Examples in two variables

Smyth (1981)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1)$$

Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right) \stackrel{?}{=} \frac{\mathrm{L}'(E_k,0)}{s_k} \qquad k \in \mathbb{N}$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\right) = 2\mathrm{L}'(\chi_{-4},-1)$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right) = \mathrm{L}'(E_{4\sqrt{2}},0)$$

Examples in three variables

Smyth(1981)

$$m(1+x+y+z)=\frac{7}{2\pi^2}\zeta(3)$$

D'Andrea & L (2003)

$$m\left(z(1-xy)^2-(1-x)(1-y)\right)=\frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)\pi^2}$$

Boyd & L (2005)

$$m(x^2+1+(x+1)y+(x-1)z)=\frac{\mathrm{L}(\chi_{-4},2)}{\pi}+\frac{21}{8\pi^2}\zeta(3)$$

Bertin (2003)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}-k\right)=$$

Kronecker-Eisenstein series

Examples in several variables

L (2003)

$$m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)z\right)$$

$$=\frac{24}{\pi^3}\mathrm{L}(\chi_{-4},4)+\frac{\mathrm{L}(\chi_{-4},2)}{\pi}$$

$$m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right)$$

$$=\frac{62}{\pi^4}\zeta(5)+\frac{14}{3\pi^2}\zeta(3)$$

$$m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)=\frac{24}{\pi^3}\mathrm{L}(\chi_{-4},4)$$

$$m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=\frac{93}{\pi^4}\zeta(5)$$

Philosophy of Beilinson's conjectures

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F number field)

- L-function ($\mathsf{L}_X = \zeta_F$)

- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)

- Regulator map

$$r : K \to \text{smooth differential forms}$$

  ($r = \log|\cdot|$)

Conjecture: special value of $\mathsf{L}_X \sim_{\mathbb{Q}^*} \int_\gamma r(\xi)$

(E.g. Dirichlet class number formula, F real quadratic, $\zeta_F'(0) \sim_{\mathbb{Q}^*} \log|\epsilon| \qquad \epsilon \in \mathcal{O}_F^*$)

Big picture

$$\cdots \to (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \to K_2(C, \partial\gamma) \to K_2(C) \to \cdots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

- $\eta(x,y)$ is exact, then $\{x,y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq \emptyset$ and we use Stokes' Theorem.

  $\rightsquigarrow$ dilogarithms $\rightsquigarrow$ zeta function of a number field.

- $\partial\gamma = \emptyset$, then $\{x,y\} \in K_2(C)$. We have $\eta(x,y)$ is not exact.

  $\rightsquigarrow$ L-series of a curve.

Big picture II

$$\cdots \to K_4(\partial\Gamma) \to K_3(S, \partial\Gamma) \to K_3(S) \to \cdots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

$$\cdots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \cdots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

Polylogarithmic motivic complexes

Beilinson (after work of Bloch, Deligne, Beilinson, etc) $r_{\mathcal{D}} : gr^{\gamma}_{j} K_{2j-i}(X)_{\mathbb{Q}} \to H^{i}_{\mathcal{D}}(X, \mathbb{R}(j))$

Goncharov: possible "Explicit construction" of $r_{\mathcal{D}}$ and $gr^{\gamma}_{j} K_{2j-i}(X)_{\mathbb{Q}}$.

F field, define $\mathcal{R}_n(F) \subset \mathbb{Z}[\mathbb{P}^1_F]$ and

$$\mathbf{B}_n(F) := \mathbb{Z}[\mathbb{P}^1_F]/\mathcal{R}_n(F)$$

$$\mathcal{R}_1(F) := \langle \{x\} + \{y\} - \{xy\}, \quad x, y \in F^*, \{0\}, \{\infty\} \rangle$$

$$\mathbf{B}_n(F) := \mathbb{Z}[\mathbb{P}^1_F]/\mathcal{R}_n(F)$$

$$\mathcal{R}_1(F) := \langle \{x\} + \{y\} - \{xy\}, \quad x, y \in F^*, \{0\}, \{\infty\} \rangle$$

$$\mathbb{Z}[\mathbb{P}^1_F] \xrightarrow{\delta_n} \begin{cases} \mathbf{B}_{n-1}(F) \otimes F^* & \text{if } n \geq 3 \\ \bigwedge^2 F^* & \text{if } n = 2 \end{cases}$$

$$\delta_n(\{x\}) = \begin{cases} \{x\}_{n-1} \otimes x & \text{if } n \geq 3 \\ (1-x) \wedge x & \text{if } n = 2 \\ 0 & \text{if } \{x\} = \{0\}, \{1\}, \{\infty\} \end{cases}$$

$$\mathcal{A}_n(F) := \ker \delta_n$$

$$\mathcal{R}_n(F) := \langle \alpha(0) - \alpha(1), \alpha(t) \in \mathcal{A}_n(F(t)) \rangle$$

$$\delta_n(\mathcal{R}_n(F)) = 0$$

$\mathbf{B}_F(n)$:

$$\mathbf{B}_n(F) \xrightarrow{\delta} \mathbf{B}_{n-1}(F) \otimes F^* \xrightarrow{\delta} \ldots \xrightarrow{\delta} \mathbf{B}_2(F) \otimes \wedge^{n-2} F^* \xrightarrow{\delta} \wedge^n F^*$$

$$\delta : \{x\}_p \otimes \bigwedge_{i=1}^{n-p} y_i \to \delta_p(\{x\}_p) \wedge \bigwedge_{i=1}^{n-p} y_i .$$

$$H^n(\mathbf{B}_F(n)) \;\cong\; K_n^M(F)$$

Conjecture 1

$$H^i(\mathbf{B}_F(n) \otimes \mathbb{Q}) \cong gr_n^\gamma K_{2n-i}(F)_{\mathbb{Q}}$$

(Goncharov) X complex algebraic variety. There exist $\eta_n(m)$ inducing a homomorphism of complexes

$$
\begin{array}{ccccccc}
\mathbf{B}_n(\mathbb{C}(X)) & \stackrel{\delta}{\to} & \mathbf{B}_{n-1}(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* & \stackrel{\delta}{\to} & \dots & \stackrel{\delta}{\to} & \bigwedge^n \mathbb{C}(X)^* \\
\downarrow \eta_n(1) & & \downarrow \eta_n(2) & & & & \downarrow \eta_n(n) \\
\mathcal{A}^0(X)(n-1) & \stackrel{\mathrm{d}}{\to} & \mathcal{A}^1(X)(n-1) & \stackrel{\mathrm{d}}{\to} & \dots & \stackrel{\mathrm{d}}{\to} & \mathcal{A}^{n-1}(X)(n-1)
\end{array}
$$

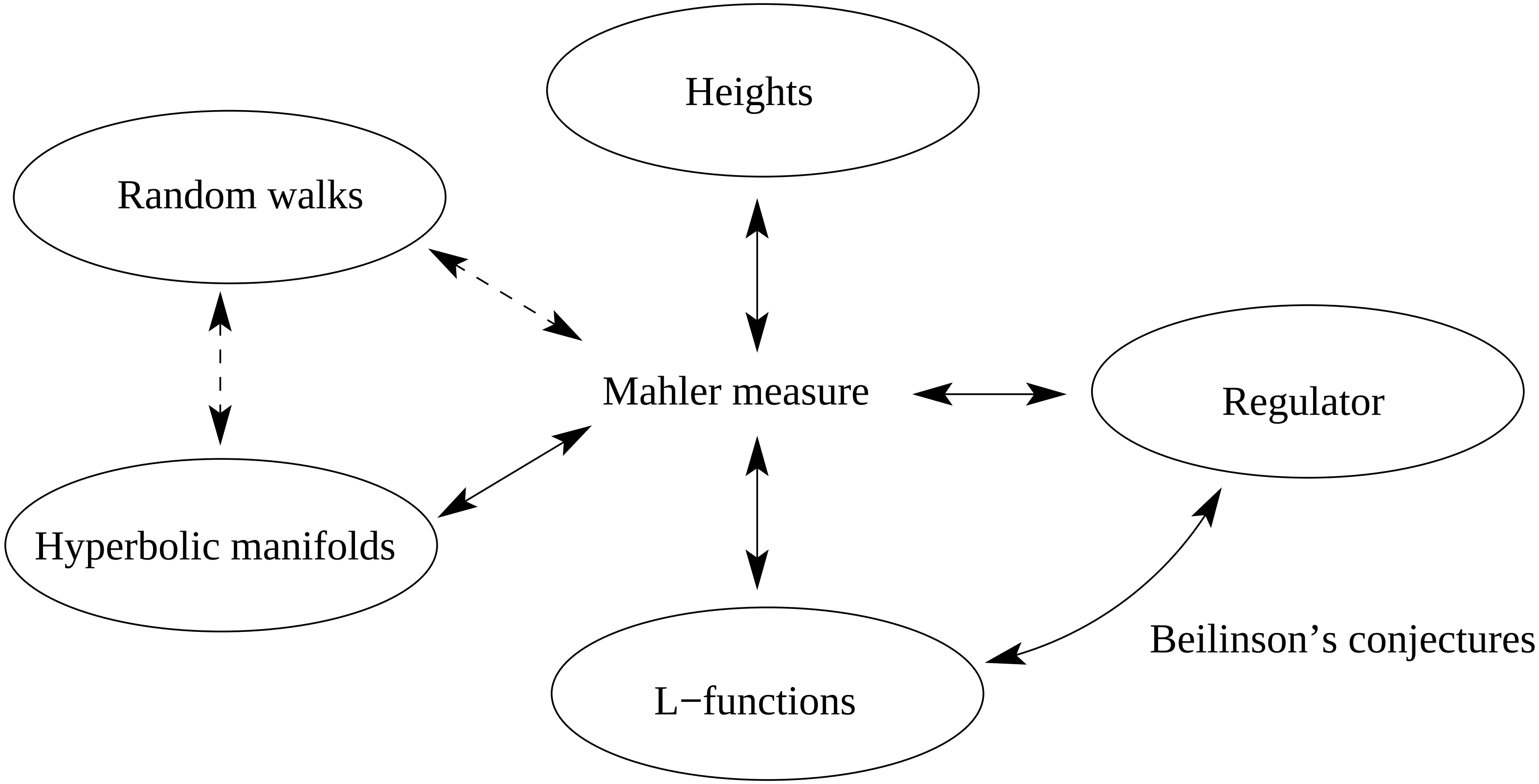
($\mathcal{A}^i(X)(j)$ = smooth i-forms with values in $(2\pi\mathrm{i})^j\mathbb{R}$) such that

- $\eta_n(1)(\{x\}_n) = \widehat{\mathcal{L}}_n(x)$.

- $\mathrm{d}\eta_n(n)(x_1 \wedge \ldots \wedge x_n) = \pi_n \left(\frac{\mathrm{d}x_1}{x_1} \wedge \ldots \wedge \frac{\mathrm{d}x_n}{x_n}\right)$.

- $\eta_n(m)$ compatible with residues (residues are given by tame symbols).

Conjecture 2 *"Image of $\eta_n(i)$" coincides with image of regulator*



Heights
Random walks
Mahler measure
Regulator
Hyperbolic manifolds
L−functions
Beilinson’s conjectures