Q1. [10 pts]. (1.1) Find the greatest common divisor of \( a = 726 \) and \( b = 275 \).

(1.2) Write \( \text{gcd}(726, 275) \) as an integer combination of \( a \) and \( b \).

(1.3) Let \( A \) be the set consisting of all integer combination of \( a \) and \( b \):

\[
A = \{ua + vb : \text{for all integers } u, v\}.
\]

Prove that \( A \) is an ideal of \( I \).

(1.4) Is \( A \) a principal ideal of \( I \)? If so, find a positive integer \( n \) such that \( A = (n) \).

Q2. [10 pts]. Do only one of (2.1) or (2.2):

(2.1) Let \( a, b, c \) be non-zero integers, and let \( (b, c) = d \) and \( (ab, c) = e \). Assume that \( (a, c) = 1 \). Prove that \( d = e \).

(2.2) Let \( R \) be an integral domain, and let \( p \) be a prime element of \( R \). Prove that \( p \) is irreducible in \( R \).

Q3. [10 pts]. Prove, by induction on \( n \), that

\[
n^2 + (n + 1)^2 + \ldots + (2n)^2 = \frac{n(n + 1)(14n + 1)}{6}
\]

for all \( n \geq 1 \). (The left-hand side is the sum of the squares of all integers from \( n \) to \( 2n \), inclusive, i.e., \( \sum_{i=n}^{2n} i^2 \).)
Q4. [20 pts]. Let \( I \) be the ring of integers, and let \( R = I/(3) \) be the quotient ring.

(4.1) Write down the distinct elements of \( R \). (You do not have to use brackets around the elements of \( R \) if this does not confuse you.)

(4.2) Calculate \([2] + [2]\) and \([2]^2\) in \( R \).

(4.3) Is \( R \) an integral domain? Why or why not. (Do not simply repeat the definition of an integral domain.)

Next, consider the polynomial ring \( R[x] \), the polynomial \( f(x) = x^2 + 1 \), the quotient ring \( S = R[x]/(f) \), and \( \alpha = [x] \).

(4.4) Show that no element of \( R \) is a root of \( f \).

(4.5) Prove that \( f \) is an irreducible polynomial in \( R[x] \).

(4.6) Prove that \( \alpha^2 = [2] \) in the quotient ring \( S \).

(4.7) How many elements does \( S \) have?

(4.8) Show that \( \alpha^4 = [1] \).

(4.9) Put \( \beta = [1] + \alpha \). Show that \( \beta^2 = [2] \alpha \).

(4.10) Prove that \( \beta^8 = [1] \).

(4.11) Is every non-zero element of \( S \) a unit? Give a reason for your answer.