Examples of Mahler Measures as special values of the Riemann Zeta function and L-series

Matilde N. Lalín

University of Texas at Austin

mlalin@math.utexas.edu

http://www.ma.utexas.edu/users/mlalin

Mahler Measure

For $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is:

$$m(P) = \int_0^1 \ldots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n$$

$$= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n}$$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

Some of Smyth's Examples with several variables

The simplest example in two variables:

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathsf{L}(\chi_{-3}, 2) = \mathsf{L}'(\chi_{-3}, -1)$$

$$\mathsf{L}(\chi_{-3}, s) := \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s}$$

is the L-series in the character of conductor 3:

$$\chi_{-3}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \bmod 3 \\ -1 & \text{if} \quad n \equiv -1 \bmod 3 \\ 0 & \text{otherwise} \end{cases}$$

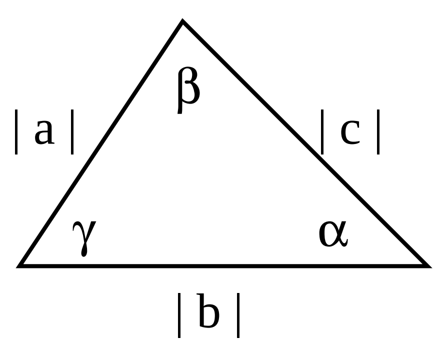
Another example in three variables

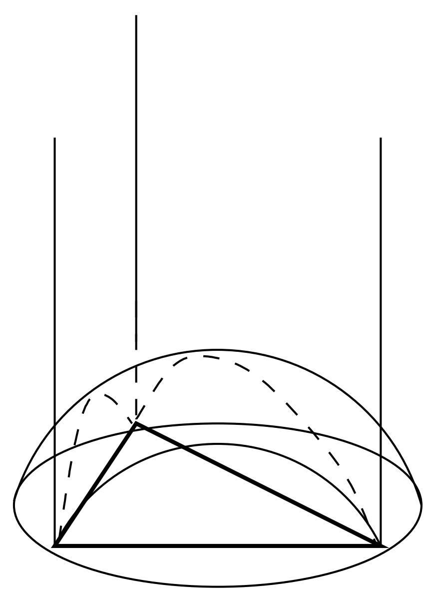
$$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

Cassaigne and Maillot's Example

$$\pi m(a + bx + cy) =$$

$$\begin{cases} D\left(\left|\frac{a}{b}\right| \mathrm{e}^{\mathrm{i}\gamma}\right) + \alpha \log|a| + \beta \log|b| + \gamma \log|c| & \triangle \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle \end{cases}$$





where

$$D(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z| \arg(1-z) \quad z \in \mathbb{C} \setminus [1, \infty)$$

Results (I)

$\pi^2 m\left(\frac{1-x_1}{1+x_1}+\alpha\frac{1-y_1}{1+y_1}z\right)$ $\pi^4 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}z\right)$ $\pi^6 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}\frac{1-x_3}{1+x_3}+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}\frac{1-y_3}{1+y_3}z\right)$	$7\,\zeta(3)$ $62\zeta(5)+28\zeta(2)\zeta(3)$ $381\zeta(7)+372\zeta(2)\zeta(5)+336\zeta(4)\zeta(3)$
$\pi m\left((1+y)+\alpha(1-y)z\right)$ $\pi^3 m\left(\frac{1-x_1}{1+x_1}(1+y)+\alpha\frac{1-y_1}{1+y_1}(1-y)z\right)$ $\pi^5 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}(1+y)+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}(1-y)z\right)$	$2\mathrm{L}(\chi_{-4},2)$ $24\mathrm{L}(\chi_{-4},4)+6\zeta(2)\mathrm{L}(\chi_{-4},2)$ $160\mathrm{L}(\chi_{-4},6)+120\zeta(2)\mathrm{L}(\chi_{-4},4)+\frac{135}{2}\zeta(4)\mathrm{L}(\chi_{-4},2)$

$$\chi_{-4}(n)=\begin{cases} 1 & \text{if} \quad n\equiv 1 \bmod 4 \\ -1 & \text{if} \quad n\equiv -1 \bmod 4 \\ 0 & \text{otherwise}\end{cases}$$

In the first column, $\alpha\in\mathbb{C}$. The values in the second column are the Mahler measures for the $\alpha=1$ case.

Results (II)

$\pi^2 m\left((1+x)+\alpha(1+y)z\right)$	$\frac{7}{2}\zeta(3)$
$\pi^4 m\left(\frac{1-x_1}{1+x_1}(1+x)+\alpha\frac{1-y_1}{1+y_1}(1+y)z\right)$	$93\zeta(5)$
$\pi^6 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}(1+x)+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}(1+y)z\right)$	$\frac{15\cdot 127}{2}\zeta(7)+186\zeta(2)\zeta(5)$
$\pi^8 m\left(\frac{1-x_1}{1+x_1}\frac{1-x_2}{1+x_2}\frac{1-x_3}{1+x_3}(1+x)+\frac{1-y_1}{1+y_1}\frac{1-y_2}{1+y_2}\frac{1-y_3}{1+y_3}(1+y)z\right)$	$14\cdot 511\zeta(9)+30\cdot 127\zeta(2)\zeta(7)+48\cdot 31\zeta(4)\zeta(5)$
$\pi^3 m\left((1+w)(1+x)+\alpha(1-w)(1+y)z\right)$	$12\zeta(2)\mathrm{L}(\chi_{-4},2)+2\mathrm{i}\mathcal{L}_{3,1}(\mathrm{i},\mathrm{i})$
$\pi^5 m\left(\frac{1-x_1}{1+x_1}(1+w)(1+x)+\frac{1-y_1}{1+y_1}(1-w)(1+y)z\right)$	$144\zeta(2)\mathrm{L}(\chi_{-4},4)+90\zeta(4)\mathrm{L}(\chi_{-4},2)+16\mathrm{i}\mathcal{L}_{3,3}(\mathrm{i},\mathrm{i})+24\mathrm{i}\zeta(2)\mathcal{L}_{3,1}(\mathrm{i},\mathrm{i})$

$\pi^2 m((1+w)(1+y)+(1-w)(x-y))$	$\frac{7}{2}\zeta(3)+\frac{\pi^2}{2}\log 2$

Polylogarithms

Multiple polylogarithms:

$$\mathrm{Li}_{k_1,\ldots,k_m}(x_1,\ldots,x_m) := \sum_{0<n_1<n_2<\ldots<n_m} \frac{x_1^{n_1}x_2^{n_2}\ldots x_m^{n_m}}{n_1^{k_1}n_2^{k_2}\ldots n_m^{k_m}}$$

(convergent for $|x_i| < 1$)

Hyperlogarithms:

$$\mathrm{I}_{k_1,\ldots,k_m}(a_1 : \ldots : a_m : a_{m+1}) :=$$

$$\int_0^{a_{m+1}} \underbrace{\frac{\mathrm{d}t}{t-a_1}\circ\frac{\mathrm{d}t}{t}\circ\ldots\circ\frac{\mathrm{d}t}{t}}_{k_1}\circ\ldots\circ\underbrace{\frac{\mathrm{d}t}{t-a_m}\circ\frac{\mathrm{d}t}{t}\circ\ldots\circ\frac{\mathrm{d}t}{t}}_{k_m}$$

k_i are integers, a_i are complex numbers, and

$$\int_0^{b_{l+1}} \frac{\mathrm{d}t}{t-b_1}\circ\ldots\circ\frac{\mathrm{d}t}{t-b_l} =$$

$$\int_{0\le t_1\le\ldots\le t_l\le b_{l+1}} \frac{\mathrm{d}t_1}{t_1-b_1}\cdots\frac{\mathrm{d}t_l}{t_l-b_l}$$

The value of the integral above depends on the homotopy class of the path connecting 0 and a_{m+1} on $\mathbb{C} \setminus \{a_1, \ldots, a_m\}$.

Proposition 1

$$\mathrm{I}_{k_1,\ldots,k_m}(a_1 : \ldots : a_m : a_{m+1}) =$$

$$(-1)^m \mathrm{Li}_{k_1,\ldots,k_m}\left(\frac{a_2}{a_1}, \frac{a_3}{a_2}, \ldots, \frac{a_m}{a_{m-1}}, \frac{a_{m+1}}{a_m}\right)$$

$$\mathrm{Li}_{k_1,\ldots,k_m}(x_1, \ldots, x_m) =$$

$$(-1)^m \mathrm{I}_{k_1,\ldots,k_m}\left(\frac{1}{x_1 \ldots x_m} : \ldots : \frac{1}{x_m} : 1\right)$$

(gives an analytic continuation to multiple polylogarithms)

Method (I)

1. Let $P_\alpha \in \mathbb{C}[x_1, \dots, x_n]$ whose coefficients depend polynomially on $\alpha \in \mathbb{C}$.

   For example, $P_\alpha(x) = 1 + \alpha x$.

   $m(P_\alpha) = \log^+ |\alpha|$.

2. Replace α by $\alpha \frac{1-y}{1+y}$. We obtain a polynomial $\tilde{P}_\alpha \in \mathbb{C}[x_1, \dots, x_n, y]$.

   In the example,
   $\tilde{P}_\alpha(x, y) = 1 + y + \alpha(1 - y)x$.

3. The Mahler measure of $\tilde{P}_\alpha$ is a certain integral of the Mahler measure of P_α:

$$m(\tilde{P}_\alpha) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} m\left(P_{\alpha \frac{1-y}{1+y}}\right) \frac{\mathrm{d}y}{y}$$

4. If the Mahler measure depends just on $|\alpha|$, make $u = \left|\alpha \frac{1-y}{1+y}\right|$.

First $y = \mathrm{e}^{\mathrm{i}\theta}$, then set $u = |\alpha| \tan\left(\frac{\theta}{2}\right)$.

$$
\begin{aligned}
m(\tilde{P}_\alpha) &= \frac{2}{\pi} \int_0^\infty m\left(P_u\right) \frac{|\alpha|\,\mathrm{d}u}{u^2 + |\alpha|^2} \\
&= \frac{\mathrm{i}}{\pi} \int_0^\infty m\left(P_u\right) \left(\frac{1}{u + \mathrm{i}|\alpha|} - \frac{1}{u - \mathrm{i}|\alpha|} \right) \mathrm{d}u
\end{aligned}
$$

In the example,

$$m(1+y+\alpha(1-y)x)$$

$$\begin{aligned}
&= \frac{\mathrm{i}}{\pi}\int_0^\infty \log^+ u\left(\frac{1}{u+\mathrm{i}|\alpha|}-\frac{1}{u-\mathrm{i}|\alpha|}\right)\mathrm{d}u \\
&= \frac{\mathrm{i}}{\pi}\int_0^1\int_s^1 \frac{\mathrm{d}t}{t}\left(\frac{1}{s+\frac{\mathrm{i}}{|\alpha|}}-\frac{1}{s-\frac{\mathrm{i}}{|\alpha|}}\right)\mathrm{d}s \\
&= \frac{\mathrm{i}}{\pi}\left(\mathrm{I}_2\left(-\frac{\mathrm{i}}{|\alpha|}:1\right)-\mathrm{I}_2\left(\frac{\mathrm{i}}{|\alpha|}:1\right)\right) \\
&= -\frac{\mathrm{i}}{\pi}(\mathrm{Li}_2(\mathrm{i}|\alpha|)-\mathrm{Li}_2(-\mathrm{i}|\alpha|))
\end{aligned}$$

Method (II)

1. Let $P(\mathbf{x}, z) = p(\mathbf{x}) + \alpha q(\mathbf{x}) z \in \mathbb{C}[\mathbf{x}, z]$ whose Mahler measure is a function of $\alpha \in \mathbb{C}$.

   For instance, $P = (1+x) + \alpha(1+y)z$.

   $\pi^2 m(P) = 2(\mathrm{Li}_3(\alpha) - \mathrm{Li}_3(-\alpha))$ for $\alpha \leq 1$.

2. We compute the Mahler measure of

$$\tilde{P} = \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_m}{1+x_m}\right) p(\mathbf{x})$$

$$+ \left(\frac{1-y_1}{1+y_1}\right) \dots \left(\frac{1-y_n}{1+y_n}\right) q(\mathbf{x}) z$$

   For example,
   $\tilde{P} = \frac{1-x_1}{1+x_1}(1+x) + \frac{1-y_1}{1+y_1}(1+y)z.$

3. We get combinations of

$$\int_{\mathbb{T}^*}\int_0^\infty\int_0^\infty \log\max\{x|p(\mathbf{x})|, y|q(\mathbf{x})|\}$$

$$\log^j x\frac{\mathrm{d}x}{x^2\pm 1}\log^k y\frac{\mathrm{d}y}{y^2\pm 1}\frac{\mathrm{d}\mathbf{x}}{\mathbf{x}}$$

by using

$$\int_0^\infty \frac{x\log^k x\mathrm{d}x}{(x^2+a^2)(x^2\pm 1)} = \frac{T_k(\log a)}{(a^2\mp 1)} \qquad T_k[x]\in\mathbb{Q}[x]$$

In the example,

$$\pi^4 m\left(\frac{1-x_1}{1+x_1}(1+x)+\frac{1-y_1}{1+y_1}(1+y)z\right)=$$

$$=\frac{8}{(2\mathrm{i})^2}\int_{\mathbb{T}^2}\int_0^\infty\int_0^{x_1}\log\max\{x_1|1+x|, y_1|1+y|\}$$

$$\frac{\mathrm{d}x_1}{x_1^2+1}\frac{\mathrm{d}y_1}{y_1^2+1}\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

4. But we know

$$\frac{1}{(2\mathrm{i})^*}\int_{\mathbb{T}^*}\log\max\{|p(\mathbf{x})|,\alpha|q(\mathbf{x})|\}\,\frac{\mathrm{d}\mathbf{x}}{\mathbf{x}}$$

$$=\pi^* m(p(\mathbf{x})+\alpha q(\mathbf{x})z)$$

Following the example,

$$\frac{1}{(2\mathrm{i})^2}\int_{\mathbb{T}^2}\log\max\{|1+x|,\alpha|1+y|\}\frac{\mathrm{d}x}{x}\,\frac{\mathrm{d}y}{y}$$

$$=-4\int_0^\alpha \frac{\mathrm{d}s}{s^2-1}\circ\frac{\mathrm{d}s}{s}\circ\frac{\mathrm{d}s}{s}$$

Then, we set $z=\frac{y_1}{x_1}$ and get $x_1\mathrm{d}z=\mathrm{d}y_1$.

$$=\frac{8}{(2\mathrm{i})^2}\int_{\mathbb{T}^2}\int_0^\infty\int_0^1(\log x_1+\log\max\{|1+x|,z|1+y|\})$$

$$\frac{x_1\mathrm{d}x_1}{x_1^2+1}\,\frac{\mathrm{d}z}{z^2x_1^2+1}\,\frac{\mathrm{d}x}{x}\,\frac{\mathrm{d}y}{y}$$

$$= 8 \int_0^\infty \int_0^1 \left(\log x_1 - 4 \int_0^z \frac{\mathsf{d}s}{s^2 - 1} \circ \frac{\mathsf{d}s}{s} \circ \frac{\mathsf{d}s}{s} \right)$$

$$\frac{x_1 \mathsf{d} x_1}{x_1^2 + 1} \frac{\mathsf{d}z}{z^2 x_1^2 + 1}$$

$$= 8\pi^2 \int_0^1 \frac{\log^2 z}{2(1 - z^2)} \mathsf{d}z$$

$$+ 8 \int_0^1 \left(-4 \int_0^z \frac{\mathsf{d}s}{s^2 - 1} \circ \frac{\mathsf{d}s}{s} \circ \frac{\mathsf{d}s}{s} \right) \frac{-\log z}{1 - z^2} \mathsf{d}z$$

$$= 7\pi^2 \zeta(3) + 8(\mathsf{Li}_{3,2}(1,1) - \mathsf{Li}_{3,2}(-1,1))$$

$$+ 8(\mathsf{Li}_{3,2}(1,-1) - \mathsf{Li}_{3,2}(-1,-1))$$

How does $\zeta(5)$ show up?

$\mathrm{Li}_{3,2}(\pm 1, \pm 1)$ are alternating Euler sums.

Use the formula (Borwein, Bradley and Broadhurst)

$$\begin{aligned}\mathrm{Li}_{3,2}(x,y) &= -\frac{1}{2}\mathrm{Li}_5(x\,y) + \mathrm{Li}_3(x)\,\mathrm{Li}_2(y) + 3\mathrm{Li}_5(x) \\ &\quad +2\mathrm{Li}_5(y) - \mathrm{Li}_2(x\,y)(\mathrm{Li}_3(x) + 2\mathrm{Li}_3(y))\end{aligned}$$

for $x, y = \pm 1$, together with

$$\mathrm{Li}_k(1) = \zeta(k) \quad \text{and} \quad \mathrm{Li}_k(-1) = \left(\frac{1}{2^{k-1}} - 1\right)\zeta(k)$$

We get

$$\mathrm{Li}_{3,2}(1,1)-\mathrm{Li}_{3,2}(-1,1)+\mathrm{Li}_{3,2}(1,-1)-\mathrm{Li}_{3,2}(-1,-1)$$

$$=-\frac{21}{4}\zeta(2)\zeta(3)+\frac{93}{8}\zeta(5)$$

We obtain the result by using that $\zeta(2)=\frac{\pi^2}{6}$