Sujets spéciaux en théorie des nombres - formes modulaires/ Special Topics in Number Theory - Modular Forms.
MAT 6684w
Homework 5. Due November 27, 2017
To get full credit solve **4** of the following problems (you are welcome to do them all). The answers may be submitted in English or French.

1. (a) Let $k, N \in \mathbb{Z}_{>0}$, and let $f \in S_k(\Gamma_1(N))$ be a normalized Hecke eigenform with q-expansion $\sum_{n=1}^{\infty} a_n q^n$ (at the cusp ∞) and character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. Prove the identity

$$\overline{a_m} = \chi(m)^{-1} a_m \qquad \forall m \geq 1 \text{ with } (m, N) = 1.$$

Deduce that the quantity $a_m^2/\chi(m)$ is real for all $m \geq 1$ with $(m, N) = 1$.

(b) Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized eigenform, and let p be a prime number. Then $a_p(f)$ is real. (Hint: treat Eisenstein series and cusp forms separately.)

Solution: (a) Since $T_m^\dagger = \langle m \rangle^{-1} T_m$, we have

$$\overline{a_m}\langle f, f\rangle = \langle f, T_m f\rangle = \langle T_m^\dagger f, f\rangle = \langle \langle m\rangle^{-1} T_m f, f\rangle = \chi(m)^{-1} a_m \langle f, f\rangle.$$

Therefore

$$\frac{a_m^2}{\chi(m)} = a_m \overline{a_m} \in \mathbb{R}.$$

(b) For a cusp, part (a) implies that

$$\overline{a_m} = a_m,$$

and therefore, it is real.

For an Eisenstein series, we consider products of $E_j(z)$. Since all the coefficients are real, then the a_p of the products are real.

For a general eigenform, we write $f = \alpha_1 f_1 + \alpha_2 f_2$ with f_1 a cusp and f_2 an Eisenstein series and we choose $a_1(f_i) = 1$. Since f_2 is a normalized eigenform and so is f, we get that f_1 is an eigenform. Then $a_m(f) = \alpha_1 a_m(f_1) + \alpha_2 a_m(f_2)$, $a_m(f_i) \in \mathbb{R}$, $\alpha_1 + \alpha_2 = 1$. If $\alpha_1 = 0$, then $\alpha_2 = 1$ and there is nothing to prove. Now assume that $\alpha_1 \neq 0$. Then $\langle f_1, f\rangle \neq 0$.

Now

$$\overline{a_m(f)}\langle f_1, f\rangle = \langle f_1, T_m f\rangle = \langle T_m^\dagger f_1, f\rangle = a_m(f_1)\langle f_1, f\rangle,$$

from where we deduce that $a_m(f) \in \mathbb{R}$.

2. Let V be the space $S_2(\Gamma_1(16))$ of cusp forms of weight 2 for $\Gamma_1(16)$. You may use the

following fact without proof: a basis for V, expressed in q-expansions at the cusp ∞, is

$$\begin{aligned} f_1 =& q - 2q^3 - 2q^4 + 2q^6 + 2q^7 + 4q^8 - q^9 + O(q^{10}), \\ f_2 =& q^2 - q^3 - 2q^4 + q^5 + 2q^7 + 2q^8 - q^9 + O(q^{10}). \end{aligned}$$

(a) Show that $S_2(\Gamma_1(8)) = \{0\}$ and $V = S_2(\Gamma_1(16))_{new}$. (Hint: consider the map $i_2^{8,16}$ on q-expansions.)

(b) Compute the matrix of the Hecke operator T_2 on V with respect to the basis (f_1, f_2).

(c) Compute a basis (g_1, g_2) of V consisting of eigenforms for T_2.

(Do the computations by hand; you may use a computer to check your results.)

Solution: (a) If there is a nonzero $g \in S_2(\Gamma_1(8))$, then $i_2^{8,16}(g) \in V$. Therefore $g(2z) = c_1 f_1(z) + c_2 f_2(z)$. But $g(2z)$ has a Fourier expansion with only even powers of q and that must be also true for $c_1 f_1(z) + c_2 f_2(z)$. By looking at the coefficient for q, we get $c_1 = 0$. But f_2 does not satify the condition either, so we get a contradiction. Therefore $S_2(\Gamma_1(8)) = \{0\}$.

Now this implies that $i_e^{8,16}(S_2(\Gamma_1(8))) = \{0\}$ for any $e \mid 2$ and $S_2(\Gamma_1(16))_{old} = \{0\}$. Since $S_2(\Gamma_1(16))_{new}$ is its orthogonal complement, we conclude that it must be the whole space, i.e., $V = S_2(\Gamma_1(16))_{new}$

(b) We have $a_n(T_2 f) = a_{2n}(f) + 2a_{n/2}(\langle 2 \rangle f)$.

Thus

$$\begin{aligned} T_2 f_1 =& * q^2 + 2q^3 + O(q^4), \\ T_2 f_2 =& q + *q^2 + O(q^4) \end{aligned}$$

and therefore,

$$T_2 f_1 = -2f_2, \qquad T_2 f_2 = f_1 - 2f_2$$

and the matrix is

$$\begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}.$$

(c) The characteristic polynomial of T_2 is $x^2 + 2x + 2$. We get eigenvalues $-1 \pm i$ and eigenvectors $(1, -1 \pm i)$. Thus we take

$$\begin{aligned} g_1 =& f_1 + (i-1) f_2 \\ =& q - (1-i)q^2 - (1+i)q^3 - 2iq^4 - (1-i)q^5 + 2q^6 + 2iq^7 + (2+2i)q^8 - iq^9 + O(q^{10}) \\ g_2 =& f_1 + (-i-1) f_2 \\ =& q - (1+i)q^2 - (1-i)q^3 + 2iq^4 - (1+i)q^5 + 2q^6 - 2iq^7 + (2-2i)q^8 + iq^9 + O(q^{10}) \end{aligned}$$

3. Let M and e be positive integers, let l be a prime number not dividing M, and let $N = l^e M$. Let f be a Hecke eigenform in $S_k(\Gamma_1(M))$ with character χ. Let V_f be the $\mathbb{C}$-linear subspace of $S_k(\Gamma_1(N))$ spanned by the forms $f_j = i_{l^j}^{M,N}(f)$ for $0 \le j \le e$.

   (a) Prove that the forms $f_0, \dots, f_e$ are $\mathbb{C}$-linearly independent.

   (b) Show that the Hecke operator T_l on $S_k(\Gamma_1(N))$ preserves the subspace V_f, and compute the matrix of T_l on V_f with respect to the basis $(f_0, \dots, f_e)$.

Solution: (a) Let

$$f(z) = \sum_{n=1}^{\infty} a_n q^n.$$

Since $f_j(z) = f(l^j z)$,

$$f_j(z) = \sum_{n=1}^{\infty} a_n q^{l^j n}.$$

Suppose that $c_0 f_0 + \cdots + c_e f_e = 0$. Let j_0 be the minimal index such that $c_{j_0} \neq 0$. Then

$$0 = \sum_{j=j_0}^{e} c_j \sum_{n=1}^{\infty} a_n q^{l^j n} = \sum_{m=1} \sum_{j=j_0, l^j \mid m}^{e} c_j a_{m/l^j} q^m.$$

Now consider the coefficient of $q^{l^{j_0}}$. It is $c_{j_0} a_1 \neq 0$ (because $a_1 \neq 0$ since f is a Hecke eigenform). This gives a contradiction.

(b) We write for $j \ge 1$,

$$i_{l^j}^{M,N}(f) = i_l^{N/l,N} i_{l^{j-1}}^{M,N/l}(f).$$

Thus,

$$T_l(f_j) = T_l(i_l^{N/l,N} i_{l^{j-1}}^{M,N/l}(f)) = i_1^{N/l,N} i_{l^{j-1}}^{M,N/l}(f) = i_{l^{j-1}}^{M,N}(f) = f_{j-1}.$$

For $j = 0$ we have $l \mid N/l, \dots, N/l^{e-1}$ but $l \nmid N/l^e = M$.

$$\begin{aligned}
T_l(f_0) = T_l(i_1^{N/l,N} i_1^{M,N/l}(f)) &= i_1^{N/l,N}(T_l i_1^{M,N/l}(f)) \\
&= i_1^{N/l^2,N}(T_l i_1^{M,N/l^2}(f)) \\
&= \cdots \\
&= i_1^{N/l^{e-1},N}(T_l i_1^{M,N/l^{e-1}}(f)) \\
&= i_1^{N/l^e,N}(T_l i_1^{M,N/l^e}(f)) - l^{k-1} i_l^{N/l^e,N}(\langle l \rangle i_1^{M,N/l^e}(f)) \\
&= i_1^{M,N}(T_l f) - l^{k-1} i_l^{M,N}(\langle l \rangle f) \\
&= a_l i_1^{M,N}(f) - \chi(l) l^{k-1} i_l^{M,N}(f) \\
&= a_l f_0 - \chi(l) l^{k-1} f_1
\end{aligned}$$

Therefore T_l preserves V_f and the matrix for the base $(f_0, \dots, f_e)$ is

$$\begin{pmatrix} a_l & 1 & 0 & 0 & \cdots & 0 \\ -\chi(l)l^{k-1} & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

4. Let N be odd. Suppose that $S_k(\Gamma_0(N))$ contains some normalized eigenform f. Write $g = f^2 \in S_{2k}(\Gamma_0(N))$. Calculate the first two terms of the q-expansions of g and $T_2 g$, and deduce that the dimension of $S_{2k}(\Gamma_0(N))$ is at least 2.

Solution: Write $f = q + a_2q^2 + a_3q^3 + a_4q^4 + O(q^5)$. Then $g = q^2 + 2a_2q^3 + (a_2^2 + 2a_3)q^4 + O(q^5)$ and

We get

$$\begin{aligned} T_2g(z) =& \frac{1}{2}\left(g(z/2) + g((z+1)/2)\right) + 2^{k-1}(\langle 2\rangle g)(2z) \\ =& q + (a_2^2 + 2a_3)q^2 + O(q^3) + 2^{k-1}(\langle 2\rangle g)(2z) \end{aligned}$$

For $\langle 2\rangle$ take $\alpha = \begin{pmatrix} (N+1)/2 & 1 \\ N & 2 \end{pmatrix}$.

$$(\langle 2\rangle g)(z) = T_\alpha g(z) = g(z)$$

since $g(z) \in S_{2k}(\Gamma_0(N))$. Thus,

$$\begin{aligned} T_2g(z) =& q + (a_2^2 + 2a_3)q^2 + O(q^3) + 2^{k-1}g(2z) \\ =& q + (a_2^2 + 2a_3)q^2 + O(q^3) \end{aligned}$$

The dimension is at least 2 because g has coefficient 0 for q but T_2g has nonzero coefficient for q.

5. Let Γ be a congruence subgroup, and let f be a modular form of weight k for Γ. Define a function $f^* : \mathbb{H} \to \mathbb{C}$ by

$$f^*(z) = \overline{f(-\overline{z})}.$$

(a) Prove that f^* is a modular form of weight k for the group $\sigma^{-1}\Gamma\sigma$, where $\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) Suppose (for simplicity) that both Γ and $\sigma^{-1}\Gamma\sigma$ contain the subgroup $\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{Z} \right\}$. Show that the standard q-expansions at ∞ of f and f^* in the variable $q = \exp(2\pi i z)$ are related by

$$a_n(f^*) = \overline{a_n(f)} \qquad \forall n \geq 0.$$

(c) Show that if $\Gamma = \Gamma_0(N)$ or $\Gamma = \Gamma_1(N)$ for some $N \geq 1$, then $\sigma^{-1}\Gamma\sigma = \Gamma$.

Solution: (a) Let $\gamma \in \Gamma$. Then

$$\begin{aligned} f^*|_k\sigma^{-1}\gamma\sigma =& (-cz+d)^{-k} f^*\left(\frac{az-b}{-cz+d}\right) \\ =& \overline{(-cz+d)^{-k} f\left(-\overline{\frac{az-b}{-cz+d}}\right)} \\ =& \overline{(-c\overline{z}+d)^{-k} f\left(-\frac{a\overline{z}-b}{-c\overline{z}+d}\right)} \\ =& \overline{f|_k\gamma(-\overline{z})} \\ =& \overline{f(-\overline{z})} \\ =& f^*(z). \end{aligned}$$

The holomorphicity does not change with this construction, so we get a modular form.

(b) The conditions imply that $h = 1$ and the q-series is a power series on $e^{2\pi i z}$. Since

$$\overline{e^{-2\pi i \overline{z}}} = e^{2\pi i z},$$

we get the result.

(c) We have that

$$\sigma^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sigma = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

the result is obvious since $N \mid c$ iff $N \mid -c$ and the congruences of a, d modulo N does not change.

6. Let g_1 and g_2 be the eigenforms for the operator T_2 on $S_2(\Gamma_1(16))$ found in Problem 2 of this list.

(a) Prove that g_1 and g_2 are in fact eigenforms for the full Hecke algebra $\mathbb{T}(S_2(\Gamma_1(16)))$. (Hint: first show that $S_2(\Gamma_1(16))$ admits a basis of eigenforms for the full Hecke algebra.)

(b) Compute the eigenvalues of the diamond operator $\langle 3\rangle$ on g_1 and g_2. (Hint: use T_3 and T_9.)

(c) Prove that the characters of g_1 and g_2 are given by

$$\langle d\rangle g_j = \chi_j(d) g_j \qquad \forall d \in (\mathbb{Z}/16\mathbb{Z})^\times, \quad j = 1, 2,$$

where χ_1, χ_2 are the two group homomorphisms $(\mathbb{Z}/16\mathbb{Z})^\times \to \mathbb{C}^\times$ with kernel $\{\pm 1\}$. (Do the computations by hand; you may use a computer to check your results.)

Solution: (a) The space $S_k(\Gamma_1(N))$ admits a basis consisting of simultaneous eigenvectors for T_m for $(m, N) = 1$ and $\langle d\rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. But such forms, when they are in $S_k(\Gamma_1(N))_{new}$, they are Hecke eigenforms (not restrictions on $(m, N) = 1$). Since we saw that $S_2(\Gamma_1(16)) = S_2(\Gamma_1(16))_{new}$, we conclude that there is a basis of eigenforms $\{h_1, h_2\}$ for the full Hecke algebra. Now, looking at T_2, since $\{h_1, h_2\}$ and $\{g_1, g_2\}$ are basis of eigenvectors for T_2, we conclude that $g_1 = \alpha_1 h_1$ and $g_2 = \alpha_2 h_2$ or viceversa.

(b) We have that $T_9 = T_3^2 - 3\langle 3\rangle T_1$. In addition, we know that $T_n g = \frac{a_n(g)}{a_1(g)} g$. In particular, we have $T_3 g_1 = -(1+i)g_1$, $T_3 g_2 = -(1-i)g_2$, $T_9 g_1 = -ig_1$ and $T_9 g_2 = ig_2$. Thus

$$\langle 3\rangle g_1 = \frac{1}{3}(T_3^2 - T_9)g_1 = \frac{(1+i)^2 + i}{3} g_1 = ig_1$$

and similarly

$$\langle 3\rangle g_2 = -ig_2.$$

(c) $(\mathbb{Z}/16\mathbb{Z})^\times \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle 3, -1\rangle$. It suffices to check things with $\langle -1\rangle$. However, since $\langle 1\rangle = \langle -1\rangle^2$, we must have that $\langle -1\rangle g_i = \pm g_i$. We can take the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ to define $\langle -1\rangle$. This gives $((\langle -1\rangle g_i)(z) = (-1)^k g_i(z)$, and since $k = 2$, we get the identity and $\{\pm 1\}$ is in the kernel.

7. For $f \in S_k(\Gamma_1(N))$, let $f^* \in S_k(\Gamma_1(N))$ be the form defined by $f^*(z) = \overline{f(-\overline{z})}$ (see Problem 5).

(a) Show that the map $S_k(\Gamma_1(N)) \to S_k(\Gamma_1(N))$ sending f to f^* preserves the subspaces $S_k(\Gamma_1(N))_{old}$ and $S_k(\Gamma_1(N))_{new}$.

(b) Let $f \in S_k(\Gamma_1(N))_{new}$ be a primitive form. Show that the form f^*, which by part (a) is in $S_k(\Gamma_1(N))_{new}$, is also a primitive form, and determine the eigenvalues of the operators $\langle d\rangle$ (for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$) and T_m (for $m \geq 1$) on f^*.

Solution: (a) We have $f \in S_k(\Gamma_1(N))_{old}$ implies $f(z) = g(ez)$ for some $g \in S_k(\Gamma_1(M))$ and $e \mid N/M$. But then

$$f^*(z) = \overline{f(-\overline{z})} = \overline{g(-e\overline{z})} = \overline{g(-\overline{ez})} = g^*(ez)$$

thus, $f^* \in S_k(\Gamma_1(N))_{old}$. The new case follows from $f^{**} = f$.

(b) We have that $(T_n f)(z) = a_n(f)f(z)$. Therefore, $(T_n f^*)(z) = \overline{(T_n f)(-\overline{z})} = \overline{a_n(f)}f^*(z)$.

Similarly $(\langle d\rangle f)(z) = \chi(d)f(z)$ and we get $(\langle d\rangle f^*)(z) = \overline{(\langle d\rangle f)(-\overline{z})} = \overline{\chi(d)}f^*(z)$.

We get f is normalized iff f^* is normalized, and by (a), they are primitive at the same time.

8. Recall that the Fricke (or Atkin–Lehner) operator w_N on $S_k(\Gamma_1(N))$ is the operator T_{α_N} with $\alpha_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$.

   (a) Show that $w_N^2 = (-N)^k \cdot id$ and that the adjoint of w_N equals $(-1)^k w_N$.

   (b) Show that for every $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, the diamond operator $\langle d\rangle$ on $S_k(\Gamma_1(N))$ satisfies $w_N^{-1}\langle d\rangle w_N = \langle d\rangle^{-1}$.

   (c) Show that for every positive integer m such that $(m, N) = 1$, the Hecke operator T_m satisfies $w_N^{-1}T_m w_N = \langle m\rangle^{-1}T_m$.

Solution: (a) First notice that

$$\alpha_N^2 = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & -N \\ 0 & -N \end{pmatrix}.$$

Therefore,

$$(w_N^2 f)(z) = (f|_k\alpha_N^2)(z) = \frac{(N^2)^k}{(-N)^k} f(\alpha_N^2 z) = (-N)^k f(z)$$

We now that $(T_\alpha)^\dagger = T_{\alpha^*}$. Thus $w_N^\dagger = T_{\alpha_N^*}$. And $\alpha_N^* = \begin{pmatrix} 0 & -1 \\ -N & 0 \end{pmatrix}$. Thus, we have $w_N^\dagger = (-1)^k w_N$.

(b) We have

$$\begin{aligned}(w_N^{-1}\langle d\rangle w_N f)(z) &= (f|_k\alpha_N \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \alpha_N^{-1})(z) \\ &= (f|_k \begin{pmatrix} d & -c \\ -bN & a \end{pmatrix})(z) \\ &= (\langle a\rangle f)(z)\end{aligned}$$

and this proves the result, since $ad \equiv 1 \bmod N$.

(c) If $m = p \nmid N$, we have

$$\begin{aligned}(w_N^{-1}T_pw_Nf)(z) =&\frac{1}{p}(f|_k\alpha_N\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\alpha_N^{-1})(z)\\ =&\frac{1}{p}(f|_k\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right))(z)\\ =&(T_p^\dagger f)(z)\\ =&(\langle p\rangle^{-1}T_pf)(z).\end{aligned}$$

When $m = p^r$ use induction and point (b). Assume true for $r-1, r-2$,

$$\begin{aligned}w_N^{-1}T_{p^r}w_N =&w_N^{-1}T_pw_Nw_N^{-1}T_{p^r-1}w_N - p^{k-1}w_N^{-1}\langle p\rangle w_Nw_N^{-1}T_{p^{r-2}}w_N\\ =&\langle p\rangle^{-1}T_p\langle p^{r-1}\rangle^{-1}T_{p^r-1} - p^{k-1}\langle p\rangle^{-1}\langle p^{r-2}\rangle^{-1}T_{p^{r-2}}\\ =&\langle p^r\rangle^{-1}(T_pT_{p^r-1} - p^{k-1}\langle p\rangle T_{p^{r-2}})\\ =&\langle p^r\rangle^{-1}T_{p^r}.\end{aligned}$$

Finally, use that when $(m,n) = 1$, we have that $T_{mn} = T_mT_n$.

9. Let w_N be the Fricke operator on $S_k(\Gamma_1(N))$; recall that this preserves the new subspace $S_k(\Gamma_1(N))_{new}$. Let $f \in S_k(\Gamma_1(N))_{new}$ be a primitive form.

(a) Show that the form w_Nf is an eigenform for the operators $\langle d\rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ and T_m for $m \geq 1$ with $(m,N) = 1$, and determine the eigenvalues of these operators on w_Nf.

(b) Deduce that $w_Nf = \eta_f f^*$ for some $\eta_f \in \mathbb{C}$, with f^* as in Problem 7.

(Hint: use Problem 1.)

(c) Prove the identities $\eta_f\eta_{f^*} = (-N)^k$, $\eta_{f^*} = (-1)^k\overline{\eta_f}$ and $|\eta_f| = N^{k/2}$. (Hint: consider $\langle w_Nf, f^*\rangle_{\Gamma_1(N)}$.)

(The complex number η_f is called the Atkin–Lehner pseudo-eigenvalue of f.)

Solution: (a) By the previous problem,

$$\langle d\rangle w_Nf = w_N\langle d\rangle^{-1}f = w_N\chi(d)^{-1}f = \chi(d)^{-1}w_Nf.$$

$$T_mw_Nf = w_N\langle m\rangle^{-1}T_mf = w_N\langle m\rangle^{-1}a_mf = a_m\chi(m)^{-1}w_Nf = \overline{a_m}w_Nf$$

where we used problem 1 from 10 in the last equality.

(b) Since $T_mf = \frac{a_m(f)}{a_1(f)}f$ and we have $T_mw_Nf = \overline{a_m}w_Nf$, we deduce that

$$a_m(w_Nf) = \eta_f\overline{a_m(f)}$$

for some $\eta_f \in \mathbb{C}$.

Therefore, by problem 3, $w_N f = \eta_f f^*$.

(c) We have $w_N^2 f = w_N \eta_f f^* = \eta_f \eta_{f^*} f$. On the other hand, $w_N^2 = (-N)^k$. Thus we get

$$\eta_f \eta_{f^*} = (-N)^k.$$

We have that $\alpha_N^* = -\alpha_N$. Therefore,

$$(-1)^k \overline{\eta_{f^*}} \langle f, f \rangle = \langle f, w_N^* f^* \rangle = \langle w_N f, f^* \rangle_{\Gamma_1(N)} = \eta_f \langle f^*, f^* \rangle_{\Gamma_1(N)}$$

Since $\langle f, f \rangle = \langle f^*, f^* \rangle$ by construction, we get

$$\eta_{f^*} = (-1)^k \overline{\eta_f}.$$

Finally,

$$(-N)^k = \eta_f \eta_{f^*} = \eta_f (-1)^k \overline{\eta_f} = (-1)^k |\eta_f|^2$$

and we deduce

$$|\eta_f| = N^k.$$