On the expected discounted penalty function for a perturbed risk process driven by a subordinator

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Abstract


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1. Introduction

We consider a generalization of the perturbed risk model of Dufresne and Gerber (1991). We substitute the compound Poisson process by a subordinator. Subordinators are generalizations of compound Poisson processes, as we will see later on in the introduction. This substitution yields

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\[ U(t) = u + c t - S(t) + W(t), \quad t \geq 0, \]  

where \( S \) is a subordinator with zero drift and Lévy measure \( \nu \) and \( W \) is a zero-drift Brownian motion with infinitesimal variance \( \sigma^2 \). The parameter \( u \) is the initial surplus and \( c \) is a constant premium rate defined as \( c = (1 + \theta) \mathbb{E}[S(1)] \), where \( \theta \) is the security loading factor. For an account on the classical risk model we refer to Asmussen (2000) or Kaas et al. (2001).

The purpose of this note is to show that the Expected Discounted Penalty Function (EDPF) for the risk model in (1) satisfies a Defective Renewal Equation (DRE) that generalizes equivalent results in Gerber and Landry (1998) and Tsai and Willmot (2002). We note that (1) is a process with independent and stationary increments, i.e. a Lévy process. Hitting times, over- and under-shoots for Lévy processes have been extensively studied in an insurance context (or at least with insurance modeling in view). For example, the model in (1) is a subclass of the family of risk models discussed in Bertoin and Doney (1994), Yang and Zhang (2001), Morales and Schoutens (2003), Huzak et al. (2004) and Doney and Kyprianou (2006). It is slightly more general than models discussed in Dufresne et al. (1991) and Morales (2004) since we are adding a Brownian motion to perturb the aggregate claim process. Yet, expressions for the EDPF equivalent to those found in Tsai and Willmot (2002) remain unexplored in setting (1).

In Garrido and Morales (2006) we studied the EDPF for risk models where the compound Poisson sum in the aggregate claims process has been substituted by a subordinator. Here, we work out an extension for a risk process driven by a subordinator and perturbed by a Brownian motion.

We start by giving a characterization of a subordinator. Subordinators form the subclass of nondecreasing Lévy processes [see Bertoin (1996) or Sato (1999) for accounts on Lévy processes and subordinators]. The Laplace transform of a subordinator \( \phi_t(s) = \mathbb{E}(e^{-sX(t)}) \) is of the form \( e^{-\psi(s)} \), where \( \psi \) is the so-called Laplace exponent, which can be written as

\[ \psi(s) = a s + \int_0^\infty (1 - e^{-s x}) \nu(dx), \quad s > 0, \tag{2} \]

with \( a \in \mathbb{R} \) and \( \nu \) a positive measure on \( \mathbb{R}_+ \) satisfying \( \int_0^\infty (1 \wedge x) \nu(dx) < \infty \).

Eq. (2) characterizes the family of all subordinators. Alternatively, if we define the tail of the Lévy measure as \( \Pi(x) = \int_x^\infty \nu(dx) \), we can rewrite (2) as [see Bertoin (1996)]

\[ \frac{\psi(s)}{s} = a + \int_0^\infty e^{-s x} \Pi(x) dx, \quad s > 0. \tag{3} \]

One example of a subordinator which has always been present in ruin theory is the compound Poisson process which can be obtained by setting \( \nu(dx) = \lambda dF(x) \) and \( a = 0 \) in (2). This yields a compound Poisson aggregate claims process with rate arrival \( \lambda \) and severity distribution \( F \). Other examples of subordinators are the gamma, inverse Gaussian and generalized inverse Gaussian processes [see Morales (2004) and Garrido and Morales (2006) for further discussion].

Before continuing with our discussion we need to state some definitions.

**Definition 1.1.** We define the ruin time \( \tau \) associated with the risk process (1) as \( \tau = \inf\{t \geq 0 ; \; U(t) < \infty\} \).

**Definition 1.2.** We define the ruin probability \( \psi \) associated with the risk process (1) as \( \psi(u) = \mathbb{P}\{\tau < \infty \mid U(0) = u\} \).

Now, since our model (1) belongs to the class studied in Huzak et al. (2004) we have that, by Theorem 3.2 in Garrido and Morales (2006), its associated ruin probability satisfies

\[ 1 - \psi(u) = \frac{\theta}{1 + \theta} \sum_{n=0}^{\infty} \left( \frac{1}{1 + \theta} \right)^n M^{*n} \ast K^{*(\alpha + 1)}(u), \quad u \geq 0, \tag{4} \]

where \( M \) is a sort of ladder-height distribution, with Laplace transform given by \( \hat{m}(s) = \int_0^\infty e^{-s x} dM(x) = (s \mathbb{E}[S(1)])^{-1} \psi_S(s) \), and \( K \) is the distribution with Laplace transform given by \( \hat{k}(s) = \int_0^\infty e^{-s x} dK(x) = c s / \psi_{c+W}(s) \). The functions \( \psi_S \) and \( \psi_{c+W} \) are, respectively, the Laplace exponents of the subordinator \( S \) and of the process \( U(t) - u + S(t) = ct + W(t) \) in (1).
From (3), we see that
\[
\hat{m}(s) = \frac{\psi_S(s)}{s \mathbb{E}[S(1) + 1]} = \int_0^\infty e^{-sx} \frac{\Pi(x)}{\int_0^\infty \Pi(t) \, dt} \, dx, \quad s > 0,
\]
which allows us to identify the ladder-height-like density
\[
m(x) = \frac{\Pi(x)}{\int_0^\infty \Pi(t) \, dt}.
\]
Eq. (5) expresses \( \hat{m} \) in terms of the tail of the Lévy measure \( \Pi \) of the subordinator \( S \). Notice that if \( S \) is a compound Poisson process then \( M \) in (4) is the ladder-height transformation of the severity distribution. Notice also that \( K \) in (4) is an exponential distribution with mean \( \sigma^2/2c \).

Gerber and Shiu (1997, 1998a) introduced the concept of discounted penalty function as a mean to study the distribution of the time to ruin, the amount at and immediately prior to ruin. Later, Gerber and Landry (1998) and Tsai and Willmot (2002) considered a generalized EDPF defined by
\[
\phi_D(u) = w_0 \phi_d(u) + \phi_c(u), \quad u \geq 0,
\]
where
\[
\phi_d(u) = \mathbb{E} \left[ e^{-\delta \tau} \mathbb{I}_{\{\tau < \infty, U(\tau) = 0\}} | U(0) = u \right]
\]
and
\[
\phi_c(u) = \mathbb{E} \left[ w(\tau_{\tau\downarrow}, |U(\tau)|) e^{-\delta \tau} \mathbb{I}_{\{\tau < \infty, U(\tau) < 0\}} | U(0) = u \right].
\]
The function \( w \) is a non-negative function defined on \( \mathbb{R}_+^2 \cup \{(0, 0)\} \) and \( \tau \) is the time of ruin. In this definition there is a constant penalty \( w_0 \) if ruin is caused by the perturbation; in fact \( w_0 = w(0, 0) \). We refer to the original papers of Gerber and Shiu (1998a, b) for a thorough discussion on all the relevant features of the EDPF.

When the aggregate claims in (1) form a compound Poisson process, i.e., \( S(t) = \sum_{n=0}^{N(t)} Y_n \), where \( N \) is a Poisson point process with intensity \( \lambda \) and \( Y_i \sim F \), Tsai and Willmot (2002) showed that the EDPF \( \phi_D \) follows a DRE [Theorem 2 in their paper].

**Remark 1.1.** Recently, Sarkar and Sen (2005) used a weak convergence argument to show that the conditions on \( \phi \) in Theorem 2 in Tsai and Willmot (2002) can be replaced by a condition on the function \( w \), namely
\[
|w(x, y) - w_0| \leq \alpha (x + y)^\beta \quad \text{for } x, y \geq 0 \text{ and some } \alpha > 0 \text{ and } \beta > 1.
\]

In this note, we show that the ruin probability associated with a risk model as in (1) satisfies a DRE that generalizes the equivalent equation in Theorem 2 in Tsai and Willmot (2002). We obtained this result by constructing a family of compound Poisson processes that converges weakly to the subordinator \( S \). The main result and its derivation are presented in Section 2.

### 2. DRE for the perturbed subordinator model

We start our discussion by constructing a family of compound Poisson processes \( \{S_\epsilon\}_{\epsilon > 0} \) converging weakly to any given subordinator \( S \). This will be the stepping stone to produce our main result.

#### 2.1. Constructing a weakly converging family to any given subordinator

Let \( S \) be a subordinator, as defined by (2), with zero drift and Lévy measure \( \nu \). Now, although \( S \) has an infinite number of small jumps, its jumps larger than a certain threshold form a compound Poisson process. Therefore we can construct a family of classical risk processes \( \{U_\epsilon\}_{\epsilon > 0} \) as follows:
\[
U_\epsilon(t) = u + c_\epsilon t - S_\epsilon(t), \quad t \geq 0,
\]
where $S_\varepsilon$ is a compound Poisson process with jump density

$$f_\varepsilon(x) = \frac{v(x)}{\int_\varepsilon^\infty v(s)ds}I_{(\varepsilon,\infty)}(x), \tag{8}$$

and rate

$$\lambda_\varepsilon = \int_\varepsilon^\infty v(s)ds. \tag{9}$$

Clearly, its Lévy measure is given by $v_\varepsilon(dx) = \lambda_\varepsilon f_\varepsilon(x)dx = v(dx)I_{(\varepsilon,\infty)}(x)$, i.e. it is simply the restriction of the Lévy measure of the original process $S$ to the interval $(\varepsilon, \infty)$. The loaded premium is

$$c_\varepsilon = (1 + \theta)\lambda_\varepsilon \mu_\varepsilon = (1 + \theta) \int_\varepsilon^\infty s v(s)ds = (1 + \theta) \left[ s \Pi(s) + \int_\varepsilon^\infty \Pi(s)ds \right]. \tag{10}$$

where $\mu_\varepsilon$ is the mean of $f_\varepsilon$ and $\Pi(s)$ is the integrated tail of the Lévy measure $v$. Notice that the integral in the right-hand side of the above equation is finite because $S$ is a subordinator and therefore $\int_0^\infty (1 \wedge x) v(dx) < \infty$.

**Lemma 2.1.** The family of risk processes $\{U_\varepsilon\}_{\varepsilon>0}$ defined in (7) converges weakly to the process $U$, i.e. we have that $U_\varepsilon(t) \xrightarrow{d} U(t)$ for all $t \geq 0$ and therefore $U_\varepsilon \xrightarrow{d} U$ as $\varepsilon \to 0$.

**Proof.** The proof is straightforward. Consider the Laplace exponent of $S_\varepsilon$ denoted by $\hat{\Psi}_{S_\varepsilon}$; since $S_\varepsilon$ is a subordinator with zero drift it has the form given in (2), i.e.

$$\hat{\Psi}_{S_\varepsilon}(s) = \int_0^\infty (e^{-sx} - 1) v_\varepsilon(dx), \quad s \geq 0.$$ 

We can see that $\{e^{-sx} - 1\}_{n>0}$ form a decreasing sequence of Lebesgue-measurable functions and by monotone convergence

$$\hat{\Psi}_{S_\varepsilon}(s) \to \int_0^\infty (e^{-sx} - 1) v(dx) = \hat{\Psi}_S(s),$$

for all $s \geq 0$, where $\hat{\Psi}_S$ is the Laplace exponent of the subordinator $S$.

Now since the Laplace transform of a subordinator $S$ is given by $\hat{S}(t)(s) = \mathbb{E}[e^{-sS(t)}] = e^{t\hat{\Psi}_S(t)}$ for all $t \geq 0$, we can see that $\hat{S}_\varepsilon(t) \to \hat{S}(t)$. This implies that $S_\varepsilon(t) \xrightarrow{d} S(t)$ for all $t \geq 0$ and therefore $S_\varepsilon \xrightarrow{d} S$. Now, since $c_\varepsilon \to c$ we have that $U_\varepsilon(t) \xrightarrow{d} U(t)$ for all $t \geq 0$ and therefore $U_\varepsilon \xrightarrow{d} U$. \quad \square

Now, we use this converging family of classical risk processes to derive a DRE for the EDPF in the perturbed subordinator case.

### 2.2. Results for the generalized EDPF

In order to prove our main theorem we need the following result stated here as a lemma. Recall that in this paper we denote by $f$ the Laplace transform of a function $f$.

**Lemma 2.2.** For $\delta \geq 0$, when $\varepsilon \to 0$ the equation

$$c_\varepsilon r + \frac{\sigma^2}{2} r^2 + \lambda_\varepsilon \int_0^\infty e^{-ry} f_\varepsilon(y) = \lambda_\varepsilon + \delta,$$

converges to the generalized Lundberg equation associated with the risk model in (1), namely

$$cr + \hat{\Psi}_{S-W}(r) = \delta,$$

where $\hat{\Psi}_{S-W}$ is the Laplace exponent of the perturbed aggregate claims process $S - W$ in (1).
Proof. Notice first that, in view of Eqs. (8) and (9), we can write
\[ c_\varepsilon r + \frac{\sigma^2}{2} r^2 + \lambda_\varepsilon \int_0^\infty \left[ e^{-ry} - 1 \right] f_\varepsilon(y) \, dy = c_\varepsilon r + \Psi_{S_r-W}(r) - \delta. \]

The right-hand side is obtained by noticing that the Laplace exponent of \( S_r-W \) is precisely \( \Psi_{S_r-W}(r) = \frac{\sigma^2}{2} r^2 + \lambda_\varepsilon \int_0^\infty \left[ e^{-ry} - 1 \right] f_\varepsilon(y) \, dy. \)

Now, since we have established that \( S_r \to S \), therefore
\[ c_\varepsilon r + \Psi_{S_r-W}(r) - \delta \to c_\varepsilon r + \Psi_{S-W}(r) - \delta, \]
where \( \Psi_{S-W}(r) = \frac{\sigma^2}{2} r^2 + \int_0^\infty \left[ e^{-ry} - 1 \right] \nu(dy). \]

The following is our main result:

**Theorem 2.1.** Consider the risk process defined in (1) and let \( \Phi_D \) denote its associated EDPF. If \( |w(x, y) - w_0| \leq \alpha(x + y)^\beta \) for \( x, y \geq 0 \) and for some \( \alpha > 0 \) and \( \beta > 1 \), then \( \Phi_D \) satisfies the DRE
\[ \Phi_D(u) = \int_0^u \Phi_D(u-y) g_D(y) \, dy + w_0 e^{-\rho u} \left[ 1 - K(u) \right] + H_w(u), \quad u \geq 0, \]
where
\[ g_D(y) = \frac{1}{1 + \theta} \int_0^y e^{-\rho(y-s)} k(y-s) \int_s^\infty e^{-\rho(x-s)} m(x) \, dx, \]
\[ H_w(u) = \frac{1}{1 + \theta} \int_0^u e^{-\rho(u-s)} k(u-s) \int_s^\infty e^{-\rho(x-s)} \chi(x) \, dx \, ds, \]
\[ \chi(x) = \int_x^\infty w(x, y-x) m(y), \quad x \geq 0, \]
\[ m(x) = \int_x^\infty \frac{\nu(s)}{\kappa(z)} \, dx, \quad x \geq 0, \]
\( \rho \) is the unique non-negative solution of the generalized Lundberg equation
\[ cr + \Psi_{S-W}(r) = \delta \quad \text{with} \quad \rho = 0 \quad \text{when} \quad \delta = 0, \]
and \( K(k) \) is an exponential distribution (density) with mean \( \sigma^2/2c \), i.e. \( K(x) = 1 - e^{-(2c/\sigma^2)u} \).

**Proof.** Let \( \{U_\varepsilon\}_{\varepsilon>0} \) be the family of classical risk processes as defined in (7) and let their associated family of EDPF’s be denoted by \( \{\Phi_{D_\varepsilon}\}_{\varepsilon>0} \). Lemma 2.1 shows that \( U_\varepsilon \overset{d}{\to} U \) as \( \varepsilon \to 0 \). We now show that the family of EDPF’s \( \{\Phi_{D_\varepsilon}\}_{\varepsilon>0} \) converge to the EDPF in Theorem 2.1, i.e. \( \Phi_{D_\varepsilon} \to \Phi_D \) as \( \varepsilon \to 0 \). In order to show this we use the Laplace transforms.

Now, we start by showing that the Laplace transforms of the functions in Theorem 2.1 are given by
\[ \hat{\Phi}_D(s) = \frac{\hat{H}_w(s)}{1 - \hat{g}_D(s)} + \frac{w_0}{\rho + \frac{2c}{\sigma^2}} \left[ 1 - \hat{g}_D(s) \right], \quad s \geq 0, \]
\[ \hat{g}_D(s) = \frac{1}{1 + \theta} \left[ s \hat{m}(s) - \rho \hat{m}(\rho) \right] \frac{2c}{\sigma^2}, \quad s \geq 0, \]
\[ \hat{H}_w(s) = \frac{1}{1 + \theta} \left[ \hat{\chi}(s) - \hat{\chi}(\rho) \right] \frac{2c}{\sigma^2}, \quad s \geq 0, \]
and
\[ \hat{\chi}(s) = \int_0^\infty e^{-sx} \left[ \int_x^\infty w(x, y-x) m(y) \right] \, dx, \quad s \geq 0. \]

These are obtained by a straightforward calculation. In a bottom-to-top order we have:
(i) The expression for \( \hat{\chi} \) holds by the definition of the Laplace transform of \( \chi \).
(ii) Notice that \( H_w \) can be written as the convolution of two functions,
\[ H_w(x) = \frac{1}{1 + \theta} e^{-\theta x} \gamma * \eta(x), \quad x \geq 0, \]

where \( \gamma(x) = e^{-\rho x} k(x) \) and \( \eta(x) = \int_{-\infty}^{\infty} e^{-\rho(z-x)} \chi(z) dz \). This implies that

\[ \mathcal{H}_w(s) = \frac{1}{1 + \theta} \mathcal{P}(s) \mathcal{T}(s), \quad s \geq 0. \tag{11} \]

Now, we only have to compute the Laplace transforms of these functions. By changing the order of integration we can see that

\[
\begin{align*}
\widehat{\eta}(s) &= \int_0^{\infty} e^{-sx} \left[ \int_x^{\infty} e^{-\rho(z-x)} \chi(z) dz \right] dx \\
&= \int_0^{\infty} \left[ \int_0^{z} e^{-x(s-\rho)} dx \right] e^{-\rho z} \chi(z) dz = \frac{\widehat{\xi}(s) - \widehat{\rho}(\rho)}{\rho - s}, \quad s \geq 0.
\end{align*}
\]

Similarly, using the fact that \( k \) is an exponential density we have that

\[
\begin{align*}
\widehat{\gamma}(s) &= \int_0^{\infty} e^{-sx} \left[ e^{-\rho x} k(x) \right] dx = \widehat{k}(s + \rho) \\
&= \frac{2e}{\alpha^2 + \rho + s}, \quad s \geq 0.
\end{align*}
\]

If we substitute \( \widehat{\gamma} \) and \( \widehat{\eta} \) into (11) we find the expression for \( \mathcal{H}_w \).

(iii) Notice that \( g_D \) can also be written as the convolution of two functions,

\[ g_D(x) = \frac{1}{1 + \theta} \gamma * \xi(x), \quad x \geq 0, \]

where \( \gamma(x) \) is the one defined in (ii) and \( \xi(x) = \int_{-\infty}^{\infty} e^{-\rho(x-s)} dm(x) \). This implies that

\[ \widehat{g}_D(s) = \frac{1}{1 + \theta} \widehat{\gamma}(s) \widehat{\xi}(s), \quad s \geq 0. \tag{12} \]

Now, we only have to compute the Laplace transforms of function \( \xi \). By changing the order of integration we can see that

\[
\begin{align*}
\widehat{\xi}(s) &= \int_0^{\infty} e^{-sx} \left[ \int_x^{\infty} e^{-\rho(z-x)} dm(z) \right] dx \\
&= \int_0^{\infty} \left[ \int_0^{z} e^{-x(s-\rho)} dx \right] e^{-\rho z} dm(z) = \frac{s\widehat{m}(s) - \rho\widehat{m}(\rho)}{\rho - s}, \quad s \geq 0.
\end{align*}
\]

If we substitute \( \widehat{\gamma} \) and \( \widehat{\xi} \) into (12) we find the expression for \( g_D \).

(iv) Finally, notice that \( \phi_D \) can also be written in terms of convolutions. We first remark that the DRE for \( \phi_D \) in Theorem 2.1 has a unique solution given by

\[ \phi_D(x) = \left[ H_w(x) + w_0 e^{-\rho x} [1 - K(x)] \right] * \sum_{n=0}^{\infty} g_D^{sn}(x), \quad x \geq 0, \]

i.e. it is the convolution of two functions \( a(x) = H_w(x) + w_0 e^{-\rho x} [1 - K(x)] \) and \( b(x) = \sum_{n=0}^{\infty} g_D^{sn}(x) \). This implies that

\[ \widehat{\phi}_D(s) = \widehat{a}(s) \widehat{b}(s), \quad s \geq 0. \tag{13} \]

Now, we only have to compute the Laplace transforms of the functions \( a \) and \( b \). Clearly, since \( K \) is an exponential distribution we have

\[
\begin{align*}
\widehat{a}(s) &= \int_0^{\infty} e^{-sx} \left[ H_w(x) + w_0 e^{-\rho x} [1 - K(x)] \right] dx \\
&= \mathcal{H}_w(s) + \frac{\sigma^2 w_0}{2\alpha} \int_0^{\infty} e^{-sx} \left[ e^{-\rho x} k(x) \right] dx
\end{align*}
\]
\[
\hat{H}_w(s) + \frac{\sigma^2 w_0}{2c} k(s + \rho) = \hat{H}_w(s) + \frac{w_0}{\rho + \frac{2c}{\sigma^2} + s}, \quad s \geq 0,
\]

and
\[
\hat{b}(s) = \int_0^\infty e^{-sx} \left[ \sum_{n=0}^\infty \hat{g}_{D}^{en}(x) \right] dx = \sum_{n=0}^\infty \left[ \hat{g}_D(s) \right]^n = \frac{1}{1 - \hat{g}_{D}(s)}, \quad s \geq 0.
\]

If we substitute \( \hat{a} \) and \( \hat{b} \) into (13) we find the expression for \( \hat{\phi}_D \).

Now, we need to show that \( \hat{\chi}_\varepsilon \to \hat{\chi} \), \( \hat{H}_w \to \hat{H}_w \), \( \hat{g}_{D,e} \to \hat{g}_{D} \) and finally \( \hat{\phi}_{D,e} \to \hat{\phi}_D \) as \( \varepsilon \to 0 \). This would prove that \( \phi_{D,e} \to \phi_D \) as \( \varepsilon \to 0 \).

We start with the function \( \hat{\chi} \). We can apply Lemma 2 in Sarkar and Sen (2005) to the elements \( S_e \) of the family (7) since these are compound Poisson processes. From Eqs. (8) and (9) we have
\[
\frac{\lambda_{c,e}}{c_e} \hat{\chi}_{\varepsilon}(s) = \frac{1}{1 + \theta} \int_0^\infty e^{-sx} \left[ \int_x^\infty w(x, y - x) \frac{\lambda_{c} f_{\varepsilon}(y)}{\int_0^\infty \Pi(z)dz} dy \right] dx, \quad s \geq 0.
\]

By Lemma 2.1 and by substituting Eqs. (8)–(10) into (14) we have that
\[
\lim_{\varepsilon \to 0} \frac{\lambda_{c}}{c_e} \hat{\chi}_{\varepsilon}(s) = \frac{1}{1 + \theta} \int_0^\infty e^{-sx} \left[ \int_x^\infty w(x, y - x) \frac{\psi(y)}{\int_0^\infty \Pi(z)dz} dy \right] dx, \quad s \geq 0.
\]

From (5), we have
\[
\lim_{\varepsilon \to 0} \frac{\lambda_{c}}{c_e} \hat{\chi}_{\varepsilon}(s) = \frac{1}{1 + \theta} \int_0^\infty e^{-sx} \left[ \int_x^\infty w(x, y - x) dm(y) \right] dx = \hat{\chi}(s), \quad s \geq 0.
\]

We move on to function \( \hat{H}_w \), applying again Lemma 2 in Sarkar and Sen (2005) to the elements \( S_e \) and from Eqs. (8) and (9) we have that
\[
\hat{H}_w(s) = \frac{\lambda_{c} \left[ \hat{\chi}_{\varepsilon}(s) - \hat{\chi}_{\varepsilon}(\rho_e) \right]}{1 + \theta} \left[ \frac{\sigma^2}{\tau^2} (\rho_e - s)(\rho_e + \frac{2c}{\sigma^2} + s) \right], \quad s \geq 0.
\]

By Lemmas 2.1 and 2.2 and our previous result on \( \hat{\chi}_{\varepsilon} \) we have
\[
\lim_{\varepsilon \to 0} \hat{H}_w(s) = \frac{\lambda_{c}}{1 + \theta} \left[ \frac{\sigma^2}{\tau^2} (\rho_e - s)(\rho_e + \frac{2c}{\sigma^2} + s) \right] \hat{H}_w(s), \quad s \geq 0.
\]

We now keep working our way up to function \( \hat{g}_{D,e} \), applying one more time Lemma 2 in Sarkar and Sen (2005) to the elements \( S_e \), and from Eqs. (8) and (9) we can write
\[
\hat{g}_{D,e}(s) = \frac{\lambda_{c} \left[ f_{\varepsilon}(s) - 1 - f_{\varepsilon}(\rho_e) + 1 \right]}{1 + \theta} \left[ \frac{\sigma^2}{\tau^2} (\rho_e - s)(\rho_e + \frac{2c}{\sigma^2} + s) \right] = \frac{1}{1 + \theta} \left[ \frac{\psi_{S,e}(s) - \psi_{S,e}(\rho_e)}{\sigma^2 (\rho_e - s)(\rho_e + \frac{2c}{\sigma^2} + s)} \right], \quad s \geq 0.
\]

Let \( m_e \) be the ladder-height density for the process \( S_e \); then from representation (4) we have that \( \hat{m}_e(s) = \frac{\psi_{S,e}(s)}{s \hat{E}[S_e(1)]} \).

Substituting this expression for \( \hat{m}_e \) and (10) yields
\[
\hat{g}_{D,e}(s) = \frac{1}{1 + \theta} \left[ \frac{\sigma^2}{\tau^2} (\rho_e - s)(\rho_e + \frac{2c}{\sigma^2} + s) \right] \hat{m}_e(s), \quad s \geq 0.
\]

By Lemmas 2.1 and 2.2 and by substituting (8)–(10) into (15) we have that
\[
\lim_{\varepsilon \to 0} \hat{g}_{D,e}(s) = \frac{1}{1 + \theta} \left[ \frac{\sigma^2}{\tau^2} (\rho_e - s)(\rho_e + \frac{2c}{\sigma^2} + s) \right] \hat{g}_{D}(s), \quad s \geq 0.
\]
Finally, using Lemma 2 in Sarkar and Sen (2005) to the elements $S_\epsilon$ we can write
\[
\hat{\phi}_{D_\epsilon}(s) = \frac{\hat{H}_{w_\epsilon}(s)}{1 - \hat{g}_{D_\epsilon}(s)} + \frac{w_0}{(\rho_\epsilon + \frac{2c}{\sigma^2} + s)[1 - \hat{g}_{D_\epsilon}(s)]}, \quad s \geq 0.
\]

By Lemmas 2.1 and 2.2 and by our previous results on $\hat{\chi}_\epsilon$, $\hat{H}_{w_\epsilon}$ and $\hat{g}_{D_\epsilon}$ we have that
\[
\lim_{\epsilon \to 0} \hat{\phi}_{D_\epsilon}(s) = \frac{\hat{H}_w(s)}{1 - \hat{g}_D(s)} + \frac{w_0}{(\rho + \frac{2c}{\sigma^2} + s)[1 - \hat{g}_D(s)]} = \hat{\phi}_D(s), \quad s \geq 0.
\]
This completes the proof. □

The following is an interesting corollary that brings insight on the role played by the functions $K$ and $M$ in the DRE of Theorem 2.1.

**Corollary 2.1.** Consider the risk process defined in (1) and let $\phi_D$ denote its associated EDPF. Then, the Laplace transforms of the functions $\phi_D$, $g_D$ and $H_w$ in Theorem 2.1 are given respectively by
\[
\hat{\phi}_D(s) = \frac{\hat{H}_w(s) + w_0\hat{A}(s)}{1 - \hat{g}_D(s)}, \quad s \geq 0,
\]
\[
\hat{g}_D(s) = \frac{[\Psi_S(s) - \Psi_S(\rho)](s + \rho)}{(\rho - s)\Psi_{c+W}(s + \rho)}, \quad s \geq 0,
\]
\[
\hat{H}_w(s) = \frac{[\hat{\chi}_v(s) - \hat{\chi}_v(\rho)](s + \rho)}{(\rho - s)\Psi_{c+W}(s + \rho)}, \quad s \geq 0,
\]
where
\[
\hat{\chi}_v(s) = \int_0^\infty e^{-sx} \left[ \int_x^\infty w(x, y - x)v(dy) \right] dx, \quad s \geq 0,
\]
\[
\hat{A}(x) = e^{-\rho x}[1 - K(x)], \quad x \geq 0,
\]
$\Psi_S$ is the Laplace exponent of $S$, $\Psi_{c+W}$ is the Laplace exponent of $c+W$ and $v$ is the Lévy measure of $S$. Moreover, the functions $k$ and $m$ in Theorem 2.1 are the density functions appearing in the ruin probability decomposition (4).

This corollary allows us to identify the EDPF in terms of the Laplace exponent of the processes that form the risk model. This is relevant since this exponent characterizes any model of the form (1).

**Proof.** In the proof of Theorem 2.1 we already linked the function $m$ to that appearing in (4). In order to make the role of $K$ visible, recall that $\hat{k}(s + \rho) = \frac{2c}{\rho + s + 2\sigma^2}$. By Theorem 3.2 in Garrido and Morales (2006), the function $K$ appearing in the ruin probability decomposition is an exponential distribution with mean $\sigma^2/2c$. This can be seen from the characterization of $K$ in (4), i.e.
\[
\hat{k}(s) = \frac{cs}{\Psi_{c+W}(s)} = \frac{cs}{cs + \frac{\sigma^4}{2}s^2} = \frac{2c}{\sigma^2 + s}.
\]
Thus, the Laplace transform $\hat{k}$ evaluated at $\rho + s$ is given by
\[
\hat{k}(s + \rho) = \frac{c(s + \rho)}{\Psi_{c+W}(s + \rho)} = \frac{2c}{\sigma^2 + s + \rho},
\]
and therefore $K$ is given by Theorem 3.2 in Garrido and Morales (2006) when the perturbation is a Brownian motion.

As for the new expressions for the Laplace transforms in Corollary 2.1 we encountered in the proof of Theorem 2.1 that
\[
\hat{A}(s) = \int_0^\infty e^{-sx} e^{-\rho x}[1 - K(x)]dx = \frac{1}{\rho + \frac{2c}{\sigma^2} + s},
\]
which proves the expression for $\phi_D$. The other two expressions are obtained by a simple substitution of

$$\hat{m}(s) = \frac{\Psi_S(s)}{sE[S(1)]} = \frac{(1 + \theta) \Psi_S(s)}{sc},$$

$$\hat{k}(s) = \frac{cs}{\Psi_{c+W}(s)},$$

and

$$dm(x) = \frac{\nu(dx)}{\int_0^\infty \Pi(s)ds} = \frac{1 + \theta}{c} \nu(dx),$$

into the corresponding expressions for $\hat{g}_D$, $\hat{H}_w$ and $\hat{\chi}$ appearing in the proof of Theorem 2.1. □

### 3. Conclusions

We have generalized results in Tsai and Willmot (2002) for the EDPF in the perturbed case. We substituted the compound Poisson process by a subordinator and showed that it still satisfies a DRE (Theorem 2.1). Our results are more general and allow for a wider range of models for the aggregate claims process, in particular those for which closed-form expressions are available like the gamma and inverse Gaussian processes. These results are possible due to the Lévy structure of the perturbed risk process. An interesting topic of further research is the generalization of Theorem 2.1 and Corollary 2.1 to a model where the perturbation is a spectrally negative Lévy process. We have reasons to conjecture that a more general version of Theorem 2.1 holds for such a general perturbed model where the functions $K$ and $M$ are those appearing in Theorem 3.2 in Garrido and Morales (2006).

### References