Sujets spéciaux en théorie des nombres - formes modulaires/ Special Topics in Number Theory - Modular Forms.

MAT 6684w

Homework 5. Due November 27, 2017

To get full credit solve **4** of the following problems (you are welcome to do them all). The answers may be submitted in English or French.

1. (a) Let $k, N \in \mathbb{Z}_{>0}$, and let $f \in S_k(\Gamma_1(N))$ be a normalized Hecke eigenform with q-expansion $\sum_{n=1}^{\infty} a_n q^n$ (at the cusp ∞) and character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. Prove the identity
$$\overline{a_m} = \chi(m)^{-1} a_m \qquad \forall m \geq 1 \text{ with } (m, N) = 1.$$
Deduce that the quantity $a_m^2/\chi(m)$ is real for all $m \geq 1$ with $(m, N) = 1$.

   (b) Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized eigenform, and let p be a prime number. Then $a_p(f)$ is real. (Hint: treat Eisenstein series and cusp forms separately.)

2. Let V be the space $S_2(\Gamma_1(16))$ of cusp forms of weight 2 for $\Gamma_1(16)$. You may use the following fact without proof: a basis for V, expressed in q-expansions at the cusp ∞, is
$$\begin{aligned} f_1 =& q - 2q^3 - 2q^4 + 2q^6 + 2q^7 + 4q^8 - q^9 + O(q^{10}), \\ f_2 =& q^2 - q^3 - 2q^4 + q^5 + 2q^7 + 2q^8 - q^9 + O(q^{10}). \end{aligned}$$

   (a) Show that $S_2(\Gamma_1(8)) = \{0\}$ and $V = S_2(\Gamma_1(16))_{new}$. (Hint: consider the map $i_2^{8,16}$ on q-expansions.)

   (b) Compute the matrix of the Hecke operator T_2 on V with respect to the basis (f_1, f_2).

   (c) Compute a basis (g_1, g_2) of V consisting of eigenforms for T_2.

   (Do the computations by hand; you may use a computer to check your results.)

3. Let M and e be positive integers, let l be a prime number not dividing M, and let $N = l^e M$. Let f be a Hecke eigenform in $S_k(\Gamma_1(M))$ with character χ. Let V_f be the $\mathbb{C}$-linear subspace of $S_k(\Gamma_1(N))$ spanned by the forms $f_j = i_{l^j}^{M,N}(f)$ for $0 \leq j \leq e$.

   (a) Prove that the forms $f_0, \ldots, f_e$ are $\mathbb{C}$-linearly independent.

   (b) Show that the Hecke operator T_l on $S_k(\Gamma_1(N))$ preserves the subspace V_f, and compute the matrix of T_l on V_f with respect to the basis $(f_0, \ldots, f_e)$.

4. Let N be odd. Suppose that $S_k(\Gamma_0(N))$ contains some normalized eigenform f. Write $g = f^2 \in S_{2k}(\Gamma_0(N))$. Calculate the first two terms of the q-expansions of g and $T_2 g$, and deduce that the dimension of $S_{2k}(\Gamma_0(N))$ is at least 2.

5. Let Γ be a congruence subgroup, and let f be a modular form of weight k for Γ. Define a function $f^* : \mathbb{H} \to \mathbb{C}$ by
$$f^*(z) = \overline{f(-\overline{z})}.$$

(a) Prove that f^* is a modular form of weight k for the group $\sigma^{-1}\Gamma\sigma$, where $\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) Suppose (for simplicity) that both Γ and $\sigma^{-1}\Gamma\sigma$ contain the subgroup $\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{Z} \right\}$. Show that the standard q-expansions at ∞ of f and f^* in the variable $q = \exp(2\pi i z)$ are related by

$$a_n(f^*) = \overline{a_n(f)} \qquad \forall n \geq 0.$$

(c) Show that if $\Gamma = \Gamma_0(N)$ or $\Gamma = \Gamma_1(N)$ for some $N \geq 1$, then $\sigma^{-1}\Gamma\sigma = \Gamma$.

6. Let g_1 and g_2 be the eigenforms for the operator T_2 on $S_2(\Gamma_1(16))$ found in Problem 2 of this list.

   (a) Prove that g_1 and g_2 are in fact eigenforms for the full Hecke algebra $\mathbb{T}(S_2(\Gamma_1(16)))$. (Hint: first show that $S_2(\Gamma_1(16))$ admits a basis of eigenforms for the full Hecke algebra.)

   (b) Compute the eigenvalues of the diamond operator $\langle 3 \rangle$ on g_1 and g_2. (Hint: use T_3 and T_9.)

   (c) Prove that the characters of g_1 and g_2 are given by

$$\langle d \rangle g_j = \chi_j(d) g_j \qquad \forall d \in (\mathbb{Z}/16\mathbb{Z})^\times, \quad j = 1, 2,$$

   where χ_1, χ_2 are the two group homomorphisms $(\mathbb{Z}/16\mathbb{Z})^\times \to \mathbb{C}^\times$ with kernel $\{\pm 1\}$. (Do the computations by hand; you may use a computer to check your results.)

7. For $f \in S_k(\Gamma_1(N))$, let $f^* \in S_k(\Gamma_1(N))$ be the form defined by $f^*(z) = \overline{f(-\overline{z})}$ (see Problem 5).

   (a) Show that the map $S_k(\Gamma_1(N)) \to S_k(\Gamma_1(N))$ sending f to f^* preserves the subspaces $S_k(\Gamma_1(N))_{old}$ and $S_k(\Gamma_1(N))_{new}$.

   (b) Let $f \in S_k(\Gamma_1(N))_{new}$ be a primitive form. Show that the form f^*, which by part (a) is in $S_k(\Gamma_1(N))_{new}$, is also a primitive form, and determine the eigenvalues of the operators $\langle d \rangle$ (for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$) and T_m (for $m \geq 1$) on f^*.

8. Recall that the Fricke (or Atkin–Lehner) operator w_N on $S_k(\Gamma_1(N))$ is the operator T_{α_N} with $\alpha_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$.

   (a) Show that $w_N^2 = (-N)^k \cdot id$ and that the adjoint of w_N equals $(-1)^k w_N$.

   (b) Show that for every $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, the diamond operator $\langle d \rangle$ on $S_k(\Gamma_1(N))$ satisfies $w_N^{-1} \langle d \rangle w_N = \langle d \rangle^{-1}$.

   (c) Show that for every positive integer m such that $(m, N) = 1$, the Hecke operator T_m satisfies $w_N^{-1} T_m w_N = \langle m \rangle^{-1} T_m$.

9. Let w_N be the Fricke operator on $S_k(\Gamma_1(N))$; recall that this preserves the new subspace $S_k(\Gamma_1(N))_{new}$. Let $f \in S_k(\Gamma_1(N))_{new}$ be a primitive form.

(a) Show that the form $w_N f$ is an eigenform for the operators $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ and T_m for $m \geq 1$ with $(m, N) = 1$, and determine the eigenvalues of these operators on $w_N f$.

(b) Deduce that $w_N f = \eta_f f^*$ for some $\eta_f \in \mathbb{C}$, with f^* as in Problem 7.

(Hint: use Problem 1.)

(c) Prove the identities $\eta_f \eta_{f^*} = (-N)^k$, $\eta_{f^*} = (-1)^k \overline{\eta_f}$ and $|\eta_f| = N^{k/2}$. (Hint: consider $\langle w_N f, f^* \rangle_{\Gamma_1(N)}$.)

(The complex number η_f is called the Atkin–Lehner pseudo-eigenvalue of f.)