Sujets spéciaux en théorie des nombres - formes modulaires/ Special Topics in Number Theory - Modular Forms.

MAT 6684w

Homework 3. Due October 30, 2017

To get full credit solve **3** of the following problems (you are welcome to do them all). The answers may be submitted in English or French.

1. Show that the cusps of $\Gamma_1(4)$, viewed as $\Gamma_1(4)$-orbits in $\mathbb{P}^1(\mathbb{Q})$, are represented by the elements 0, 1/2 and ∞ of $\mathbb{P}^1(\mathbb{Q})$. For each of these cusps $\mathfrak{c}$, determine whether $\mathfrak{c}$ is regular or irregular, and compute its width $h_\Gamma(\mathfrak{c})$.

2. Let p be an odd prime number. Determine a set of representatives for the $\Gamma_1(p)$-orbits in $\mathbb{P}^1(\mathbb{Q})$. For each of the corresponding cusps $\mathfrak{c}$ of $\Gamma_1(p)$, compute its width $h_\Gamma(\mathfrak{c})$.

3. (a) Let χ be a Dirichlet character modulo N. Prove that

$$\sum_{j=0}^{N-1} \chi(j) = \begin{cases} \phi(N) & \chi = \mathbf{1}_N, \\ 0 & \text{otherwise,} \end{cases}$$

where ϕ denotes Euler's totient function.

(b) Let j be an integer. Prove that

$$\sum_{\chi \mod N} \chi(j) = \begin{cases} \phi(N) & j \equiv 1 \pmod N \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is over all Dirichlet characters modulo N.

4. For integers $k > 0$ and $n \geq 0$, write

$$r_k(n) = \#\{(x_1, \dots, x_k) \in \mathbb{Z}^k \mid x_1^2 + \cdots + x_k^2 = n\}.$$

Let χ be the unique non-trivial Dirichlet character modulo 4. Assume without proof that there exist modular forms $E_1^{\mathbf{1},\chi} \in M_1(\Gamma_1(4))$, and $E_3^{\mathbf{1},\chi}, E_3^{\chi,\mathbf{1}} \in M_3(\Gamma_1(4))$ with q-expansions

$$E_1^{\mathbf{1},\chi} = \frac{1}{4} + \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(d) \right) q^n,$$

$$E_3^{\mathbf{1},\chi} = -\frac{1}{4} + \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(d) d^2 \right) q^n,$$

$$E_3^{\chi,\mathbf{1}} = \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(n/d) d^2 \right) q^n.$$

(a) Prove that

$$r_2(n) = 4\sum_{d|n}\chi(d) \qquad \forall n \geq 1.$$

(b) Prove that

$$r_6(n) = \sum_{d|n}(16\chi(n/d) - 4\chi(d))d^2 \qquad \forall n \geq 1.$$

5. Let $\chi : \mathbb{Z} \to \mathbb{C}$ be a Dirichlet character modulo N. The L-function of χ is the holomorphic function $L(\chi, s)$ (of the variable s) defined by

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

(a) Prove that the sum converges absolutely and uniformly on every right half-plane of the form $\{s \in \mathbb{C} | \operatorname{Re} s \geq \sigma\}$ with $\sigma > 1$.

(b) Prove the identity

$$L(\chi, s) = \prod_{p\text{prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \qquad \operatorname{Re} s > 1$$

(Hint: expand each factor in a power series...)

Note: The functions $L(\chi, s)$ were introduced by Dirichlet in the proof of his famous theorem on primes in arithmetic progressions: Let N and a be coprime positive integers. Then there exist infinitely many prime numbers p with $p \equiv a \bmod N$.

6. Let χ be a Dirichlet character modulo N. We consider the function $\mathbb{Z} \to \mathbb{C}$ sending an integer m to the complex number

$$\tau(\chi, m) = \sum_{n=0}^{N-1} \chi(n) \exp(2\pi i m n/N)$$

(This can be viewed as a discrete Fourier transform of χ.) The case $m = 1$ is known as the Gauss sum attached to χ.

$$\tau(\chi) = \sum_{n=0}^{N-1} \chi(n) \exp(2\pi i n/N)$$

(a) Compute $\tau(\chi)$ for all non-trivial Dirichlet characters χ modulo 4 and modulo 5, respectively.

(b) Suppose that χ is primitive. Prove that for all $m \in \mathbb{Z}$ we have

$$\tau(\chi, m) = \overline{\chi}(m)\tau(\chi)$$

(Hint: writing $d = (m, N)$, distinguish the cases $d = 1$ and $d > 1$. If $N_1 = N/d$, prove There is an integer c such that $c \equiv 1 \bmod N_1$, $(c, N) = 1$, and $\chi(c) \neq 1$.)

(c) Deduce that if χ is primitive, we have

$$\tau(\chi)\tau(\overline{\chi}) = \chi(-1)N$$

and

$$\tau(\chi)\overline{\tau(\chi)} = N.$$

The goal of the following questions is to construct Eisenstein series with character. In each question you may use the results of all preceding questions.

For the problems 7 and 9, I only have partial solutions written. The computations may be long. Please think of the following problems as "just for fun".

7. Let χ be a primitive Dirichlet character modulo N. The generalized Bernoulli numbers attached to χ are the complex numbers $B_k(\chi)$ for $k \geq 0$ defined by the identity

$$\sum_{k=0}^{\infty} \frac{B_k(\chi)}{k!} t^k = \frac{t}{\exp(Nt) - 1} \sum_{j=1}^{N} \chi(j) \exp(jt)$$

in the ring $\mathbb{C}[[t]]$ of formal power series in t.

(a) Let ω_N be a primitive N-th root of unity in $\mathbb{C}$. Prove that if χ is non-trivial (i.e. $N > 1$), then we have

$$\sum_{j=0}^{N-1} \chi(j) \frac{x + \omega_N^j}{x - \omega_N^j} = \frac{2N}{\tau(\overline{\chi})(x^N - 1)} \sum_{m=0}^{N-1} \overline{\chi}(m) x^m$$

in the field $\mathbb{C}(x)$ of rational functions in the variable x. (Hint: compute residues.)

(b) Prove that for every integer $k \geq 2$ such that $(-1)^k = \chi(-1)$, the special value of the Dirichlet L-function of χ at k is

$$L(\chi, k) = -\frac{(2\pi i)^k B_k(\overline{\chi})}{2\tau(\overline{\chi}) N^{k-1} k!}.$$

8. Let $k \geq 3$, and let α and β be Dirichlet characters modulo M and N, respectively. For all $k \geq 3$, we define a function $G_k^{\alpha,\beta} : \mathbb{H} \to \mathbb{C}$ by

$$G_k^{\alpha,\beta}(z) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{\alpha(m)\overline{\beta}(n)}{(mz + n)^k}.$$

(a) Prove that the function $G_k^{\alpha,\beta}$ is weakly modular of weight k for the congruence subgroup

$$\Gamma_1(M, N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 \bmod [M, N], c \equiv 0 \bmod M, b \equiv 0 \bmod N \right\}.$$

(b) Show that $G_k^{\alpha,\beta}$ is the zero function unless $\alpha(-1)\beta(-1) = (-1)^k$.

(c) Prove the identity

$$G_k^{\alpha,\beta}(z) = 2\alpha(0)\sum_{n>0}\frac{\overline{\beta}(n)}{n^k} + 2\sum_{m>0}\alpha(m)\sum_{n\in\mathbb{Z}}\frac{\overline{\beta}(n)}{(mz+n)^k}.$$

9. Keeping the notation of the previous question, assume in addition that $\alpha(-1)\beta(-1) = (-1)^k$ and that the character β is primitive.

(a) Prove that for all $w \in \mathbb{H}$ we have

$$\sum_{n\in\mathbb{Z}}\frac{\overline{\beta}(n)}{(w+n)^k} = \frac{(-2\pi i)^k\tau(\overline{\beta})}{N^k(k-1)!}\sum_{d=1}^{\infty}\beta(d)d^{k-1}\exp(2\pi i d w/N)$$

(b) Deduce the formula

$$\begin{aligned} G_k^{\alpha,\beta}(z) = &-\alpha(0)\frac{(2\pi i)^k B_k(\beta)}{\tau(\beta)N^{k-1}k!} \\ &+\frac{2(-2\pi i)^k\tau(\overline{\beta})}{N^k(k-1)!}\sum_{n=1}^{\infty}\left(\sum_{d\mid n}\alpha(n/d)\beta(d)d^{k-1}\right)\exp(2\pi i n z/N) \end{aligned}$$

(c) Let $E_k^{\alpha,\beta}(z)$ be the unique scalar multiple of $G_k^{\alpha,\beta}(Nz)$ such that the coefficient of q in the q-expansion of $E_k^{\alpha,\beta}$ equals 1. Prove the identity

$$E_k^{\alpha,\beta}(z) = -\alpha(0)\frac{B_k(\beta)}{2k} + \sum_{n=1}^{\infty}\left(\sum_{d\mid n}\alpha(n/d)\beta(d)d^{k-1}\right)q^n.$$

(d) Prove that $E_k^{\alpha,\beta}(z)$ is a modular form of weight k for $\Gamma_1(MN)$.