Mahler measure of multivariable polynomials

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Matilde N. Lalín– Clay Mathematics Institute, University of Bristish Columbia and Pacific Insitute for Mathematical Sciences[1]

Mahler measure and Lehmer's question

Looking for large primes, Pierce [11] proposed the following idea in 1918.
Consider $P \in \mathbb{Z}[x]$ monic, and write

$$P(x) = \prod_i (x - \alpha_i).$$

Then, let us look at

$$\Delta_n = \prod_i (\alpha_i^n - 1).$$

The α_i are integers over $\mathbb{Z}$. By applying Galois theory, it is easy to see that $\Delta_n \in \mathbb{Z}$. Note that if $P(x) = x - 2$, we get the sequence $\Delta_n = 2^n - 1$. Thus, we recover the example of Mersenne numbers. The idea is to look for primes among the factors of Δ_n. The prime divisors of such integers must satisfy some congruence conditions that are quite restrictive, hence they are easier to factorize than a randomly given number of the same size.

In order to minimize the number of trial divisions, the sequence Δ_n should grow slowly. Lehmer [10] studied $\frac{\Delta_{n+1}}{\Delta_n}$, observed that

$$\lim_{n\to\infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

and suggested the following definition.

Definition 1 *Given $P \in \mathbb{C}[x]$, such that*

$$P(x) = a \prod_i (x - \alpha_i),$$

define the measure [2] of P as

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}. \tag{1}$$

The logarithmic measure is defined as[3]

$$m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|. \tag{2}$$

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[1] `mlalin@math.ubc.ca`– `http://www.math.ubc.ca/~mlalin`

[2] The name Mahler came later after the person who successfully extended this definition to the several-variable case.

[3] $\log^+ x = \log \max\{1, x\}$ for $x \in \mathbb{R}_{\geq 0}$

When does $M(P) = 1$ for $P \in \mathbb{Z}[x]$? We have

Lemma 2 *(Kronecker) Let $P = \prod_i (x - \alpha_i) \in \mathbb{Z}[x]$, if $|\alpha_i| \leq 1$, then the α_i are zero or roots of the unity.*

By Kronecker's Lemma, $P \in \mathbb{Z}[x]$, $P \neq 0$, then $M(P) = 1$ if and only if P is the product of powers of x and cyclotomic polynomials. This statement characterizes integral polynomials whose Mahler measure is 1.

Lehmer found the example

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = \log(1.176280818\ldots) = 0.162357612\ldots$$

and asked the following (Lehmer's question, 1933):

Is there a constant $C > 1$ such that for every polynomial $P \in \mathbb{Z}[x]$ with $M(P) > 1$, then $M(P) \geq C$?

Lehmer's question remains open nowadays. His 10-degree polynomial remains the best possible result.

The use of this polynomial has led to the discovery of the prime number $1,794,327,140,357$ but bigger primes were discovered with the use of other polynomials.

Mahler measure of several-variable polynomials

Definition 3 *For $P \in \mathbb{C}[x_1, \ldots, x_n]$, the (logarithmic)* Mahler measure *is defined by*

$$m(P) := \int_0^1 \cdots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \,\mathrm{d}\theta_1 \ldots \,\mathrm{d}\theta_n = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n} \tag{3}$$

Because of Jensen's equality $\int_0^1 \log |\mathrm{e}^{2\pi\mathrm{i}\theta} - \alpha| \,\mathrm{d}\theta = \log^+ |\alpha| = \log \max\{1, |\alpha|\}$, we recover the one-varible case.

It is possible to prove that this integral is not singular and that $m(P)$ always exists.

Some properties

Proposition 4 *For $P, Q \in \mathbb{C}[x_1, \ldots, x_n]$*

$$m(P \cdot Q) = m(P) + m(Q). \tag{4}$$

It is also true that $m(P) \geq 0$ if P has integral coefficients.

Mahler measure is related to heights. Indeed, if α is an algebraic number, and P_α is its minimal polynomial over $\mathbb{Q}$, then

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha),$$

where h is the logarithmic Weil height. This identity also extends to several-variable polynomials and heights in hypersurfaces.

Let us also mention the following result:

Theorem 5 *(Boyd–Lawton) For $P \in \mathbb{C}[x_1, \ldots, x_n]$*

$$\lim_{k_2 \to \infty} \cdots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \ldots, x^{k_n})) = m(P(x_1, \ldots, x_n)). \tag{5}$$

In particular Lehmer's problem in several variables reduces to the one-variable case.

Examples

For the several-variable case, it seems that there is no simpler general formula than the integral defining the measure. However, many examples have been found relating the Mahler measure of polynomials in two variables to special values of L-functions in quadratic characters, L-functions on elliptic curves and dilogarithms.

- Smyth [13]

$$m(x + y + 1) = \frac{3\sqrt{3}}{4\pi} \mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1), \tag{6}$$

where

$$\mathrm{L}(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \quad \text{and} \quad \chi_{-3}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \bmod 3 \\ -1 & \text{if} \quad n \equiv -1 \bmod 3 \\ 0 & \text{if} \quad n \equiv 0 \bmod 3 \end{cases}$$

- Smyth [1]

$$m(x + y + z + 1) = \frac{7}{2\pi^2} \zeta(3). \tag{7}$$

- Boyd [2], Deninger [5], Rodriguez-Villegas [12]

$$\begin{aligned} m\left(x + \frac{1}{x} + y + \frac{1}{y} - k\right) &\stackrel{?}{=} \frac{\mathrm{L}'(E_k, 0)}{B_k} \qquad k \in \mathbb{N} \\ m\left(x + \frac{1}{x} + y + \frac{1}{y} - 4\right) &= 2\mathrm{L}'(\chi_{-4}, -1) \\ m\left(x + \frac{1}{x} + y + \frac{1}{y} - 4\sqrt{2}\right) &= \mathrm{L}'(A, 0) \end{aligned}$$

Where B_k is a rational number, and E_k is the elliptic curve with corresponds to the zero set of the polynomial. When $k = 4$ the curve has genus zero. When $k = 4\sqrt{2}$ the elliptic curve is

$$A : y^2 = x^3 - 44x + 112,$$

which has complex multiplication.

- Condon (2003):

$$\pi^2 m \left(z - \left(\frac{1 - x}{1 + x}\right)(1 + y)\right) = \frac{28}{5} \zeta(3)$$

- D'Andrea & L. (2003):

$$\pi^2 m \left(z(1 - xy)^2 - (1 - x)(1 - y)\right) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)}$$

- Boyd & L. (2005):

$$m(x^2+1+(x+1)y+(x-1)z) = \frac{\mathrm{L}(\chi_{-4},2)}{\pi} + \frac{21}{8\pi^2}\zeta(3)$$

Theorem 6 *We have the following identities:* [4]

For $n \geq 1$:

$$\pi^{2n} m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_{2n}}{1+x_{2n}}\right)z\right)$$

$$= \frac{1}{(2n-1)!}\sum_{h=1}^{n} s_{n-h}(2^2,\dots,(2n-2)^2)\frac{(2h)!(2^{2h+1}-1)}{2}\pi^{2n-2h}\zeta(2h+1). \tag{8}$$

For $n \geq 0$:

$$\pi^{2n+1} m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_{2n+1}}{1+x_{2n+1}}\right)z\right)$$

$$= \frac{1}{(2n)!}\sum_{h=0}^{n} s_{n-h}(1^2,\dots,(2n-1)^2)(2h+1)!2^{2h+1}\pi^{2n-2h}\mathrm{L}(\chi_{-4},2h+2). \tag{9}$$

ζ is the Riemann zeta function,

$$\mathrm{L}(\chi_{-4},s) := \sum_{n=1}^{\infty}\frac{\chi_{-4}(n)}{n^s}, \quad \chi_{-4}(n) = \begin{cases} \left(\frac{-1}{n}\right) & \text{if } n \text{ odd},\\ \\ 0 & \text{if } n \text{ even}\end{cases}$$

Also,

$$s_l(a_1,\dots,a_k) = \begin{cases} 1 & \text{if} \quad l=0,\\ \sum_{i_1<\dots<i_l} a_{i_1}\dots a_{i_l} & \text{if} \quad 0<l\leq k,\\ 0 & \text{if} \quad k<l \end{cases} \tag{10}$$

are the elementary symmetric polynomials, i.e.,

$$\prod_{i=1}^{k}(x+a_i) = \sum_{l=0}^{k} s_l(a_1,\dots,a_k)x^{k-l}. \tag{11}$$

For concreteness, we list the first values for each family in the following table:

[4] In order to simplify notation, we describe the polynomials as rational functions, writing $1+\frac{1-x}{1+x}z$ instead of $1+x+(1-x)z$, and so on. The Mahler measure does not change since the denominators are product of cyclotomic polynomials.

$\pi^2 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)z\right)$	$7\,\zeta(3)$
$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right)$	$62\zeta(5)+\frac{14\pi^2}{3}\zeta(3)$
$\pi^6 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_6}{1+x_6}\right)z\right)$	$381\zeta(7)+62\pi^2\zeta(5)+\frac{56\pi^4}{15}\zeta(3)$
$\pi^8 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_8}{1+x_8}\right)z\right)$	$2044\zeta(9)+508\pi^2\zeta(7)+\frac{868\pi^4}{15}\zeta(5)+\frac{16\pi^6}{5}\zeta(3)$
$\pi m\left(1+\left(\frac{1-x_1}{1+x_1}\right)z\right)$	$2\mathrm{L}(\chi_{-4},2)$
$\pi^3 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_3}{1+x_3}\right)z\right)$	$24\mathrm{L}(\chi_{-4},4)+\pi^2\mathrm{L}(\chi_{-4},2)$
$\pi^5 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_5}{1+x_5}\right)z\right)$	$160\mathrm{L}(\chi_{-4},6)+20\pi^2\mathrm{L}(\chi_{-4},4)+\frac{3\pi^4}{4}\mathrm{L}(\chi_{-4},2)$
$\pi^7 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_7}{1+x_7}\right)z\right)$	$896\mathrm{L}(\chi_{-4},8)+\frac{560}{3}\pi^2\mathrm{L}(\chi_{-4},6)+$ $\frac{259}{15}\pi^4\mathrm{L}(\chi_{-4},4)+\frac{5}{8}\pi^6\mathrm{L}(\chi_{-4},2)$

There are similar but more complicated results with

$$m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)(1+y)z\right)$$

and

$$m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)x+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)\right)y\right)$$

Integrals and Polylogarithms

The following definitions and notations may be found in Goncharov's works, [6, 7]:

Definition 7 *Multiple polylogarithms are defined as the power series*

$$\mathrm{Li}_{n_1,\dots,n_m}(x_1,\dots,x_m):=\sum_{0<k_1<k_2<\dots<k_m}\frac{x_1^{k_1}x_2^{k_2}\dots x_m^{k_m}}{k_1^{n_1}k_2^{n_2}\dots k_m^{n_m}}$$

which are convergent for $|x_i|<1$. The length of a polylogarithm function is the number m and its weight is the number $w=n_1+\dots+n_m$.

Definition 8 *Hyperlogarithms are defined as the iterated integrals*

$$\mathrm{I}_{n_1,\dots,n_m}(a_1:\dots:a_m:a_{m+1}):=$$

$$\int_0^{a_{m+1}} \underbrace{\frac{\mathrm{d}t}{t-a_1}\circ\frac{\mathrm{d}t}{t}\circ\ldots\circ\frac{\mathrm{d}t}{t}}_{n_1}\circ\underbrace{\frac{\mathrm{d}t}{t-a_2}\circ\frac{\mathrm{d}t}{t}\circ\ldots\circ\frac{\mathrm{d}t}{t}}_{n_2}\circ\ldots\circ\underbrace{\frac{\mathrm{d}t}{t-a_m}\circ\frac{\mathrm{d}t}{t}\circ\ldots\circ\frac{\mathrm{d}t}{t}}_{n_m}$$

where n_i are integers, a_i are complex numbers, and

$$\int_0^{b_{k+1}} \frac{\mathrm{d}t}{t-b_1}\circ\ldots\circ\frac{\mathrm{d}t}{t-b_k} = \int_{0\le t_1\le\ldots\le t_k\le b_{k+1}} \frac{\mathrm{d}t_1}{t_1-b_1}\cdots\frac{\mathrm{d}t_k}{t_k-b_k}.$$

The value of the integral above only depends on the homotopy class of the path connecting 0 and a_{m+1} on $\mathbb{C}\setminus\{a_1,\ldots,a_m\}$.

It is easy to see (for instance, in [7]) that,

$$\begin{aligned}\mathrm{I}_{n_1,\ldots,n_m}(a_1:\ldots:a_m:a_{m+1}) &= (-1)^m \mathrm{Li}_{n_1,\ldots,n_m}\left(\frac{a_2}{a_1},\frac{a_3}{a_2},\ldots,\frac{a_m}{a_{m-1}},\frac{a_{m+1}}{a_m}\right),\\ \mathrm{Li}_{n_1,\ldots,n_m}(x_1,\ldots,x_m) &= (-1)^m \mathrm{I}_{n_1,\ldots,n_m}((x_1\ldots x_m)^{-1}:\ldots:x_m^{-1}:1),\end{aligned}$$

which gives an analytic continuation of multiple polylogarithms. Observe that we recover the special value of the Riemann zeta function $\zeta(k)$ for $k\ge 2$ as $\mathrm{Li}_k(1)$, as well as $\mathrm{L}(\chi_{-4},k) = -\frac{\mathrm{i}}{2}(\mathrm{Li}_k(\mathrm{i}) - \mathrm{Li}_k(-\mathrm{i}))$.

Lemma 9 *We have the following length-one identities:*

$$\int_0^1 \log^j x \frac{\mathrm{d}x}{x^2-1} = (-1)^{j+1} j!\left(1-\frac{1}{2^{j+1}}\right)\zeta(j+1), \tag{12}$$

$$\int_0^1 \log^j x \frac{\mathrm{d}x}{x^2+1} = (-1)^{j} j!\mathrm{L}(\chi_{-4},j+1), \tag{13}$$

PROOF. The idea is to translate the integral into hyperlogarithms. We use the fact that $\int_x^1 \frac{\mathrm{d}s}{s} = -\log x$.□

An important integral

We will need to compute the integral $\int_0^\infty \frac{x\log^k x\,\mathrm{d}x}{(x^2+a^2)(x^2+b^2)}$. The following Lemma will help:

Lemma 10 *We have the following integral:*

$$\int_0^\infty \frac{x^\alpha\,\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} = \frac{\pi(a^{\alpha-1}-b^{\alpha-1})}{2\cos\frac{\pi\alpha}{2}(b^2-a^2)} \qquad \text{for} \quad 0<\alpha<1. \tag{14}$$

PROOF. We write the integral as a difference of two integrals:

$$\int_0^\infty \frac{x^\alpha\,\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} = \int_0^\infty \left(\frac{1}{x^2+a^2}-\frac{1}{x^2+b^2}\right)\frac{x^\alpha\,\mathrm{d}x}{b^2-a^2}. \tag{15}$$

and use residues in order to evaluate the integral.□

By continuity, the formula in the statement is true for $\alpha=1$, in fact the integral converges for $\alpha<3$.

Next, we will define some polynomials that will be used in the formula for $\int_0^\infty \frac{x\log^k x\,\mathrm{d}x}{(x^2+a^2)(x^2+b^2)}$.

Definition 11 *Let $P_k(x) \in \mathbb{Q}[x]$, $k \geq 0$, be defined recursively as follows:*

$$P_k(x) = \frac{x^{k+1}}{k+1} + \frac{1}{k+1} \sum_{j>1\,(\mathrm{odd})}^{k+1} (-1)^{\frac{j+1}{2}} \binom{k+1}{j} P_{k+1-j}(x). \tag{16}$$

For instance, the first $P_k(x)$ are: $P_0(x) = x$, $P_1(x) = \frac{x^2}{2}$, $P_2(x) = \frac{x^3}{3} + \frac{x}{3}$, $P_3(x) = \frac{x^4}{4} + \frac{x^2}{2}$, etc.

Lemma 12 *The following properties are true*

1. $\deg P_k = k + 1$.
2. *Every monomial of $P_k(x)$ has degree odd (even) for k even (odd).*
3. $P_k(0) = 0$.
4. $P_{2l}(\mathrm{i}) = 0$ *for* $l > 0$.
5. $(2l + 1)P_{2l}(x) = \frac{\partial}{\partial x} P_{2l+1}(x)$.
6. $2lP_{2l-1}(x) \equiv \frac{\partial}{\partial x} P_{2l}(x) \bmod x$.

The above properties can be easily proved by induction. These polynomials are related to Bernoulli polynomials.

Proposition 13 *We have:*

$$\int_0^\infty \frac{x \log^k x \, \mathrm{d}x}{(x^2 + a^2)(x^2 + b^2)} = \left(\frac{\pi}{2}\right)^{k+1} \frac{P_k\left(\frac{2\log a}{\pi}\right) - P_k\left(\frac{2\log b}{\pi}\right)}{a^2 - b^2}. \tag{17}$$

PROOF. Differentiate and play with the recurrence. $\square$

Description of the general method

We will prove our main result by first examining a general situation. Let $P_\alpha \in \mathbb{C}[\mathbf{x}]$ such that its coefficients depend polynomially on a parameter $\alpha \in \mathbb{C}$. We replace α by $\left(\frac{x_1-1}{x_1+1}\right) \dots \left(\frac{x_n-1}{x_n+1}\right)$ and obtain a new polynomial $\tilde{P} \in \mathbb{C}[\mathbf{x}, x_1, \dots, x_n]$. By definition of Mahler measure, it is easy to see that

$$m(\tilde{P}) = \frac{1}{(2\pi\mathrm{i})^n} \int_{\mathbb{T}^n} m\left(P_{\left(\frac{x_1-1}{x_1+1}\right)\dots\left(\frac{x_n-1}{x_n+1}\right)}\right) \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n}.$$

We perform a change of variables to polar coordinates, $x_j = \mathrm{e}^{\mathrm{i}\theta_j}$:

$$= \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} m\left(P_{\mathrm{i}^n \tan\left(\frac{\theta_1}{2}\right)\dots\tan\left(\frac{\theta_n}{2}\right)}\right) \, \mathrm{d}\theta_1 \dots \, \mathrm{d}\theta_n.$$

Set $x_i = \tan\left(\frac{\theta_i}{2}\right)$. We get,

$$= \frac{1}{\pi^n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} m\left(P_{\mathrm{i}^n x_1 \dots x_n}\right) \frac{\mathrm{d}x_1}{x_1^2 + 1} \dots \frac{\mathrm{d}x_n}{x_n^2 + 1}$$

$$= \frac{2^n}{\pi^n} \int_0^\infty \dots \int_0^\infty m\left(P_{\mathrm{i}^n x_1 \dots x_n}\right) \frac{\mathrm{d}x_1}{x_1^2+1} \dots \frac{\mathrm{d}x_n}{x_n^2+1}.$$

Making one more change, $\hat{x}_1 = x_1, \dots, \hat{x}_{n-1} = x_1 \dots x_{n-1}, \hat{x}_n = x_1 \dots x_n$:

$$= \frac{2^n}{\pi^n} \int_0^\infty \dots \int_0^\infty m\left(P_{\mathrm{i}^n \hat{x}_n}\right) \frac{\hat{x}_1\, \mathrm{d}\hat{x}_1}{\hat{x}_1^2+1} \frac{\hat{x}_2\, \mathrm{d}\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2} \dots \frac{\hat{x}_{n-1}\, \mathrm{d}\hat{x}_{n-1}}{\hat{x}_{n-1}^2+\hat{x}_{n-2}^2} \frac{\mathrm{d}\hat{x}_n}{\hat{x}_n^2+\hat{x}_{n-1}^2}.$$

We need to compute this integral. In most of our cases, the Mahler measure of P_α depends only on the absolute value of α.

By iterating Proposition 13, the above integral can be written as a linear combination, with coefficients that are rational numbers and powers of π in such a way that the weights are homogeneous, of integrals of the form

$$\int_0^\infty m\left(P_x\right) \log^j x \frac{\mathrm{d}x}{x^2 \pm 1}.$$

It is easy to see that j is even iff n is odd and the corresponding sign in that case is "+".

We are going to compute these coefficients.

Let us establish some convenient notation:

Definition 14 *Let $a_{n,h} \in \mathbb{Q}$ be defined for $n \geq 1$ and $h = 0, \dots, n-1$ by*

$$\int_0^\infty \dots \int_0^\infty m\left(P_{x_1}\right) \frac{x_{2n}\, \mathrm{d}x_{2n}}{x_{2n}^2+1} \frac{x_{2n-1}\, \mathrm{d}x_{2n-1}}{x_{2n-1}^2+x_{2n}^2} \dots \frac{\mathrm{d}x_1}{x_1^2+x_2^2}$$

$$= \sum_{h=1}^{n} a_{n,h-1} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty m\left(P_x\right) \log^{2h-1} x \frac{\mathrm{d}x}{x^2-1}. \tag{18}$$

Let $b_{n,h} \in \mathbb{Q}$ be defined for $n \geq 0$ and $h = 0, \dots, n$ by

$$\int_0^\infty \dots \int_0^\infty m\left(P_{x_1}\right) \frac{x_{2n+1}\, \mathrm{d}x_{2n+1}}{x_{2n+1}^2+1} \frac{x_{2n}\, \mathrm{d}x_{2n}}{x_{2n}^2+x_{2n+1}^2} \dots \frac{\mathrm{d}x_1}{x_1^2+x_2^2}$$

$$= \sum_{h=0}^{n} b_{n,h} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty m\left(P_x\right) \log^{2h} x \frac{\mathrm{d}x}{x^2+1}. \tag{19}$$

We claim:

Lemma 15

$$\sum_{h=0}^{n} b_{n,h} x^{2h} = \sum_{h=1}^{n} a_{n,h-1} \left(P_{2h-1}(x) - P_{2h-1}(\mathrm{i})\right) \tag{20}$$

$$\sum_{h=1}^{n+1} a_{n+1,h-1} x^{2h-1} = \sum_{h=0}^{n} b_{n,h} P_{2h}(x) \tag{21}$$

PROOF. First observe that

$$\sum_{h=0}^{n} b_{n,h} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty m\left(P_x\right) \log^{2h} x \frac{\mathrm{d}x}{x^2+1}$$

$$= \sum_{h=1}^{n} a_{n,h-1} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty \int_0^\infty m\left(P_x\right) y \log^{2h-1} y \frac{\mathrm{d}y}{y^2-1} \frac{\mathrm{d}x}{x^2+y^2}. \tag{22}$$

But

$$\int_0^\infty \frac{y \log^{2h-1} y \,\mathrm{d}y}{(y^2+x^2)(y^2-1)} = \left(\frac{\pi}{2}\right)^{2h} \frac{P_{2h-1}\left(\frac{2\log x}{\pi}\right) - P_{2h-1}\left(\mathrm{i}\right)}{x^2+1}$$

by applying Proposition 13 for $a = x$ and $b = \mathrm{i}$.

The right side of equation (22) becomes

$$= \sum_{h=1}^{n} a_{n,h-1} \left(\frac{\pi}{2}\right)^{2n} \int_0^\infty m\left(P_x\right) \left(P_{2h-1}\left(\frac{2\log x}{\pi}\right) - P_{2h-1}\left(\mathrm{i}\right)\right) \frac{\mathrm{d}x}{x^2+1}.$$

As a consequence, equation (22) translates into the polynomial identity (20). Equation (21) is proved in a similar way.

□

Proposition 16

$$\begin{aligned} 2n(-1)^l s_{n-l}(2^2,\dots,(2n-2)^2) &= \sum_{h=l}^{n} (-1)^h \binom{2h}{2l-1} s_{n-h}(1^2,\dots,(2n-1)^2), \\ (2n+1)(-1)^l s_{n-l}(1^2,\dots,(2n-1)^2) &= \sum_{h=l}^{n} (-1)^h \binom{2h+1}{2l} s_{n-h}(2^2,\dots,(2n)^2). \end{aligned}$$

PROOF. These equalities are easy to prove if we think of the symmetric functions as coefficients of certain polynomials, as in equation (11).

In order to prove the first equality, multiply by x^{2l} on both sides and add for $l = 1, \dots, n$. Then compare the polynomials.

□

Theorem 17 *We have:*

$$\sum_{h=0}^{n-1} a_{n,h} x^{2h} = \frac{(x^2+2^2)\dots(x^2+(2n-2)^2)}{(2n-1)!} \tag{23}$$

for $n \geq 1$ and $h = 0, \dots, n-1$, and

$$\sum_{h=0}^{n} b_{n,h} x^{2h} = \frac{(x^2+1^2)\dots(x^2+(2n-1)^2)}{(2n)!} \tag{24}$$

for $n \geq 0$ and $h = 0, \dots, n$.

In other words,

$$a_{n,h} = \frac{s_{n-1-h}(2^2,\dots,(2n-2)^2)}{(2n-1)!}, \tag{25}$$

$$b_{n,h} = \frac{s_{n-h}(1^2,\dots,(2n-1)^2)}{(2n)!}. \tag{26}$$

PROOF. For $2n+1=1$, $n=0$ and the integral becomes

$$\int_0^\infty m\left(P_x\right)\frac{\mathrm{d}x}{x^2+1}$$

so $b_{0,0}=1$.

For $2n=2$, $n=1$ and we have

$$\int_0^\infty\int_0^\infty m\left(P_x\right)\frac{y\,\mathrm{d}y}{y^2+1}\frac{\mathrm{d}x}{x^2+y^2}=\int_0^\infty m\left(P_x\right)\frac{\log x\,\mathrm{d}x}{x^2-1}$$

so $a_{1,0}=1$.

Then the statement is true for the first two cases.

We proceed by induction. Suppose that

$$a_{n,h}=\frac{s_{n-1-h}(2^2,\ldots,(2n-2)^2)}{(2n-1)!}.$$

We have to prove that

$$b_{n,h}=\frac{s_{n-h}(1^2,\ldots,(2n-1)^2)}{(2n)!}.$$

By Lemma 15, it is enough to prove that

$$\sum_{h=0}^{n}s_{n-h}(1^2,\ldots,(2n-1)^2)x^{2h}=2n\sum_{h=1}^{n}s_{n-h}(2^2,\ldots,(2n-2)^2)\left(P_{2h-1}\left(x\right)-P_{2h-1}\left(\mathrm{i}\right)\right). \tag{27}$$

But it is easy to see that from Proposition 16.

Similarly it is possible to prove.

$$\sum_{h=0}^{n}s_{n-h}(2^2,\ldots,(2n)^2)x^{2h+1}=(2n+1)\sum_{h=0}^{n}s_{n-h}(1^2,\ldots,(2n-1)^2)P_{2h}\left(x\right), \tag{28}$$

and using Lemma 15 we conclude the proof. □

Proof of the Theorem

We managed to express the Mahler measure of $\tilde{P}$ as a linear combination of functions that depend on the Mahler measure of P_α. Assume now that $P_\alpha(z)=1+\alpha z$.

$$m(1+\alpha z)=\log^+|\alpha|$$

This is the simplest possible case. For the even case we get

$$\pi^{2n}m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_{2n}}{1+x_{2n}}\right)z\right)$$

$$=2^{2n}\sum_{h=1}^{n}a_{n,h-1}\left(\frac{\pi}{2}\right)^{2n-2h}\int_0^\infty\log^+x\log^{2h-1}x\frac{\mathrm{d}x}{x^2-1}$$

$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} 2^{2h} \pi^{2n-2h} \int_1^{\infty} \log^{2h} x \frac{\mathrm{d}x}{x^2-1}.$$

Now set $y = \frac{1}{x}$,

$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} 2^{2h} \pi^{2n-2h} \int_0^{1} \log^{2h} y \frac{\mathrm{d}y}{1-y^2}.$$

If we apply Lemma 9, we obtain

$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} 2^{2h} \pi^{2n-2h} (2h)! \left(1 - \frac{1}{2^{2h+1}}\right) \zeta(2h+1)$$

$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} \frac{(2h)!(2^{2h+1}-1)}{2} \pi^{2n-2h} \zeta(2h+1).$$

The odd case is similar. □

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