1. (a) Let $R$ be a commutative ring with multiplicative identity $1 \neq 0$. Define units and zero divisors of $R$.
   (b) Find all units in $\overline{R} = \mathbb{Z}/(18)$.
   (c) Compute $15^{-1}$ in $\mathbb{Z}_{7564}$.

2. Let $f(x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1$ in $\mathbb{Z}_7[x]$. Find the greatest common divisor $d(x) = (f(x), g(x))$, as well as polynomials $h(x)$ and $k(x)$ such that
   \[ d(x) = k(x)f(x) + h(x)g(x). \]

3. Factorize the following polynomials into products of irreducibles:
   (a) $f(x) = 2x^5 + 5x^4 + 4x^3 + 7x^2 + 7x + 2 \in \mathbb{Q}[x]$;
   (b) $f(x) = x^3 + x^2 + x + 1 \in \mathbb{Z}_2[x]$;
   (c) $f(x) = x^{11} + 1 \in \mathbb{Z}_{11}[x]$.

4. (a) Define an ideal of a commutative ring with multiplicative identity.
   (b) Prove: Let $R$ be a commutative ring with multiplicative identity $1 \neq 0$, $A$ is an ideal of $R$ and "$\sim$" is a relation on $R$ defined by
      \[ x, y \in R, \ x \sim y \iff x - y \in A. \]
      Then:
      (i) "$\sim$" is an equivalence relation.
      (ii) $x \sim y, \ u \sim v \Rightarrow x + u \sim y + v, \ xu \sim yv$.

5. (a) Let $R$ be an integral domain. Define a prime element in $R$; define an irreducible element in $R$.
   (b) Determine which of the polynomials below are irreducible over $\mathbb{Q}$:
      i. $f(x) = 5x^5 + 9x^4 + 15x^3 + 3x^2 + 6x + 3$
      ii. $f(x) = x^4 + 15x^3 + 7$. Justify your answer.
   (c) Prove the following result: If $F$ is a field and $f(x) \in F[x]$, then $f(x)$ is irreducible if and only if $f(x)$ is a prime.

6. Assume $R = \mathbb{Z}_3$, $\overline{R} = \mathbb{Z}_{12}$ and $\phi : R \to \overline{R}$ is defined by $\phi(x) = 4x$, for all $x \in R$ (bars are removed).
   (a) Show that $\phi$ is well defined.
   (b) Verify that $\phi$ is a ring homomorphism.
   (c) Find Ker($\phi$).
   (d) Is $\phi(R^*) \subseteq \overline{R}^*$? Justify your answer.
7. Let $F$ be a field and let

$$A = \{ f(x) \in F[x] \mid f(1) = 0 \} \subseteq F[x].$$

(a) Show that $A \neq \emptyset$

(b) Show that $A$ is an ideal of $F[x]$

(c) Show that $A = (h(x)) = h(x)F[x]$ where $h(x) = x - 1$ and $(h(x))$ denotes the principal ideal generated by $h(x)$.

8. Consider the polynomial ring $R = \mathbb{Z}_2[x]$.

(a) List all polynomials of degree 3 in $\mathbb{Z}_2[x]$.

(b) Find all maximal ideals of $R = \mathbb{Z}_2[x]$ generated by polynomials of degree 3.

(c) Explain how to construct a field of 25 elements.