Sujets spéciaux en théorie des nombres - formes modulaires/ Special Topics in Number Theory - Modular Forms.
MAT 6684w
Homework 4. Due November 13, 2017
To get full credit solve **3** of the following problems (you are welcome to do them all). The answers may be submitted in English or French.

1. Let N be a positive integer, let p be a prime number, and let

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \quad \Gamma = \Gamma_0(N), \Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha.$$

Determine a system of coset representatives for the quotient $\Gamma' \setminus \Gamma$.

Solution: As seen in class,

$$\alpha^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\alpha = \begin{pmatrix} a & bp \\ c/p & d \end{pmatrix}.$$

Thus

$$\Gamma' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| N \mid c, p \mid b \right\}.$$

Now consider the map $\Gamma \to \mathrm{SL}_2(\mathbb{F}_p)$. When $p \mid N$, the image of the map consists of the matrices of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ and the inverse image of $\left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}$ equals Γ'. Therefore,

$$\Gamma' \setminus \Gamma \cong \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{F}_p^\times \right\} \setminus \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{F}_p^\times, b \in \mathbb{F}_p \right\}$$

The index of the quotient above is $\frac{p(p-1)}{p-1} = p$.
A system of representatives is

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| 0 \leq b \leq p-1 \right\},$$

since they are p elements and they are non-equivalent.

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b-b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

iff $b = b'$.

When $p \nmid N$, the image of the map is $\mathrm{SL}_2(\mathbb{Z})$ (since $\Gamma_1(N) \subset \Gamma_0(N)$ and we proved that the map is surjective for $\Gamma_1(N)$) and the inverse image of the matrices of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ gives Γ'. This implies

$$\Gamma' \setminus \Gamma \cong \left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \middle| a \in \mathbb{F}_p^\times, c \in \mathbb{F}_p \right\} \setminus \mathrm{SL}_2(\mathbb{F}_p)$$

This is the case that was discussed in class. Thus we get

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| 0 \le b \le p-1 \right\} \cup \left\{ \begin{pmatrix} ap & 1 \\ cN & 1 \end{pmatrix} \right\},$$

where a, c are fixed numbers such that $ap - cN = 1$.

2. Prove that for any even integer $k \geq 4$ and prime p we have

$$T_p G_k = \sigma_{k-1}(p) G_k$$

for the Eisenstein series G_k and the Hecke operator T_p on $M_k(\mathrm{SL}_2(\mathbb{Z}))$.

Solution: Since the diamond operator is trivial for $N = 1$, we have

$$\begin{aligned} T_p G_k =& \sum_{n=0}^{\infty} (a_{pn}(G_k) + p^{k-1} a_{n/p}(G_k)) q^n \\ =& -(1 + p^{k-1}) \frac{(2\pi i)^k B_k}{k!} + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} (\sigma_{k-1}(pn) + p^{k-1} \sigma_{k-1}(n/p)) q^n \\ =& -\sigma_{k-1}(p) \frac{(2\pi i)^k B_k}{k!} + 2 \frac{(2\pi i)^k}{(k-1)!} \sigma_{k-1}(p) \sum_{\substack{n \geq 1 \\ p \nmid n}} \sigma_{k-1}(n) q^n \\ &+ 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{\substack{n \geq 1 \\ p \mid n}} (\sigma_{k-1}(pn) + p^{k-1} \sigma_{k-1}(n/p)) q^n \end{aligned}$$

If $p \mid n$, we write $n = p^\ell n_1$ with $p \nmid n_1$, we have

$$\begin{aligned}
\sigma_{k-1}(pn) + p^{k-1}\sigma_{k-1}(n/p) =& \sigma_{k-1}(p^{\ell+1}n_1) + p^{k-1}\sigma_{k-1}(p^{\ell-1}n_1) \\
=& (\sigma_{k-1}(p^{\ell+1}) + p^{k-1}\sigma_{k-1}(p^{\ell-1})\sigma_{k-1}(n_1) \\
=& \left(\frac{p^{(k-1)(\ell+2)}-1}{p^{k-1}-1} + p^{k-1}\frac{p^{(k-1)\ell}-1}{p^{k-1}-1}\right)\sigma_{k-1}(n_1) \\
=& \left(\frac{(p^{(k-1)(\ell+1)}-1)(p^{k-1}+1)}{p^{k-1}-1}\right)\sigma_{k-1}(n_1) \\
=& \sigma_{k-1}(p^\ell)\sigma_{k-1}(p)\sigma_{k-1}(n_1) \\
=& \sigma_{k-1}(p)\sigma_{k-1}(n).
\end{aligned}$$

Combining with the computation above, this proves the result.

3. Let p be a prime and consider the lattice $\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where $\omega_1, \omega_2 \in \mathbb{C}^\times$ and $\omega_1/\omega_2 \notin \mathbb{R}$.

   (a) Show that there are exactly $p^2 + p + 1$ lattices $\Lambda' \subset \mathbb{C}$ satisfying $\Lambda' \supset \Lambda$ and $[\Lambda' : \Lambda] = p^2$, and give a list of these.

   (b) Try to generalize part (a) (e.g. replace $[\Lambda' : \Lambda] = p^2$ by $[\Lambda' : \Lambda] = p^k$ with $k \in \mathbb{Z}_{>0}$).

Solution: (a) Let $\Lambda' = \mathbb{Z}\nu_1 + \mathbb{Z}\nu_2$. Then $\omega_1 = a\nu_1 + b\nu_2$ and $\omega_2 = c\nu_1 + d\nu_2$ with $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = p^2$ Turning to Hermitian normal form gives $\begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix}$ with $a', d' > 0$, and $0 \le c' < d'$. Also $a'd' = p^2$. Thus, we have three possibilities. (1) $a' = p^2, d' = 1, c' = 0$ (2) $a' = p = d', 0 \le c' < p$, and (3) $a' = 1, d' = p^2, 0 \le c' < p^2$. This yields $1+p+p^2$ non-equivalent lattices of the form (1) $\mathbb{Z}\frac{\omega_1}{p^2} + \mathbb{Z}\omega_2$, (2) $\mathbb{Z}\frac{\omega_1}{p} + \mathbb{Z}\frac{c\omega_1 + p\omega_2}{p^2}$, $0 \le c < p$, (3) $\mathbb{Z}\omega_1 + \mathbb{Z}\frac{c\omega_1+\omega_2}{p^2}$, $0 \le c < p^2$.

(b) We consider the same transformation, but now we get that $a' = p^j$, $d' = p^{k-j}$ and $0 \le c' < p^{k-j}$.

We get, $1 + p + \cdots + p^k$ lattices, namely, for $j = 0, \dots, k$, $\mathbb{Z}\frac{\omega_1}{p^j} + \mathbb{Z}\frac{c\omega_1 + p^{k-j}\omega_2}{p^k}$, $0 \le c < p^{k-j}$,

4. Calculate the matrix of the Hecke operator T_2 on the space $S_{24}(\mathrm{SL}_2(\mathbb{Z}))$ with respect to a basis of your choice. Show that the characteristic polynomial of T_2 is $x^2 - 1080x - 20468736$. (You may use a computer, but not a package in which this question can be solved with a one-line command.)

Solution: There are various bases for S_{24}. A natural choice is

$$\begin{aligned} F_1 = E_4^3\Delta &= q + 696q^2 + 162252q^3 + 12831808q^4 + \cdots \\ F_2 = \Delta^2 &= q^2 - 48q^3 + 1080q^4 + \dots \end{aligned}$$

$$\begin{aligned} T_2F_1 =&(a_0(F_1) + 2^{23}a_0(F_1)) + a_2(F_1)q + (a_4(F_1) + 2^{23}a_1(F_1))q^2 + \cdots \\ =&696q + 21220416q^2 + \cdots \\ T_2F_2 =&(a_0(F_2) + 2^{23}a_0(F_2)) + a_2(F_2)q + (a_4(F_2) + 2^{23}a_1(F_2))q^2 + \cdots \\ =&q + 1080q^2 \end{aligned}$$

Then

$$\begin{aligned} T_2F_1 &= 696F_1 + 20736000F_2 \\ T_2F_2 &= F_1 + 384F_2 \end{aligned}$$

Finally,

$$\begin{pmatrix} 696 & 1 \\ 20736000 & 384 \end{pmatrix}.$$

This has the characteristic polynomial of the statement.

5. Consider the formal (so we do not worry about convergence) generating function of the Hecke operators T_n on $M_k(\Gamma_1(N))$

$$g(s) := \sum_{n=1}^{\infty} T_n n^{-s}.$$

Deduce the following formal product expansion (over all primes p):

$$g(s) = \prod_p (1 - T_p p^{-s} + \langle p \rangle p^{k-1-2s})^{-1},$$

where we assume that $\langle p \rangle = 0$ when $p \mid N$.

Solution: Using multiplicativity of T_n and the Fundamental Theorem of Arithmetics, we have

$$g(s) = \prod_p (1 + T_p p^{-s} + T_{p^2} p^{-2s} + \cdots).$$

We look at

$$\begin{aligned}(1 - T_p x + \langle p\rangle p^{k-1}x^2)\sum_{n=0}^{\infty} T_{p^n}x^n =& \sum_{n=0}^{\infty} T_{p^n}x^n - \sum_{n=0}^{\infty} T_p T_{p^n} x^{n+1} + \sum_{n=0}^{\infty} \langle p\rangle p^{k-1} T_{p^n} x^{n+2}\\ =& \sum_{n=0}^{\infty} T_{p^n}x^n - \sum_{n=1}^{\infty} T_p T_{p^{n-1}} x^{n} + \sum_{n=2}^{\infty} \langle p\rangle p^{k-1} T_{p^{n-2}} x^{n}\\ =& 1 + \sum_{n=2}^{\infty}(T_{p^n} - T_p T_{p^{n-1}} + \langle p\rangle p^{k-1} T_{p^{n-2}})x^n\\ =& 1.\end{aligned}$$

6. Let $k, N \in \mathbb{Z}_{>0}$, and let χ be a Dirichlet character modulo N.

   (a) For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, denote by d_γ the lower-right entry of γ. Show that

   $$M_k(N,\chi) = \{f \in M_k(\Gamma_1(N)) : f|_k\gamma = \chi(d_\gamma)f \text{ for all } \gamma \in \Gamma_0(N)\}$$

   and

   $$S_k(N,\chi) = \{f \in S_k(\Gamma_1(N)) : f|_k\gamma = \chi(d_\gamma)f \text{ for all } \gamma \in \Gamma_0(N)\}$$

   (b) Let $\mathbf{1}_N$ denote the trivial character modulo N. Show that

   $$M_k(N,\mathbf{1}_N) = M_k(\Gamma_0(N)) \quad \text{and} \quad S_k(N,\mathbf{1}_N) = S_k(\Gamma_0(N)).$$

Solution: (a) By definition, $M_k(N,\chi)$ is the set of $f \in M_k(\Gamma_1(N))$ such that

$$\langle d\rangle f = \chi(d)f \qquad \forall d \in (\mathbb{Z}/N\mathbb{Z})^\times.$$

But

$$(\langle d\rangle f)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f|_k\gamma_d(z),$$

and this, for the classes in $\Gamma_1(N) \setminus \Gamma_0(N)$.

Since every $\gamma \in \Gamma_0(N)$ corresponds to a $\langle d\rangle$, and every $\langle d\rangle$ gives rise to $\gamma_d \in \Gamma_0(N)$, we get the equality.

The case of cusps is similar and follows from the fact that $\langle d\rangle$ preserves cusps (which is a consequence of the fact that T_α preserves cusps).

(b) In this case we have

$$\langle d\rangle f = f \qquad \forall d \in (\mathbb{Z}/N\mathbb{Z})^\times$$

or

$$f|_k\gamma = f \qquad \forall \gamma \in \Gamma_1(N) \setminus \Gamma_0(N),$$

which is equivalent to saying that $f \in M_k(\Gamma_0(N))$.

The case of cusps is similar and follows from the fact that $\langle d\rangle$ preserves cusps.

7. Let $k \in \mathbb{Z}_{>0}$, let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ be an eigenform, normalized such that $a_1(f) = 1$, and let p be a prime number. Let $\alpha, \beta \in \mathbb{C}$ be the roots of the polynomial $t^2 - a_p(f)t + p^{k-1}$.

You may use without proof that $a_p(f)$ is real.

(a) Prove the formula

$$a_{p^r}(f) = \sum_{j=0}^{r} \alpha^j \beta^{r-j} \qquad \forall r \geq 0.$$

(b) Show that the following conditions are equivalent: (1) $|a_p(f)| \leq 2p^{(k-1)/2}$; (2) α and β are complex conjugates of absolute value $p^{(k-1)/2}$.

(c) Show that if the equivalent conditions of part (b) hold for all prime numbers p, then the q-expansion coefficients of f satisfy the bound

$$|a_n(f)| \leq \sigma_0(n) n^{(k-1)/2} \qquad \forall n \geq 1,$$

where $\sigma_0(n)$ is the number of (positive) divisors of n.

Note: If f is a cusp form, then the conditions of part (b) do hold. This follows from two very deep theorems proved by P. Deligne in 1968 and 1974.

Solution: (a) We have that

$$a_{p^r} = a_p a_{p^{r-1}} - p^{k-1} a_{p^{r-2}}$$

and $a_p = \alpha + \beta$, $p^{k-1} = \alpha\beta$. We proceed by induction. Notice that $a_1 = 1$ and the statement is also true for a_p. Suppose that

$$a_{p^\ell} = \sum_{j=0}^{\ell} \alpha^j \beta^{\ell-j} \qquad \forall 0 \leq \ell < r$$

Now

$$\begin{aligned} a_{p^r} &= a_p a_{p^{r-1}} - p^{k-1} a_{p^{r-2}} \\ &= (\alpha + \beta) \sum_{j=0}^{r-1} \alpha^j \beta^{r-1-j} - \alpha\beta \sum_{j=0}^{r-2} \alpha^j \beta^{r-2-j} \\ &= \sum_{j=0}^{r-1} \alpha^{j+1} \beta^{r-1-j} + \sum_{j=0}^{r-1} \alpha^j \beta^{r-j} - \sum_{j=0}^{r-2} \alpha^{j+1} \beta^{r-1-j} \\ &= \alpha^r + \sum_{j=0}^{r-1} \alpha^j \beta^{r-j} \\ &= \sum_{j=0}^{r} \alpha^j \beta^{r-j}. \end{aligned}$$

(b) (2) $\Rightarrow$ (1):

$$|a_p(f)| = |\alpha + \beta| \le |\alpha| + |\beta| \le 2p^{(k-1)/2}.$$

(1) $\Rightarrow$ (2): The roots of $t^2 - a_p(f)t + p^{k-1}$ are

$$\frac{a_p(f) \pm \sqrt{a_p(f)^2 - 4p^{k-1}}}{2}.$$

If $|a_p(f)| \le 2p^{(k-1)/2}$, then the discriminant is negative (since $a_p(f)$ is real), and the absolute value of each roots is

$$\frac{\sqrt{a_p(f)^2 + 4p^{k-1} - a_p(f)^2}}{2} = p^{(k-1)/2}.$$

(c) By part (a) we have

$$|a_{p^r}(f)| \le \sum_{j=0}^{r} |\alpha^j \beta^{r-j}| = (r+1)p^{r(k-1)/2} = \sigma_0(p^r)(p^r)^{(k-1)/2}.$$

For arbitrary n, let

$$n = \prod_{p \text{prime}} p^{e_p}$$

be the prime factorization of n, where $e_p \ge 0$ and $e_p = 0$ for all but finitely many p. Then

$$|a_n| = \prod_p |a_{p^{e_p}}| \le \prod_p \sigma_0(p^{e_p})(p^{e_p})^{(k-1)/2} = \sigma_0(n)n^{(k-1)/2},$$

since $\sigma_0(n)$ is multiplicative.