Mahler measure and evaluation of regulators

Matilde N. Lalín

Mathematical Sciences Research Institute
mlalin@msri.org
http://www.math.ubc.ca/~mlalin

May 10th, 2006

Mahler measure of polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Given

$$P(x) = a_d \prod_{n=1}^{d} (x - \alpha_n) \in \mathbb{C}[x]$$

$$m(P) = \log|a_d| + \sum_{n=1}^{d} \log^+ |\alpha_n|$$

Jensen's formula:

$$\int_0^1 \log|\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|$$

recovers one-variable case.

Properties

- $m(P) \geq 0$ if P has integral coefficients.
- $m(P \cdot Q) = m(P) + m(Q)$
- α algebraic number, and P_α minimal polynomial over $\mathbb{Q}$,

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$$

where h is the logarithmic Weil height.

Kronecker's Lemma
$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

Lehmer's question

Lehmer (1933)

$$m(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1)$$

$= \log(1.176280818\dots) = 0.162357612\dots$

Does there exist $C > 0$, for all $P(x) \in \mathbb{Z}[x]$

$m(P) = 0$ or $m(P) > C$??

Is the above polynomial the best possible?

Lehmer's question

Lehmer (1933)

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1)$$

$= \log(1.176280818\dots) = 0.162357612\dots$

$$\text{Does there exist} \quad C > 0, \quad \text{for all} \quad P(x) \in \mathbb{Z}[x]$$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

Is the above polynomial the best possible?

Boyd & Lawton

$P \in \mathbb{C}[x_1, \dots, x_n]$

$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) = m(P(x_1, x_2, \dots, x_n))$$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

Examples in several variables

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1)$$

- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right) \stackrel{?}{=} \frac{\mathrm{L}'(E_k,0)}{B_k} \qquad k\in\mathbb{N}, \quad k\neq 4$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\right) = 2\mathrm{L}'(\chi_{-4},-1)$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right) = \mathrm{L}'(A,0)$$

$$A: Y^2 = X^3 - 44X + 112$$

Examples in three variables

- Condon (2003):

$$\pi^2 m\left(z - \left(\frac{1-x}{1+x}\right)(1+y)\right) = \frac{28}{5}\zeta(3)$$

- D'Andrea & L. (2003):

$$\pi^2 m\left(z(1-xy)^2 - (1-x)(1-y)\right) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)}$$

- Boyd & L. (2005):

$$\pi^2 m(x^2 + 1 + (x+1)y + (x-1)z) = \pi \mathrm{L}(\chi_{-4}, 2) + \frac{21}{8}\zeta(3)$$

Examples with more than three variables

L.(2003):

- $$\pi^3 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right) = 24\mathrm{L}(\chi_{-4},4)$$
- $$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right) = 62\zeta(5)+\frac{14}{3}\pi^2\zeta(3)$$
- $$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right) = 93\zeta(5)$$

Known formulas for n.

Polylogarithms

The *kth polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Zagier:

$$\mathcal{L}_k(x) := \mathrm{Re}_k \left(\sum_{j=0}^{k-1} \frac{2^j B_j}{j!} (\log |x|)^j \mathrm{Li}_{k-j}(x) \right)$$

B_j is jth Bernoulli number

$\mathrm{Re}_k = \mathrm{Re}$ or Im if k is odd or even.

One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

Polylogarithms

The *kth polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.
Zagier:

$$\mathcal{L}_k(x) := \mathrm{Re}_k \left(\sum_{j=0}^{k-1} \frac{2^j B_j}{j!} (\log|x|)^j \mathrm{Li}_{k-j}(x) \right)$$

B_j is jth Bernoulli number
$\mathrm{Re}_k = \mathrm{Re}$ or Im if k is odd or even.
One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

$\mathcal{L}_k$ satisfies lots of functional equations

$$\mathcal{L}_k\left(\frac{1}{x}\right) = (-1)^{k-1}\mathcal{L}_k(x) \qquad \mathcal{L}_k(\bar{x}) = (-1)^{k-1}\mathcal{L}_k(x)$$

Bloch–Wigner dilogarithm ($k=2$)

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

Five-term relation

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0$$

Philosophy of Beilinson's conjectures

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F a number field)
- L-function ($\mathrm{L}_F = \zeta_F$)
- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)
- Regulator map $\mathrm{reg} : K \to \mathbb{R}$ ($\mathrm{reg} = \log|\cdot|$)

$$(K \text{ rank } 1) \qquad \mathrm{L}'_X(0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

(Dirichlet class number formula, for F real quadratic, $\zeta'_F(0) \sim_{\mathbb{Q}^*} \log|\epsilon|$, $\epsilon \in \mathcal{O}_F^*$)

An algebraic integration for Mahler measure

Deninger (1997): General framework

Rodriguez-Villegas (1997) : $P(x,y) \in \mathbb{C}[x,y]$

$$m(P) = m(P^*) - \frac{1}{2\pi}\int_\gamma \eta(x,y)$$

$$\eta(x,y) = \log|x|\,\mathrm{d}\arg y - \log|y|\,\mathrm{d}\arg x$$

$$\eta(x,1-x) = \mathrm{d}D(x) \qquad \mathrm{d}\eta(x,y) = \mathrm{Im}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}\right)$$

An algebraic integration for Mahler measure

Deninger (1997): General framework

Rodriguez-Villegas (1997) : $P(x, y) \in \mathbb{C}[x, y]$

$$m(P) = m(P^*) - \frac{1}{2\pi} \int_\gamma \eta(x, y)$$

$$\eta(x, y) = \log|x| \mathrm{d} \arg y - \log|y| \mathrm{d} \arg x$$

$$\eta(x, 1-x) = \mathrm{d} D(x) \qquad \mathrm{d}\eta(x, y) = \mathrm{Im}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}\right)$$

The three-variable case

$$P(x,y,z) = (1-x)-(1-y)z \quad X = \{P(x,y,z)=0\}$$

$$m(P) = m(1-y) + \frac{1}{(2\pi\mathrm{i})^3}\int_{\mathbb{T}^3} \log\left|z - \frac{1-x}{1-y}\right| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathbb{T}^2} \log^+\left|\frac{1-x}{1-y}\right| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma} \log|z| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$\Gamma = X \cap \{|x|=|y|=1, |z|\geq 1\}$$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma} \eta(x,y,z)$$

The three-variable case

$$P(x,y,z) = (1-x) - (1-y)z \quad X = \{P(x,y,z) = 0\}$$

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} \frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \log |z| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x,y,z)$$

The three-variable case

$$P(x,y,z) = (1-x) - (1-y)z \quad X = \{P(x,y,z) = 0\}$$

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} \frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \log|z| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x,y,z)$$

The three-variable case

$$P(x,y,z) = (1-x) - (1-y)z \quad X = \{P(x,y,z) = 0\}$$

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} \frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \log |z| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x,y,z)$$

$$\eta(x,y,z) = \log|x|\left(\frac{1}{3}\mathrm{d}\log|y|\wedge\mathrm{d}\log|z| - \mathrm{d}\arg y\wedge\mathrm{d}\arg z\right)$$
$$+\log|y|\left(\frac{1}{3}\mathrm{d}\log|z|\wedge\mathrm{d}\log|x| - \mathrm{d}\arg z\wedge\mathrm{d}\arg x\right)$$
$$+\log|z|\left(\frac{1}{3}\mathrm{d}\log|x|\wedge\mathrm{d}\log|y| - \mathrm{d}\arg x\wedge\mathrm{d}\arg y\right)$$

$$\mathrm{d}\eta(x,y,z) = \mathrm{Re}\left(\frac{\mathrm{d}x}{x}\wedge\frac{\mathrm{d}y}{y}\wedge\frac{\mathrm{d}z}{z}\right)$$

$$\eta(x, 1-x, y) = \mathrm{d}\,\omega(x, y)$$

where

$$\omega(x, y) = -D(x)\mathrm{d}\arg y$$

$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$z = \frac{1-x}{1-y}$$

$$\eta(x, y, z) = -\eta(x, 1-x, y) - \eta(y, 1-y, x)$$

$$m(P) = \frac{1}{(2\pi)^2}\int_\Gamma \eta(x, 1-x, y) + \eta(y, 1-y, x) = \frac{1}{(2\pi)^2}\int_{\partial\Gamma} \omega(x, y) + \omega(y, x)$$

$$\eta(x, 1-x, y) = \mathrm{d}\,\omega(x, y)$$

where

$$\omega(x, y) = -D(x)\mathrm{d}\arg y$$

$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$z = \frac{1-x}{1-y}$$

$$\eta(x, y, z) = -\eta(x, 1-x, y) - \eta(y, 1-y, x)$$

$$m(P) = \frac{1}{(2\pi)^2}\int_{\Gamma}\eta(x, 1-x, y) + \eta(y, 1-y, x) = \frac{1}{(2\pi)^2}\int_{\partial\Gamma}\omega(x, y) + \omega(y, x)$$

$$\omega(x,x) = \mathrm{d}\mathcal{L}_3(x)$$

$$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$$

Maillot: if $P \in \mathbb{R}[x,y,z]$,

$$\partial\Gamma = \gamma = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$$

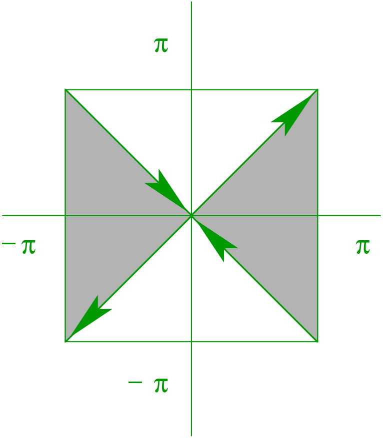
ω defined in

$$C = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\}$$

Want to apply Stokes' Theorem again.

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$



$$m((1-x)-(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y)+\omega(y,x)$$

$$\omega(x,x) = \mathrm{d}\mathcal{L}_3(x)$$

$$= \frac{1}{4\pi^2} 8(\mathcal{L}_3(1) - \mathcal{L}_3(-1)) = \frac{7}{2\pi^2}\zeta(3)$$

In general

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2} \int_\Gamma \eta(x,y,z)$$

Need

$$x \wedge y \wedge z = \sum r_i \;\; x_i \wedge (1-x_i) \wedge y_i$$

in $\bigwedge^3(\mathbb{C}(X)^*) \otimes \mathbb{Q}$,

($\{x,y,z\} = 0$ in $K_3^M(\mathbb{C}(X)) \otimes \mathbb{Q}$) then

$$\int_\Gamma \eta(x,y,z) = \sum r_i \int_\Gamma \eta(x_i, 1-x_i, y_i)$$

$$= \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i)$$

$$\omega(x,y) = -D(x)\,\mathrm{d}\arg y$$
$$+\frac{1}{3}\log|y|(\log|1-x|\,\mathrm{d}\log|x| - \log|x|\,\mathrm{d}\log|1-x|)$$

Let
$$R_2(x,y) = [x] + [y] + [1-xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right]$$
in $\mathbb{Z}\left[\mathbb{P}^1_{\mathbb{C}(C)}\right]$.
F field,
$$B_2(F) := \mathbb{Z}[\mathbb{P}^1_F]/\langle [0], [\infty], R_2(x,y)\rangle$$
Need
$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$
in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$.
Then
$$\int_\gamma \omega(x,y) = \sum r_i\, \mathcal{L}_3(x_i)|_{\partial\gamma}$$

$$\omega(x,y) = -D(x)\,\mathrm{d}\arg y$$
$$+\frac{1}{3}\log|y|(\log|1-x|\,\mathrm{d}\log|x| - \log|x|\,\mathrm{d}\log|1-x|)$$

Let

$$R_2(x,y) = [x] + [y] + [1-xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right]$$

in $\mathbb{Z}\left[\mathbb{P}^1_{\mathbb{C}(C)}\right]$.

F field,

$$B_2(F) := \mathbb{Z}[\mathbb{P}^1_F]/\left\langle [0], [\infty], R_2(x,y)\right\rangle$$

Need

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$.

Then

$$\int_\gamma \omega(x,y) = \sum r_i\, \mathcal{L}_3(x_i)|_{\partial\gamma}$$

Big picture in three variables

$$\cdots \to K_4(\partial\Gamma) \to K_3(X, \partial\Gamma) \to K_3(X) \to \dots$$

$$\partial\Gamma = X \cap \mathbb{T}^3$$

$$\cdots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \dots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

Polylogarithmic motivic complexes

Beilinson (after work of Bloch, Deligne, Beilinson, etc)
$r_{\mathcal{D}} : gr_j^{\gamma} K_{2j-i}(X)_{\mathbb{Q}} \to H^i_{\mathcal{D}}(X, \mathbb{R}(j))$

Goncharov: possible "Explicit construction" of $r_{\mathcal{D}}$ and $gr_j^{\gamma} K_{2j-i}(X)_{\mathbb{Q}}$.
F field, define $\mathcal{R}_n(F) \subset \mathbb{Z}[\mathbb{P}^1_F]$ and

$$\mathbf{B}_n(F) := \mathbb{Z}[\mathbb{P}^1_F]/\mathcal{R}_n(F)$$

$$\mathcal{R}_1(F) := \langle \{x\} + \{y\} - \{xy\}, \quad x, y \in F^*, \{0\}, \{\infty\} \rangle$$

$$\mathbf{B}_n(F) := \mathbb{Z}[\mathbb{P}^1_F]/\mathcal{R}_n(F)$$

$$\mathcal{R}_1(F) := \langle \{x\} + \{y\} - \{xy\}, \quad x, y \in F^*, \{0\}, \{\infty\} \rangle$$

$$\mathbb{Z}[\mathbb{P}^1_F] \xrightarrow{\delta_n} \begin{cases} \mathbf{B}_{n-1}(F) \otimes F^* & \text{if } n \geq 3 \\ \bigwedge^2 F^* & \text{if } n = 2 \end{cases}$$

$$\delta_n(\{x\}) = \begin{cases} \{x\}_{n-1} \otimes x & \text{if } n \geq 3 \\ (1-x) \wedge x & \text{if } n = 2 \\ 0 & \text{if } \{x\} = \{0\}, \{1\}, \{\infty\} \end{cases}$$

$$\mathcal{A}_n(F) := \ker \delta_n$$

$$\mathcal{R}_n(F) := \langle \alpha(0) - \alpha(1), \alpha(t) \in \mathcal{A}_n(F(t)) \rangle$$

$$\delta_n(\mathcal{R}_n(F)) = 0$$

$\mathbf{B}_F(n)$:

$$\mathbf{B}_n(F) \xrightarrow{\delta} \mathbf{B}_{n-1}(F) \otimes F^* \xrightarrow{\delta} \ldots \xrightarrow{\delta} \mathbf{B}_2(F) \otimes \wedge^{n-2} F^* \xrightarrow{\delta} \wedge^n F^*$$

$$\delta : \{x\}_p \otimes \bigwedge_{i=1}^{n-p} y_i \longrightarrow \delta_p(\{x\}_p) \wedge \bigwedge_{i=1}^{n-p} y_i.$$

$$H^n(\mathbf{B}_F(n)) \quad \cong \quad K_n^M(F)$$

Conjecture

$$H^i(\mathbf{B}_F(n) \otimes \mathbb{Q}) \cong gr_n^\gamma K_{2n-i}(F)_\mathbb{Q}$$

(Goncharov) X complex algebraic variety. There exist $\eta_n(m)$ inducing a homomorphism of complexes

$$\begin{array}{ccccccc}
\mathbf{B}_n(\mathbb{C}(X)) & \xrightarrow{\delta} & \mathbf{B}_{n-1}(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & \bigwedge^n \mathbb{C}(X)^* \\
\downarrow \eta_n(1) & & \downarrow \eta_n(2) & & & & \downarrow \eta_n(n) \\
\mathcal{A}^0(X)(n-1) & \xrightarrow{\mathrm{d}} & \mathcal{A}^1(X)(n-1) & \xrightarrow{\mathrm{d}} & \dots & \xrightarrow{\mathrm{d}} & \mathcal{A}^{n-1}(X)(n-1)
\end{array}$$

($\mathcal{A}^i(X)(j)$ = smooth i-forms with values in $(2\pi \mathrm{i})^j\mathbb{R}$) such that

- $\eta_n(1)(\{x\}_n) = \widehat{\mathcal{L}}_n(x)$.
- $\mathrm{d}\eta_n(n)(x_1 \wedge \cdots \wedge x_n) = \pi_n\left(\frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}\right)$.
- $\eta_n(m)$ compatible with residues (residues are given by tame symbols).

Conjecture

"Image of $\eta_n(i)$" coincides with image of regulator

Deninger(1997)

$$m(P) = m(P^*) + \frac{1}{(-2i\pi)^{n-1}} \int_\Gamma \eta_n(n)(x_1, \dots, x_n)$$

where

$$\Gamma = \{P(x_1, \dots, x_n) = 0\} \cap \{|x_1| = \dots = |x_{n-1}| = 1, |x_n| \geq 1\}$$

$$\pi^{2n} m\left(1 + \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_{2n}}{1+x_{2n}}\right) z\right)$$

$$= \sum_{h=1}^{n} c_{n,h} \pi^{2n-2h} \zeta(2h+1)$$

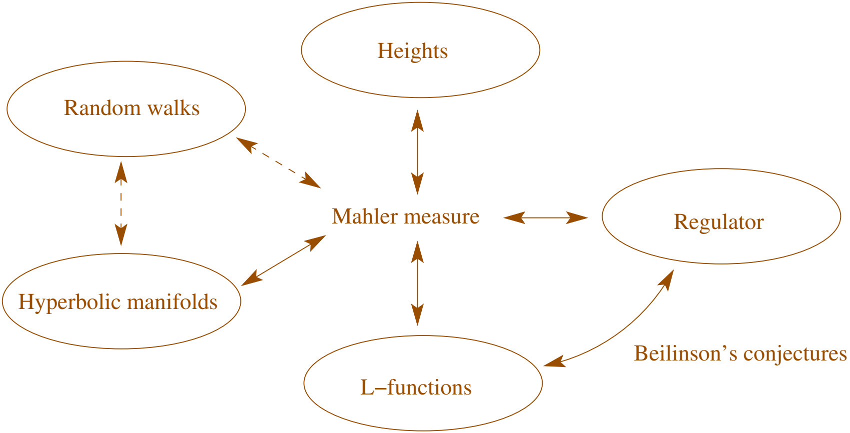
Deninger(1997)

$$m(P) = m(P^*) + \frac{1}{(-2i\pi)^{n-1}} \int_\Gamma \eta_n(n)(x_1, \dots, x_n)$$

where

$$\Gamma = \{P(x_1, \dots, x_n) = 0\} \cap \{|x_1| = \dots = |x_{n-1}| = 1, |x_n| \geq 1\}$$

$$\pi^{2n} m \left(1 + \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_{2n}}{1+x_{2n}}\right) z\right)$$
$$= \sum_{h=1}^{n} c_{n,h} \pi^{2n-2h} \zeta(2h+1)$$


Heights
Random walks
Mahler measure
Regulator
Hyperbolic manifolds
L–functions
Beilinson's conjectures