Sujets spéciaux en théorie des nombres - formes modulaires/ Special Topics in Number Theory - Modular Forms.
MAT 6684w
Homework 4. Due November 13, 2017
To get full credit solve **3** of the following problems (you are welcome to do them all). The answers may be submitted in English or French.

1. Let N be a positive integer, let p be a prime number, and let
$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \quad \Gamma = \Gamma_0(N), \Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha.$$
Determine a system of coset representatives for the quotient $\Gamma' \setminus \Gamma$.

2. Prove that for any even integer $k \geq 4$ and prime p we have
$$T_p G_k = \sigma_{k-1}(p) G_k$$
for the Eisenstein series G_k and the Hecke operator T_p on $M_k(\mathrm{SL}_2(\mathbb{Z}))$.

3. Let p be a prime and consider the lattice $\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where $\omega_1, \omega_2 \in \mathbb{C}^\times$ and $\omega_1/\omega_2 \notin \mathbb{R}$.
(a) Show that there are exactly $p^2 + p + 1$ lattices $\Lambda' \subset \mathbb{C}$ satisfying $\Lambda' \supset \Lambda$ and $[\Lambda' : \Lambda] = p^2$, and give a list of these.
(b) Try to generalize part (a) (e.g. replace $[\Lambda' : \Lambda] = p^2$ by $[\Lambda' : \Lambda] = p^k$ with $k \in \mathbb{Z}_{>0}$).

4. Calculate the matrix of the Hecke operator T_2 on the space $S_{24}(\mathrm{SL}_2(\mathbb{Z}))$ with respect to a basis of your choice. Show that the characteristic polynomial of T_2 is $x^2 - 1080x - 20468736$. (You may use a computer, but not a package in which this question can be solved with a one-line command.)

5. Consider the formal (so we do not worry about convergence) generating function of the Hecke operators T_n on $M_k(\Gamma_1(N))$
$$g(s) := \sum_{n=1}^{\infty} T_n n^{-s}.$$
Deduce the following formal product expansion (over all primes p):
$$g(s) = \prod_p (1 - T_p p^{-s} + \langle p \rangle p^{k-1-2s})^{-1},$$
where we assume that $\langle p \rangle = 0$ when $p \mid N$.

6. Let $k, N \in \mathbb{Z}_{>0}$, and let χ be a Dirichlet character modulo N.

(a) For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, denote by d_γ the lower-right entry of γ. Show that

$$M_k(N, \chi) = \{f \in M_k(\Gamma_1(N)) : f|_k\gamma = \chi(d_\gamma)f \text{ for all } \gamma \in \Gamma_0(N)\}$$

and

$$S_k(N, \chi) = \{f \in S_k(\Gamma_1(N)) : f|_k\gamma = \chi(d_\gamma)f \text{ for all } \gamma \in \Gamma_0(N)\}$$

(b) Let $\mathbf{1}_N$ denote the trivial character modulo N. Show that

$$M_k(N, \mathbf{1}_N) = M_k(\Gamma_0(N)) \quad \text{and} \quad S_k(N, \mathbf{1}_N) = S_k(\Gamma_0(N)).$$

7. Let $k \in \mathbb{Z}_{>0}$, let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ be an eigenform, normalized such that $a_1(f) = 1$, and let p be a prime number. Let $\alpha, \beta \in \mathbb{C}$ be the roots of the polynomial $t^2 - a_p(f)t + p^{k-1}$.

You may use without proof that $a_p(f)$ is real.

(a) Prove the formula

$$a_{p^r}(f) = \sum_{j=0}^{r} \alpha^j \beta^{r-j} \qquad \forall r \geq 0.$$

(b) Show that the following conditions are equivalent: (1) $|a_p(f)| \leq 2p^{(k-1)/2}$; (2) α and β are complex conjugates of absolute value $p^{(k-1)/2}$.

(c) Show that if the equivalent conditions of part (b) hold for all prime numbers p, then the q-expansion coefficients of f satisfy the bound

$$|a_n(f)| \leq \sigma_0(n) n^{(k-1)/2} \qquad \forall n \geq 1,$$

where $\sigma_0(n)$ is the number of (positive) divisors of n.

Note: If f is a cusp form, then the conditions of part (b) do hold. This follows from two very deep theorems proved by P. Deligne in 1968 and 1974.