mathbf{1},\chi} = -\frac{1}{4} + q + q^2 + \cdots$$

$$E_3^{\chi,\mathbf{1}} = q + 4q^2 + \cdots$$

Solving $-c_1/4 = 1$ and $c_1 + c_2 = 12$ yields $c_1 = -4$ and $c_2 = 16$ and

$$r_6(n) = \sum_{d|n} (16\chi(n/d) - 4\chi(d))d^2.$$

5. Let $\chi : \mathbb{Z} \to \mathbb{C}$ be a Dirichlet character modulo N. The L-function of χ is the holomorphic function $L(\chi, s)$ (of the variable s) defined by

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

(a) Prove that the sum converges absolutely and uniformly on every right half-plane of the form $\{s \in \mathbb{C} | \operatorname{Re} s \geq \sigma\}$ with $\sigma > 1$.

(b) Prove the identity

$$L(\chi, s) = \prod_{p\text{prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \qquad \operatorname{Re} s > 1$$

(Hint: expand each factor in a power series...)

Note: The functions $L(\chi, s)$ were introduced by Dirichlet in the proof of his famous theorem on primes in arithmetic progressions: Let N and a be coprime positive integers. Then there exist infinitely many prime numbers p with $p \equiv a \bmod N$.

Solution: (a) Let s be such that $\operatorname{Re} s \leq \sigma$. Then

$$\begin{aligned} |L(\chi, s)| &\leq \sum_{n=1}^{\infty} \left| \frac{\chi(n)}{n^s} \right| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} < \infty \qquad \text{for } \sigma > 1. \end{aligned}$$

Thus, by the Weierstrass M-test, we get that on the region $\{s \in \mathbb{C} | \operatorname{Re} s \geq \sigma\}$ the series converges absolutely and uniformily.

(b) Since χ is completely multiplicative, we have that

$$\left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \sum_{\ell=0}^{\infty} \left(\frac{\chi(p)}{p^s}\right)^{\ell} = \sum_{\ell=0}^{\infty} \frac{\chi(p^{\ell})}{p^{\ell s}}.$$

The equality is then a consequence of the Fundamental Theorem of Arithmetic and the fact that we can rearrange the sum and the products in the domain of convergence.

6. Let χ be a Dirichlet character modulo N. We consider the function $\mathbb{Z} \to \mathbb{C}$ sending an integer m to the complex number

$$\tau(\chi, m) = \sum_{n=0}^{N-1} \chi(n) \exp(2\pi i m n/N)$$

(This can be viewed as a discrete Fourier transform of χ.) The case $m = 1$ is known as the Gauss sum attached to χ.

$$\tau(\chi) = \sum_{n=0}^{N-1} \chi(n) \exp(2\pi i n/N)$$

(a) Compute $\tau(\chi)$ for all non-trivial Dirichlet characters χ modulo 4 and modulo 5, respectively.

(b) Suppose that χ is primitive. Prove that for all $m \in \mathbb{Z}$ we have

$$\tau(\chi, m) = \overline{\chi}(m)\tau(\chi)$$

(Hint: writing $d = (m, N)$, distinguish the cases $d = 1$ and $d > 1$. If $N_1 = N/d$, prove There is an integer c such that $c \equiv 1 \bmod N_1$, $(c, N) = 1$, and $\chi(c) \neq 1$.)

(c) Deduce that if χ is primitive, we have

$$\tau(\chi)\tau(\overline{\chi}) = \chi(-1)N$$

and

$$\tau(\chi)\overline{\tau(\chi)} = N.$$

Solution: (a) Modulo 4: the only nontrivial character is $\chi_{-1}(n) = \left(\frac{n}{4}\right)$.

$$\tau(\chi_{-1}) = \chi_{-1}(0) + \chi_{-1}(1)i + \chi_{-1}(2)(-1) + \chi_{-1}(3)(-i) = 2i$$

Modulo 5: there are three nontrivial characters. Take

$$\chi_1(m) = \begin{cases} i^n & m \equiv 2^n \bmod 5, \\ 0 & 5 \mid m. \end{cases}$$

Then the non-trivial characters are $\chi_\ell = \chi_1^\ell$ for $\ell = 1, 2, 3$. Set $\omega_5 = \exp(2\pi i/5)$ We have

$$\begin{aligned}\tau(\chi_\ell) =& \chi_\ell(0) + \chi_\ell(1)\omega_5 + \chi_\ell(2)\omega_5^2 + \chi_\ell(3)\omega_5^3 + \chi_\ell(4)\omega_5^4 \\ =& \omega_5 + i^\ell\omega_5^2 + (-i)^\ell\omega_5^3 + (-1)^\ell\omega_5^4 \\ =& i^\ell(\omega_5^2 + (-1)^\ell\omega_5^3) + (\omega_5 + (-1)^\ell\omega_5^4) \\ =& \begin{cases} (-1)^m 2\cos(4\pi/5) + 2\cos(2\pi/5) & \ell = 2m \\ \\ (-1)^{m+1} 2\sin(4\pi/5) + 2i\sin(2\pi/5) & \ell = 2m+1 \end{cases}\end{aligned}$$

We have that $\omega_5 = \frac{-1+\sqrt{5}}{4} + i\sqrt{\frac{5+\sqrt{5}}{8}}$ and that $\omega_5^2 = \frac{-1-\sqrt{5}}{4} + i\sqrt{\frac{5-\sqrt{5}}{8}}$. We further obtain

$$\begin{aligned}\tau(\chi_\ell) =& \begin{cases} (-1)^m \frac{-1-\sqrt{5}}{2} + \frac{-1+\sqrt{5}}{2} & \ell = 2m \\ \\ (-1)^{m+1}\sqrt{\frac{5-\sqrt{5}}{2}} + i\sqrt{\frac{5+\sqrt{5}}{2}} & \ell = 2m+1 \end{cases} \\ =& \begin{cases} -1 & \ell = 0 \\ \\ -\sqrt{\frac{5-\sqrt{5}}{2}} + i\sqrt{\frac{5+\sqrt{5}}{2}} & \ell = 1 \\ \\ \sqrt{5} & \ell = 2 \\ \\ \sqrt{\frac{5-\sqrt{5}}{2}} + i\sqrt{\frac{5+\sqrt{5}}{2}} & \ell = 3 \end{cases}\end{aligned}$$

(The first line corresponds to the trivial character.)

(b) First suppose that $(m, N) = 1$.

$$\tau(\chi, m) = \sum_{n=0}^{N-1} \chi(n) \exp(2\pi imn/N)$$
$$= \overline{\chi(m)}\chi(m) \sum_{n=0}^{N-1} \chi(n) \exp(2\pi imn/N)$$
$$= \overline{\chi(m)} \sum_{n=0}^{N-1} \chi(mn) \exp(2\pi imn/N).$$

Now, going over all n modulo N is equivalent to going over all nm modulo N and we get

$$\tau(\chi, m) = \overline{\chi(m)}\tau(\chi).$$

Now suppose that $(m, N) = d > 1$. Write $m = dm_1$, $N = dN_1$. There is an integer c such that $c \equiv 1 \bmod N_1$, $(c, N) = 1$, and $\chi(c) \neq 1$. Otherwise, χ would not be primitive modulo N. Indeed, let $a \equiv b \bmod N_1$ with a, b relatively prime to N. Then $a \equiv cb \bmod N$, with some c defined modulo N. Now $\chi(a) = \chi(cb) = \chi(b)$ and the character is modulo N_1.

Take such a c,

$$\chi(c)\tau(\chi, m) = \sum_{n=0}^{N-1} \chi(cn) \exp(2\pi imn/N).$$

cn runs through a complete residue system modulo N. In addition

$$\exp(2\pi imn/N) = \exp(2\pi im_1n/N_1) = \exp(2\pi icm_1n/N_1) = \exp(2\pi icmn/N)$$

and

$$\chi(c)\tau(\chi, m) = \sum_{\ell=0}^{N-1} \chi(\ell) \exp(2\pi im\ell/N) = \tau(\chi, m).$$

Since $\chi(c) \neq 1$, we get that $\tau(\chi, m) = 0 = \overline{\chi(m)}\tau(\chi)$.

(c) Notice that $\chi(n) \neq 0$ only when $(n, N) = 1$.

$$\begin{aligned}
\tau(\overline{\chi})\tau(\chi) &= \sum_{m=0}^{N-1} \overline{\chi(m)} \exp(2\pi i m/N)\tau(\chi) \\
&= \sum_{m=0}^{N-1} \tau(\chi, m) \exp(2\pi i m/N) \\
&= \sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} \chi(n) \exp(2\pi i m n/N) \right) \exp(2\pi i m/N) \\
&= \sum_{n=0}^{N-1} \chi(n) \sum_{m=0}^{N-1} \exp(2\pi i m(n+1)/N).
\end{aligned}$$

We have $\sum_{m=0}^{N-1} \exp(2\pi i m(n+1)/N) = N$ if $n+1 = 0$ and 0 otherwise. Then

$$\tau(\overline{\chi})\tau(\chi) = \chi(N-1)N = \chi(-1)N.$$

The other equality is similar. In this case,

$$\begin{aligned}
\overline{\tau(\chi)}\tau(\chi) &= \sum_{m=0}^{N-1} \overline{\chi(m)} \exp(-2\pi i m/N)\tau(\chi) \\
&= \sum_{m=0}^{N-1} \tau(\chi, m) \exp(-2\pi i m/N) \\
&= \sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} \chi(n) \exp(2\pi i m n/N) \right) \exp(-2\pi i m/N) \\
&= \sum_{n=0}^{N-1} \chi(n) \sum_{m=0}^{N-1} \exp(2\pi i m(n-1)/N) \\
&= \chi(1)N \\
&= N
\end{aligned}$$