NORMALIZABLE, INTEGRABLE AND LINEARIZABLE SADDLE POINTS IN COMPLEX QUADRATIC SYSTEMS IN $\mathbb{C}^2$

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Abstract. In this paper we consider complex differential systems in the neighborhood of a singular point with eigenvalues in the ratio $1 : -\lambda$ with $\lambda \in \mathbb{R}^{+*}$. We address the questions of normalizability (i.e. convergence of the normalizing transformation), integrability and linearizability of the system. We introduce the notion of isochronicity of a system at an integrable saddle for general λ and prove that a system is linearizable if and only if it is isochronous. We then specialize to quadratic systems and give explicit examples of non-normalizable quadratic systems as well as quadratic systems which are integrable but not linearizable, for any λ satisfying a convenient diophantine condition. A distinction between normalizable and orbitally normalizable systems is also drawn along similar lines.

Our main interest is the global organization of the strata of those systems for which the normalizing transformations converge, or for which we have integrable or linearizable saddles as λ and the other parameters of the system vary. We give several tools for demonstrating normalizability, integrability and linearizability and apply them to the detailed study of several classes of quadratic systems. Many of the results are valid in the larger context of polynomial or analytic vector fields. We explain certain features of the *bifurcation diagram*, namely the existence of a continuous skeleton of integrable (linearizable systems) with sequences of holes filled with orbitally normalizable (normalizable) systems.

1. Introduction

In this paper we study the question of normalizability, integrability and linearizability (cf. Definitions 1.1) for the analytic system

$$\begin{aligned} \dot{x} &= x + f(x,y) = x + o(x,y) \\ \dot{y} &= -\lambda y + g(x,y) = -\lambda y + o(x,y) \end{aligned} \tag{1.1}$$

in $\mathbb{C}^2$ with a saddle at the origin i.e. $\lambda > 0$. The aim of this paper is two-fold.

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Firstly most of the known results on (1.1) study the geometry of the foliation defined by the solutions of (1.1) and ask if the foliation can be transformed to the foliation of a linear system or of a resonant system in normal form. We call this the integrability and orbital normalizability problem. These results do not take into account the time dependence of solutions of (1.1). We address in this paper the question of integrability when λ is irrational and orbital normalizability when λ is rational.

In this paper we also examine the problem of conjugacy either to the linear model or to the resonant model. We call these the linearizability and normalizability problems.

We introduce the notion of *isochronicity* expressed in terms of the integral of some *time form* and prove that it is precisely the isochronicity condition that makes an integrable system linearizable and a nonintegrable orbitally normalizable system normalizable to a particular normal form: the "resonant model". These results generalize the classical equality between isochronous systems and linearizable systems in the case of $1 : -1$ resonance.

We establish the diophantine condition on λ necessary and sufficient for linearizability of any integrable system with the fixed eigenvalue ratio $-\lambda$. We also provide explicit polynomial (usually quadratic) examples giving the difference between various notions.

We believe that the more general study of time dependence for integrable and normalizable systems enriches their understanding just as the study of isochronous centers enriches the study of the centers.

Most studies of the system (1.1) where f and g are analytic functions are concerned with the case where λ is fixed. A second goal of our paper is to understand how the strata of integrable, linearizable and normalizable saddles are organized globally when λ, f and g vary. For that purpose, we restrict some parts of our study to the study of the space of all polynomial systems (1.1) of a fixed degree with varying λ and examine the strata of integrable, linearizable or (orbitally) normalizable systems in this finite parameter space. It is well known, for instance, that for rational λ the set of integrable systems is algebraic; for good irrational λ all systems are integrable; for bad irrational λ the parameters for which the system is integrable form a very complicated set having probably some fractal structure. It seems very challenging to understand the structure of these sets.

There does not seem a priori to be any continuity with respect to λ of the set of integrable systems and the algorithm that seeks integrable or linearizable systems for $\lambda = \frac{p}{q}$ depends in an essential way of the chosen values of p and q and can in no way be made continuous. However, in examples developed here or in [Z], we observed continuous families of integrable systems depending on λ. We introduce and examine techniques leading to such continuous families : Darboux integrability and linearizability, and blow-down to a node.

Even though these techniques are simple, they are sufficiently rich to prove linearizability of all linearizable quadratic systems for $\lambda = 2$ studied in Section 7 and many of the strata thus obtained extend to other values of λ. The present work is a first step in the direction of understanding why our parameter space contains such a continuous "skeleton" of integrable and linearizable systems. We also prove a structure theorem for the space of linearizable or integrable systems.

The paper is organized as follows:

In the first section some of the main results are formulated: Theorem A gives the diophantine condition on λ assuring linearizability of an integrable system. Theorem B is a general structure theorem on the parameters giving integrability or linearizability. Theorems C and D prove the existence of quadratic systems which are nonintegrable or integrable and nonlinearizable under a convenient diophantine condition. Some open questions are formulated. These theorems are proved in Section 2.

In Section 3 (resp. 4) isochronicity condition is defined for integrable (resp. orbitally normalizable) systems. It is proved that this condition is necessary and sufficient for linearizability (resp. normalizability to the particular "resonant model"). This part might also be of independent interest, as it gives a local result characterizing relative exactness of a form in a neighborhood of a saddle point having a Darboux first integral in terms of vanishing of integrals along asymptotic cycles.

In Sections 5 and 6 two methods for proving integrability or normalizability are developed: the *Darboux method* which is here extended to treat normalizability and linearizability problems and *the blow-down to a node* method. The node is either linearizable or normalizable, yielding that the original system is either linearizable or normalizable in a neighborhood of the saddle point.

These methods are applied in Sections 7 and 8 to quadratic systems and Lotka-Volterra systems. We give in particular the complete list of linearizable quadratic systems for $\lambda = 2$ and show that many of these strata extend to strata defined for other values of λ. The necessary and sufficient conditions for integrability had been found in [FZ]. The dimension of the strata of integrable or linearizable centers drops for some rational values of λ, the problem coming from failure of integrability. In all the examples studied in quadratic systems the "missing parts" are filled with normalizable systems. Hence the known examples of normalizable and integrable systems are nicely organized as algebraic surfaces or half-surfaces (stopping at a value of λ) in the whole parameter space $\{\Lambda = (\lambda, c_{kl}, d_{kl})\}$ describing quadratic systems. A few other strata are found for $\lambda = 2$ which look isolated. Although our study of Lotka-Volterra systems is far from complete, it seems that the family of Lotka-Volterra is sufficiently rich to exhibit most general features that are expected from the general polynomial families: existence of non normalizable systems, existence of integrable and non linearizable systems, etc. We propose to go deeper in that direction in some future work.

We now give the basic definitions and known facts.

Definitions 1.1.

(1) *The system* (1.1) *is integrable at the origin if the form*
$$\omega = (-\lambda y + g(x,y))dx + (x + f(x,y))dy = 0 \tag{1.2}$$
is linearizable in a neighborhood of the origin, i.e. if and only if there exists an analytic change of coordinates
$$(X,Y) = (x + \phi(x,y), y + \psi(x,y)) = (x + o(x,y), y + o(x,y)) \tag{1.3}$$
bringing the system (1.1) *to the system*
$$\begin{aligned} \dot{X} &= Xh(X,Y) = X(1 + O(X,Y)) \\ \dot{Y} &= -\lambda Y h(X,Y) = -\lambda Y(1 + O(X,Y)). \end{aligned} \tag{1.4}$$

($X^\lambda Y$ is then a first integral of the type introduced by Dulac.) The definition is also valid for the case $\lambda = 0$.

(2) *The system is linearizable at the origin if and only if there exists an analytic change of coordinates* (1.3) *linearizing the system. The definition is also valid for the case $\lambda = 0$.*

(3) *For $\lambda = \frac{p}{q} \in \mathbb{Q}$ the system is normalizable at the origin if and only if there exists an analytic change of coordinates* (1.3) *bringing the system* (1.1) *to an analytic normal form*

$$\begin{aligned}\dot{X} &= Xk_1(U)\\ \dot{Y} &= -\lambda Y k_2(U),\end{aligned} \tag{1.5}$$

where $U = X^pY^q$ and $k_1(U), k_2(U)$ are analytic functions of U such that $k_1(0,0) = k_2(0,0) = 1$. The definition extends to the case $\lambda = 0$ by taking $U = Y$ in that case.

(4) *For $\lambda = \frac{p}{q} \in \mathbb{Q}$ the system is orbitally normalizable at the origin if the form* (1.2) *is normalizable at the origin. In practice this is the case if and only if there exists an analytic change of coordinates of the form* (1.3) *transforming* (1.1) *to the semi-normal form*

$$\begin{aligned}\dot{X} &= Xk_1(U)h(X,Y)\\ \dot{Y} &= -\lambda Y k_2(U)h(X,Y),\end{aligned} \tag{1.6}$$

where k_1, k_2 and h are analytic functions such that $h(0,0) = 1$, and $U = X^pY^q$.

(5) *For $\lambda = \frac{p}{q} \in \mathbb{Q}$ there exists a formal change of coordinates of the form* (1.3) *transforming* (1.1) *to the formal normal form*

$$\begin{aligned}\dot{X} &= X(1 + \sum_{k\geq 1} a_k U^k)\\ \dot{Y} &= -\lambda Y(1 + \sum_{k\geq 1} b_k U^k),\end{aligned} \tag{1.7}$$

where $u = X^pY^q$. The quantity $a_k - b_k$ is called the k-th saddle quantity. If $a_i = b_i$ for $i < k$ and $a_k \neq b_k$ the system is said to be non integrable of order k

Equivalent definitions 1.2.

(1) *The system* (1.1) *is integrable if and only if the holonomy of any separatrix is linearizable. This follows from the theorem of Mattei-Moussu* [MM].

(2) *The system* (1.1) *is orbitally normalizable if and only if the holonomy of any separatrix is normalizable i.e. given by the time-one flow of a vector field in a neighborhood of the origin in $\mathbb{C}$ composed with a rotation (cf. Theorem 4.3).*

Remarks 1.3.

(1) If a system is linearizable then it is integrable.

(2) If a system is normalizable, then it is orbitally normalizable.

(3) It is possible to perform a change of coordinates straightening the invariant stable and unstable manifolds of (1.1), i.e. bringing the system to the form

$$\begin{aligned}\dot{x} &= x(1+\overline{f}(x,y))\\ \dot{y} &= -\lambda y(1+\overline{g}(x,y)\end{aligned} \tag{1.8}$$

with $\overline{f},\overline{g} = O(x,y)$. Then the origin is integrable if and only if the system obtained by division by $1+\overline{f}(x,y)$ is linearizable:

$$\begin{aligned}\dot{x} &= x\\ \dot{y} &= -\lambda y\frac{1+\overline{g}(x,y)}{1+\overline{f}(x,y)}\end{aligned} \tag{1.9}$$

(If (1.9) is integrable, there is a first integral of the form $X^\lambda Y$, with $X = xh$. Choosing $y_1 = Yh^{-\lambda}$ linearizes (1.9))

(4) The system (1.8) is orbitally normalizable if and only if the corresponding system (1.9) is normalizable. (If (1.9) is orbitally normalizable then there is a first integral of the form $X^{pk(a-1)}Y^{qka}e^{-1/X^{pk}Y^{qk}}$ from the Martinet-Ramis normal form (see Corollary 4.2), which has symmetry transformations of the form (4.20). It is clear that we can choose $X = x$ using such a transformation. The corresponding transform of Y gives the normalizing change of coordinates for (1.9))

In the sequel it will be very important to distinguish how well $\lambda \notin \mathbb{Q}$ can be approximated by rational numbers. For that purpose, we introduce a set of conditions on λ.

Definition 1.4. *For $\lambda \in \mathbb{R}^+ \setminus \mathbb{Q}$ we introduce the expansion of λ in continuous fraction. This yields a sequence of approximations of λ by means of $\frac{p_n}{q_n}$ (see for instance* [Y]*).*

(1) *The number λ is a Brjuno number if and only if*

$$\sum \frac{1}{q_n}\log q_{n+1} < +\infty \tag{1.10}$$

(this is condition ω in [B]*). We denote by $\mathcal{B}$ the set of Brjuno numbers and by $\mathcal{B}^C$ its complement in $\mathbb{R}^+\setminus\mathbb{Q}$. (This is not the original form of Brjuno's definition* [B] *but he shows in his paper that his definition is equivalent to the one above.)*

(2) *The number λ satisfies a Cremer condition if and only if*

$$\limsup \frac{1}{q_n}\log q_{n+1} = +\infty. \tag{1.11}$$

(It is also the complement of $\overline{\omega}$ in [B]*). We denote by $\mathcal{C}$ the set of numbers satisfying a Cremer condition and by $\mathcal{C}^C$ its complement in $\mathbb{R}^+\setminus\mathbb{Q}$.*

(3) *The number λ satisfies a Peréz-Marco condition if and only if*

$$\sum \frac{1}{q_n}\log\log q_{n+1} < +\infty. \tag{1.12}$$

We denote by $\mathcal{P}$ the set of numbers satisfying a Pérez-Marco condition [PM].

Definition 1.5. *Let $\mathcal{I}$ (resp. $\mathcal{L}$) be the set of $\lambda \in \mathbb{R}^+$ such that any system* (1.1) *is integrable (resp. linearizable).*

Facts 1.6.

(1) *The sets $\mathcal{I}$ and $\mathcal{L}$ are invariant under $\lambda \mapsto 1/\lambda$. Indeed we just make the change $(x, y, t) \mapsto (y, x, \lambda t)$.*

(2) *The set $\mathcal{I}$ is invariant under $\lambda \mapsto \lambda + 1$: this follows from the similar property for diffeomorphisms $z \mapsto exp(2i\pi\lambda)z + o(z)$,* [Y], *and from* [MM] *and* [PMY]. *(Note that the blow up $(X, Y) = (x, y/x)$ transforms a linearizable system with parameter λ to a linearizable system with parameter $\lambda + 1$.*

(3) *For $\lambda \in \mathbb{Q}$, a system is integrable (resp. linearizable) if and only if it is formally integrable (resp. formally linearizable)* [B]. *Indeed Brjuno's Theorem II ensures the convergence under condition ω and condition A. The condition ω of Bruno is written as $-\sum_{k=1}^{\infty} \frac{\ln \omega_k}{2^k} < \infty$ where $\omega_k = \min\{|(Q, \Lambda)|; (Q, \Lambda) \neq 0, ||Q|| < 2^k, Q \in \mathbb{N}\}$. For λ irrational it is equivalent to* (1.10). *In this form it is valid for $\lambda = \frac{p}{q}$. Condition A for this case is exactly equivalent to the vanishing of all the saddle quantities, i.e the formal normal form being:*

$$\begin{aligned} \dot{X} &= Xh(U) \\ \dot{Y} &= -\frac{p}{q} Yh(U) \end{aligned} \tag{1.13}$$

with $U = X^p Y^q$ and $h(U) = 1 + \sum_{k \geq 2} a_k U^k$, which means formal integrability (and formal linearizability when all the $a_k = 0$).

(4) *The following inclusions obviously hold:*

$$\mathcal{B} \subset \mathcal{P} \subset \mathcal{C}^C \subset \mathbb{R}^+ \setminus \mathbb{Q}. \tag{1.14}$$

Moreover, all these sets have full measure.

(5) *For a given irrational λ all systems of the form* (1.1) *are integrable if and only if $\lambda \in \mathcal{B}$. This follows from* [Y] *and* [PMY]. *This means $\mathcal{I} = \mathcal{B}$.*

(6) *For $\lambda \in \mathcal{B}$, any system* (1.1) *is linearizable,* [B], *so $\mathcal{I} = \mathcal{L} = \mathcal{B}$.*

Theorem A. *Consider all integrable systems of the form* (1.1) *for a fixed λ. Every such system is linearizable if and only if $\lambda \in \mathcal{C}^C$.*

Theorem B. *Let*

$$\begin{aligned} \dot{x} &= x + \sum_{i+j=2}^{n} c_{ij} x^i y^j = P(x, y) \\ \dot{y} &= -\lambda y + \sum_{i+j=2}^{n} d_{ij} x^i y^j = Q(x, y) \end{aligned} \tag{1.15}$$

be a polynomial system of degree at most $n \geq 2$. We consider the space of coefficients $\Lambda = (\lambda, c_{ij}, d_{ij}) \subset \mathbb{R}^+ \times \mathbb{C}^{(n+4)(n-1)}$.

I. *The subset of Λ for which the system is not integrable (or not linearizable) is a G_δ set (i.e. a countable intersection of open sets) with measure zero.*

For each fixed λ the set U_λ of $(c_{ij}, d_{ij}) \subset \mathbb{C}^{(n+4)(n-1)}$ for which the system is not integrable (or not linearizable) is a G_δ set.

II. *In particular U_λ is either*
i) an open and dense set of full measure for $\lambda \in \mathbb{Q}$ (the complement of an algebraic set);
ii) a G_δ set of full measure for λ in a dense G_δ subset of $\mathbb{R}^+$;
iii) the void set for λ in a set of full measure. This is in particular the case for $\lambda \in \mathcal{B}$;

III. *Moreover, for each fixed rational $\lambda = \frac{p}{q}$, the set V_λ (resp. V'_λ) of non normalizable (resp. non orbitally normalizable) systems either is a G_δ set of full measure or the empty set. It is a G_δ set of full measure for all $\lambda = \frac{q}{2}$ and $\lambda = \frac{2}{q}$. Moreover the set of parameter values of (c_{ij}, d_{ij}) for which the system is orbitally normalizable but not integrable of a fixed order k is an analytic subvariety.*

We conjecture that all the sets U_λ (resp. V_λ and V'_λ) are of full measure for $\lambda \notin \mathcal{B}$ (resp. λ rational).

Theorem C. *For any irrational $\lambda \in \mathcal{C}$ there exists an integrable and nonlinearizable quadratic system*

$$\begin{aligned} \dot{x} &= x + c_{20}x^2 + c_{11}xy + c_{02}y^2 = x + P_2(x,y) \\ \dot{y} &= -\lambda y + d_{20}x^2 + d_{11}xy + d_{02}y^2 = y + Q_2(x,y). \end{aligned} \tag{1.16}$$

Theorem D. *For any $\lambda \in \mathcal{C}$ there exists a nonintegrable quadratic system of the form* (1.16).

Remark 1.7. The preceding theorems suggest the following organization of systems. For $\lambda \in \mathbb{Q}$ we have algebraic sets of integrable and linearizable systems which are distinct. For λ irrational but not far from $\mathbb{Q}$ (λ in $\mathcal{C}$) these sets remain distinct. When we go further from $\mathbb{Q}$, integrable and linearizable systems become indistinguishable (in $\mathcal{C}^C \setminus \mathcal{B}$). Going still further from $\mathbb{Q}$ (inside $\mathcal{B}$), all systems become integrable and linearizable. As soon as $\lambda \in \mathbb{C} \setminus \mathbb{R}$, all systems are linearizable.

A similar hierarchy is observed for diffeomorphisms with derivative $exp(2\pi i\lambda)$ at the origin, where for $\lambda \notin \mathcal{P}$ nonlinearizable diffeomorphisms may or may not have periodic orbits accumulating at the origin, while all nonlinearizable diffeomorphisms have such periodic orbits for $\lambda \in \mathcal{P} \setminus \mathcal{B}$. All diffeomorphisms are linearizable for $\lambda \in \mathcal{B}$.

Questions 1.8.

I. For fixed λ describe the structure of the set of all integrable, linearizable or (orbitally) normalizable systems. This is difficult when the condition under consideration involves convergence problems. One should expect a study similar to the study of the Mandelbrot set. Particular questions are:

i) For each rational $\lambda = \frac{p}{q}$ do there exist quadratic systems for which the normalizing transformations diverge? We have shown that the answer is yes for $\lambda = 2/n$ or $n/2$ and presumably so for other values.

ii) For each rational $\lambda = \frac{p}{q}$ do there exist non integrable orbitally normalizable (resp. normalizable) quadratic systems? We have shown that the answer is yes for $\lambda = n$ or $\lambda = 1/n$ (resp. $\lambda = \frac{nq+1}{2n}$ or $\lambda = \frac{2n}{nq+1}$), where $n, q \in \mathbb{N}$. Can we describe the structure of non integrable normalizable systems and orbitally normalizable systems for fixed $\lambda = \frac{p}{q}$?

iii) For a given irrational $\lambda \notin \mathcal{B}$ describe the structure of the set of integrable quadratic systems (1.16)*.*

iv) For a given irrational $\lambda \in \mathcal{C}$ describe the structure of the set of linearizable quadratic systems (1.16) *and how it sits inside the set of integrable systems.*

II. For any given irrational $\lambda \notin \mathcal{B}$ does there exist a nonintegrable quadratic system (1.16)*? From the study below the authors conjecture that a natural candidate is given by the system*

$$\begin{aligned} \dot{x} &= x(1+y) \\ \dot{y} &= -\lambda y(1+x+y). \end{aligned} \tag{1.17}$$

III. For $\lambda \notin \mathcal{P}$ does there exist polynomial nonintegrable systems without periodic orbits of the holonomy accumulating at the origin? Or is materialization of resonances ([IP1] *and* [IP2]*) the only obstruction to integrability of polynomial systems?*

2. Proof of Theorems A, B, C and D

Proof of Theorem A. We start with a system of the form (1.4). A linearizing transformation is given by a transformation $(u, v) = (Xf(X,Y), Yf^{-\lambda}(X,Y))$, with $f(0,0) = 1$, where $f(X,Y)$ satisfies

$$Xf_X(X,Y) - \lambda Y f_Y(X,Y) = f(X,Y)\left(\frac{1}{h(X,Y)} - 1\right). \tag{2.1}$$

We look instead for $g(X,Y) = \log(f(X,Y))$. Then

$$Xg_X(X,Y) - \lambda Y g_Y(X,Y) = \left(\frac{1}{h(X,Y)} - 1\right). \tag{2.2}$$

Let $k(X,Y) = \frac{1}{h(X,Y)} - 1 = \sum_{i+j>0} a_{ij} X^i Y^j$ and $g(X,Y) = \sum_{i+j>0} b_{ij} X^i Y^j$. Then

$$b_{ij} = \frac{a_{ij}}{i - \lambda j}. \tag{2.3}$$

Since h is convergent, there exists $\epsilon > 0$ such that $\sum |a_{ij}|\epsilon^{i+j} < +\infty$. In particular $\limsup \max_{i+j=n} |a_{ij}|^{1/n} < +\infty$. The series $g(X,Y)$ will be convergent if and only if $\limsup \max_{i+j=n} |b_{ij}|^{1/n} < +\infty$. Among all pairs (i,j) of indices appear the particular pairs (p_n, q_n), where p_n/q_n are the successive approximations of λ given in its continuous fraction. Moreover we always have the following inequality (see for instance [Y])

$$\frac{1}{2q_{n+1}} < \frac{1}{q_n + q_{n+1}} < |p_n - \lambda q_n| < \frac{1}{q_{n+1}}. \tag{2.4}$$

To show the convergence of $g(X,Y)$ it is sufficient to show that

$$\limsup_{n\to\infty} \max_{i+j=n} |b_{ij}|^{1/(i+j)} < +\infty. \tag{2.5}$$

This is automatically satisfied if $\limsup \left(\frac{1}{p_n-\lambda q_n}\right)^{\frac{1}{p_n+q_n}} < +\infty$, i.e. $\limsup \frac{1}{p_n+q_n} \log |p_n-\lambda q_n|^{-1} < +\infty$. Using (2.4) and noting that $p_n \sim \lambda q_n$ this is equivalent to condition $\lambda \in \mathcal{C}^C$. If we now suppose that $\lambda \in \mathcal{C}$ we can construct a nonlinearizable system by starting with a system in which $h(X,Y) = (1-X)(1-Y)$, i.e. $a_{ij} = 1$ for all i,j. $\square$

In the sequel we make several uses of the following general fact which was mentioned in particular cases in [I] and [IP2].

Lemma 2.1. *Let*

$$h(x,y) = \sum_{i+j=0}^{+\infty} a_{ij}(b_1,\dots,b_n)x^iy^j \tag{2.6}$$

where the a_{ij} are polynomials of degree $\leq i+j-1$ in the variables $b_1,\dots,b_n \in \mathbb{C}$. If the series (2.6) *converges in a neighborhood of $(x,y) = (0,0)$ for $b = (b_1,\dots,b_n)$ in a set of nonzero measure then the series converges in a neighborhood of $(x,y) = (0,0)$ for all $(b_1,\dots,b_n) \in \mathbb{C}^n$.*

Proof. The proof ([I] and [IP2]) is a straightforward application (with a slight adaptation) of a lemma of Nadirashvili [N]. (We also learn from a preprint of Pérez-Marco that it could be seen as a consequence of Bernstein lemma [PM2].) Indeed, for $R > 0$ sufficiently large, the series converges for $b = (b_1,\dots,b_n)$ in a subset E of nonzero measure lying inside the closed ball B_R of radius R in $\mathbb{C}^n$. Let $E_N = \{b \in B_R |\ |a_{ij}(b)| \leq N^{i+j-1}\}$.

Then $E = \cup E_N$, yielding that one of the E_N has nonzero measure. By Nadirashvili's lemma there exists a constant C, depending only on n, such that for any polynomial $g(b)$ of degree d

$$max_{b\in B_R}|g(b)| < \left(C\frac{\text{mes}\, B_R}{\text{mes}\, E_N}\right)^d \max_{b\in E_N} g(b). \tag{2.7}$$

If we call $C_1 = C\frac{\text{mes}\, B_R}{\text{mes}\, E}$, then, for $b \in B_R$: $|a_{ij}(b)| < (C_1N)^{i+j-1}$, which is sufficient to guarantee the convergence of $h(x,y)$ in a neighborhood of the origin. $\square$

Proof of Theorem B.
Proof of I and II. We make a change of coordinates $x \mapsto \overline{x} = x - h(y)$, where $x = h(y) = o(y)$ is the unstable manifold, bringing (1.15) to the system

$$\begin{aligned}\dot{\overline{x}} &= \overline{x}(1+p(\overline{x},y))\\ \dot{y} &= -\lambda y + q(\overline{x},y).\end{aligned} \tag{2.8}$$

The function $h(y) = \sum_{n\geq 2} a_ny^n$ can be calculated as a power series from the equation $P(h(y),y) = h'(y)Q(h(y),y)$. By induction we can prove that a_n is a polynomial of degree at most $n-1$ in the coefficients of (1.15). Hence the system (2.8) is obtained as

$$\begin{aligned}\dot{\overline{x}} &= P(\overline{x}+h(y),y) - h'(y)Q(\overline{x}+h(y),y)\\ \dot{y} &= Q(\overline{x}+h(y),y).\end{aligned} \tag{2.9}$$

It is easily checked that the coefficient of any monomial x^iy^j in this system is a polynomial in the original coefficients of degree at most $i+j-1$ (as it is obtained by multiplication of monomials with this property). Dividing the system by $(1+p(\overline{x},y))$ and renaming $\overline{x}=x$ yields a system

$$\begin{aligned}\dot{x} &= x\\ \dot{y} &= -\lambda y + r(x,y)\end{aligned} \tag{2.10}$$

The function $r(x,y)$ is analytic and its coefficients are polynomials in the coefficients of (1.15). Moreover, the coefficients of the monomials x^iy^j are still of degree at most $i+j-1$. The system is integrable if this last system (2.10) is linearizable. (For linearizability we work directly with system (2.8)).

In the case of integrability we look for a linearizing change of coordinates

$$Y = y + \sum_{i+j\geq 2} h_{ij}x^iy^j \tag{2.11}$$

for system (2.10). (In the case of linearizability we look for a linearizing change of coordinates $(X,Y) = (x+\sum_{i+j\geq 2} g_{ij}x^iy^j, y+\sum_{i+j\geq 2} h_{ij}x^iy^j)$ for (2.8) and the rest of the argument is similar).

The h_{ij} are rational functions of the coefficients $\Lambda = (\lambda, c_{kl}, d_{kl})$. In fact only λ can occur in the denominator. If $\lambda = p/q$ is rational then we take $h_{i,j}=0$ for $qi-p(j-1)=0$ (resp. $g_{i,j}=0$ for $q(i-1)-pj=0$).

We consider the functions $h_{i,j}$ defined with values in $\overline{\mathbb{C}}$ (the value ∞ is taken as soon as $\lambda = p/q$ and i,j correspond to a non-vanishing resonant monomial x^iy^j).

Then the system is not integrable if and only if $\limsup \max_{i+j=n} |h_{ij}|^{1/(i+j)} = +\infty$. Remark that this is the case if and only if

$$\sup_{i,j} |h_{ij}|^{1/(i+j)} = +\infty. \tag{2.12}$$

Let us call

$$M(\Lambda) = \sup_{i,j} |h_{ij}|^{1/(i+j)}. \tag{2.13}$$

This function is lower semi-continuous with values in $[0,+\infty]$. Then for all $m\in\mathbb{R}$, the set $\{\Lambda | M(\Lambda) > m\}$ is open. Let us call

$$NI = \{\Lambda | M(\Lambda) = +\infty\}. \tag{2.14}$$

Then

$$NI = \cap_{m\in\mathbb{N}}\{\Lambda | M(\Lambda) > m\}. \tag{2.15}$$

The system is not integrable on a countable intersection of open sets, i.e. on a G_δ set.

Hence for each fixed value of λ the set of non-integrable systems $NI(\lambda)$ is a G_δ. For $\lambda \notin \mathbb{Q}$ it can be:

i) a nonvoid G_δ set for λ in a dense G_δ subset of $\mathbb{R}^+$ (see Proposition 2.3 and Theorem 2.6 below);
ii) the void set for $\lambda \in \mathcal{B}$.

Moreover, when λ is irrational, applying Lemma 2.1, we have that, if for a fixed λ the set $NI(\lambda)$ is nonvoid, then it has full measure. The same holds for λ rational as the set of integrable systems is algebraic (and not the full set as shown in Proposition 2.3 below).

Similar properties hold for linearizability when dealing with the two series linearizing the first and second equations.

Proof of III. The fact that the normalizing transformations diverge either on a set of full measure or on the empty set is proved in [IP2]. The idea of the proof of [IP2] is to apply Lemma 2.1 to the normalizing transformations $(X,Y) = (x+h_1(x,y), y+h_2(x,y))$ where $h_{1,2}$ satisfy the hypothesis of Lemma 2.1. The fact that the set is a G_δ set is shown as in I and II above.

The fact that this set has full measure for $\lambda = \frac{q}{2}$ and $\lambda = \frac{2}{q}$ follows from the explicit example of non normalizable systems in Proposition 6.4 and its dual under $(x,y,t) \mapsto (y,x,\lambda t)$.

Moreover it follows from [E] and [MR] that, for fixed value of the order of non integrability, the orbital normalizability is decided by the vanishing of the Ecalle-Martinet-Ramis moduli. These have coefficients which are analytic functions of the (c_{ij}, d_{ij}). $\square$

Theorem C is a consequence of the following more explicit:

Theorem C′. *The integrable system*

$$\begin{aligned} \dot{x} &= x(1-x-y) \\ \dot{y} &= -\lambda y(1-x-y) \end{aligned} \tag{2.16}$$

is linearizable if and only if $\lambda \in \mathcal{C}^C$.

Proof. Note that $F(x,y) = x^\lambda y$ is a first integral of the system (2.16). Let $h(x,y) = 1-x-y$. Then

$$1/h(x,y) = (1-x-y)^{-1} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\binom{i+j}{i}x^i y^j. \tag{2.17}$$

Let $k(x,y) = \frac{1}{h(x,y)} - 1 = \sum_{i+j>0} a_{ij}x^i y^j$. Then, as in the proof of Theorem A, $k(x,y)$ is convergent, yielding the convergence of $g(x,y)$ defined in (2.2) as soon as $\lambda \in \mathcal{C}^C$.

Moreover $|a_{ij}| > 1$. Hence, for λ irrational, $g(x,y)$ defined in (2.2) diverges exactly when $\lambda \in \mathcal{C}$. This comes from the divergence of the subseries

$$\sum b_{p_n q_n}\epsilon^{p_n+q_n} = \sum \frac{a_{p_n q_n}}{p_n - \lambda q_n}\epsilon^{p_n+q_n}. \tag{2.18}$$

If λ is rational, then the first resonant terms cannot be eliminated not even by a formal change of variables. $\square$

Remark 2.2. An example of an integrable nonlinearizable system was given by Françoise in [F], but his example is not polynomial.

Proposition 2.3. *There exists $\epsilon \in \mathbb{R}$ such that the system*

$$\begin{aligned} \dot{x} &= x(1-x-y) \\ \dot{y} &= -\lambda y(1-x-y) + \epsilon xy \end{aligned} \tag{2.19}$$

is nonintegrable for all $\lambda = p/q$ and its first saddle quantity is nonvanishing. For that ϵ there exists a dense G_δ-set of values of λ in $\mathbb{R}^+$ for which the system is nonintegrable.

Proof. As far as integrability is concerned we can divide the system by $1-x-y$, yielding the system

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -\lambda y + \epsilon \frac{xy}{1-x-y}. \end{aligned} \tag{2.20}$$

If we let $Y = y + o(x,y)$ be the change to normal form, then the normal form is given by

$$\begin{aligned} \dot{x} &= x \\ \dot{Y} &= -p/qY + \sum_{n=1}^{\infty} A^n_{p,q}(\epsilon) x^{np} Y^{nq+1}. \end{aligned} \tag{2.21}$$

The $A^n_{p,q}(\epsilon)$ are the saddle quantities. Lemma 2.3 of [IP2] yields that they are polynomials in ϵ of degree $\le np+nq$ which are quasi-homogeneous of degree $np+nq$ in the coefficients of the original system. More precisely for a system

$$\begin{aligned} \dot{x} &= x + \sum_{i+j\ge 2} c_{ij} x^i y^j \\ \dot{y} &= -\frac{p}{q} y + \sum_{i+j\ge 2} d_{ij} x^i y^j \end{aligned} \tag{2.22}$$

we assign to the coefficients c_{ij} and d_{ij} weights $i+j-1$. Recall that a polynomial $P(z_1, \dots z_N)$ is quasi-homogeneous of degree m with weight α_j for the variable z_j if

$$P(t^{\alpha_1} z_1, \dots, t^{\alpha_N} z_N) = t^m P(z_1, \dots, z_N).$$

In our case we have $c_{ij} = 0$ for all i,j. In particular $A^1_{p,q}$ is of degree 1 in the coefficient of $x^p y^{q+1}$ which is $d_{p,q+1} = \binom{p+q-1}{p-1}\epsilon$. (This monomial cannot be removed as it is resonant and there are no resonant monomial of lower degree.) Moreover it is of degree higher than one in the d_{ij} with $i+j \le p+q$. As all $d_{ij} = O(\epsilon)$ this yields

$$A^1_{p,q} = \binom{p+q-1}{p-1}\epsilon + o(\epsilon). \tag{2.23}$$

There exists only a countable set of values of ϵ for which all the $c^1_{p,q}(\epsilon)$ vanish, from which the result follows.

The second part of the proof is as in [IP1], [IP2] and [PM]. For purpose of completeness we write explicitly the details. The first step is the following lemma:

Lemma 2.4. *Let*

$$\begin{aligned}\dot{x} &= x + P(x,y) = x + o(x,y)\\ \dot{y} &= -(\frac{p}{q}+\eta)y + Q(x,y) = -(\frac{p}{q}+\eta)y + o(x,y)\end{aligned} \tag{2.24}$$

have a nonzero first saddle quantity at the origin for $\eta = 0$. Then given $r > 0$ there exists $\delta > 0$ such that the holonomy around the y-separatrix of (2.24) *has a nonzero periodic point of period q with size $< r$, for real η satisfying $|\eta| < \delta$.*

Proof. We can of course suppose that the system is in the following prenormal form going back to Dulac [Du]

$$\begin{aligned}\dot{x} &= x\\ \dot{y} &= y[-(\frac{p}{q}+\eta) + (A+O(\eta))x^py^q + x^py^qk(x,y)],\end{aligned} \tag{2.25}$$

where $A \neq 0$ and $k(x,y) = O(x,y)$ inside a polydisc of radius 2 (a scaling in (x,y) may be necessary to obtain radius 2). Then the holonomy around the y-separatrix can be calculated as the map $h(y) = H(1,y)$, where $H(\theta,y)$ satisfies ([MM])

$$\begin{aligned}\frac{\partial H}{\partial\theta}(\theta,y) = 2i\pi H(\theta,y)\Big[&-(\frac{p}{q}+\eta) + (A+O(\eta))\exp(2ip\pi\theta)H(\theta,y)^q\\ &+ \exp(2ip\pi\theta)H(\theta,y)^q k(\exp(2i\pi\theta, H(\theta,y)))\Big].\end{aligned} \tag{2.26}$$

Letting $H(\theta,y) = \sum_{k\geq 1} c_k(\theta)y^k$, with $H(0,y) = y$, this yields (by solving linear differential equations):

$$\begin{cases} c_1(1) = \exp(-2i\pi(\frac{p}{q}+\eta)) & \\ c_i(1) = 0 & 1 < i \leq q\\ c_{q+1}(1) = A + O(\eta). & \end{cases} \tag{2.27}$$

Then, using the notation $h^{\circ q}$ for the q-th iterate of h:

$$h^{\circ q}(y) - y = (\exp(-2i\pi q\eta)-1)y + A(q+1)(\exp(2i\pi\frac{p}{q}) + O(\eta))y^{q+1} + o(y^{q+1}). \tag{2.28}$$

Hence for η sufficiently small the equation $h^{\circ q}(y) - y = 0$ has a small nonzero root.

Remark 2.5. This periodic orbit corresponds to the invariant manifold obtained in [IP1], [IP2] and called materialization of resonance.

End of proof of Proposition 2.3. Indeed, starting from the system (2.19) with $\lambda = p/q$ and given $n \in \mathbb{N}$, there exists a small neighborhood $V_n(p/q)$ of p/q such that, for all $\lambda \in V_n(p/q)$, the holonomy around the y-separatrix has a periodic orbit of period q in the ball of radius $1/n$ surrounding the origin in y-space. Then $U_n = \cup_{p/q\in\mathbb{Q}^+} V_n(p/q)$ is a dense open set in $\mathbb{R}^+$ and $\cap_{n\geq 1}U_n$ is a dense G_δ-set of values $\lambda \in \mathbb{R}^+$, for which the system (2.19) is nonintegrable, as it has a sequence of periodic orbits of the holonomy converging to the origin. □

As a by-product we get a new proof of the result of [IP2]. Let $\mathbb{S}$ be the unit sphere in $\mathbb{C}^6$ (of real dimension 11) (which we consider as the space of coefficients $\{c_{20}, \dots d_{02}\}$ of (1.16)).

Theorem 2.6 [IP2]. *There exists a subset $\mathcal{M} \subset \mathbb{S}$, of full measure, such that for any $m = (c_{20}, \ldots, d_{02}) \in \mathcal{M}$, there exists a dense G_δ-set U_m in $\mathbb{R}^+$, such that any system* (1.16) *with quadratic part given by m and with $\lambda \in U_m$ is nonintegrable.*

Proof. For each $\lambda = \frac{p}{q}$ the coefficient of the first resonant monomial is given by a polynomial $L_{p,q}(c_{20}, \ldots d_{02})$. The polynomial is nonzero since it takes a nonzero value on (2.19). Hence its set of zeros is an algebraic variety $V_{p,q}$ of $\mathbb{S}$. Then

$$\mathcal{M} = \mathbb{S} \setminus \cup_{p,q} V_{p,q} \tag{2.29}$$

is a set of full measure. It is dense because it is the intersection of a countable number of dense open sets.

Any system (1.15) with a quadratic part from $\mathcal{M}$ has a nonzero first saddle quantity for any p, q. The rest of the proof is as in the proof of Proposition 2.3. □

For λ rational, Ilyashenko and Pyartli proved in [IP2] the existence of a nonintegrable polynomial system of degree $n \geq 3$. In fact their method uses a polynomial system with two monomials, one of which is resonant. This gives $n \geq 3$ and not $n \geq 2$ as claimed in the paper. A quadratic example cannot be obtained using their method.

The proof of Theorem D will follow from the proof of the more general theorem.

Theorem D′. *For $\lambda \in \mathcal{C}$ there exists a set $E_\lambda \subset \mathbb{C}$ of full measure such that for $\epsilon \in E_\lambda$ the system* (2.19) *is nonintegrable.*

Proof. The theorem follows directly from the theorem of Ilyashenko [I] applied to (2.20). This theorem studies general systems of the form

$$\dot{z} = \Lambda z + \epsilon f(z), \quad z \in \mathbb{C}^n \tag{2.30}$$

and asserts that if $\lambda \in \mathcal{C}$ and if the Taylor coefficients of some component of the germ $f(X, Y)$ can be estimated from below in modulus by some geometric progression, then the series normalizing (2.30) diverges for almost all ϵ. The proof of this theorem uses Lemma 2.1 above. □

3. Linearizability and isochronous saddles

In this section we examine the difference between integrability and linearizability. We define the notion of isochronicity for integrable saddles and prove that an integrable saddle is linearizable if and only if it is isochronous. This generalizes the classical theorem of Poincaré which treats the center case and corresponds to $\lambda = 1$.

An integrable saddle point can be written in the form

$$\begin{aligned} \dot{x} &= xh(x, y) \\ \dot{y} &= -\lambda y h(x, y), \end{aligned} \tag{3.1}$$

where h is a holomorphic function at the origin. We can assume moreover that the coordinates are chosen in such a way that

$$h(x, y) = 1 + xyk(x, y), \tag{3.2}$$

with k analytic ([D] or [MM]). Consider a polydisc B of the form $B = \{(x,y) \in \mathbb{C}^2 : |x| < r_1, |y| < r_2\}$, $r_1, r_2 > 0$, which belongs to the polydisc of convergence of h and on which h does not vanish.

Assume first that $\lambda = \frac{p}{q}$ is rational. Then isochronicity can be defined similarly as in the center case. The system has a multivalued first integral $F(x,y) = x^\lambda y$. A univalued first integral is given by $G(x,y) = x^p y^q$. The restriction $G : B \setminus G^{-1}(0) \to G(B) \setminus \{0\}$ defines a fibration. Any fiber is diffeomorphic to an annulus. Its first homology group is one-dimensional generated by the cycle $\gamma_c \subset G^{-1}(c)$, $\gamma_c(t) = (e^{iqt}, c^{1/q}e^{-ipt})$, $t \in [0, 2\pi]$. The cycle γ_c makes q turns around the y-axis and p turns around the x-axis.

Let a differential form dt be defined by

$$dt = \frac{dx}{xh(x,y)}. \tag{3.3}$$

The *period* T is the function defined by

$$T(c) = \frac{1}{q}\int_{\gamma_c} dt, \quad c \in G(B) \setminus \{0\}. \tag{3.4}$$

Definition 3.1. *For $\lambda = \frac{p}{q} \in \mathbb{Q}$ the origin is an isochronous saddle of* (3.1) *if its period $T(c)$ is constant on $G(B) \setminus \{0\}$.*

Note that the form dt can be written as

$$dt = \frac{dx}{x} + \eta, \tag{3.5}$$

where η is a holomorphic one-form at the origin. The form η is essentially coordinate independent. More precisely, the form dt is coordinate independent and by a change of coordinates $X = x\phi(x,y)$, $Y = y\psi(x,y)$, $\phi(0,0) = \psi(0,0) = 1$, the form dx/x is transformed to $dX/X - d\phi/\phi$, with $d\phi/\phi$ exact. Now

$$\frac{1}{q}\int_{\gamma(c)} \frac{dx}{x} = 2\pi i \tag{3.6}$$

and $T(0) = 2\pi i$ as η is holomorphic and the length of the path of integration tends to zero. Hence isochronicity is equivalent to condition (3.7)

$$\int_{\gamma(c)} \eta = 0. \tag{3.7}$$

Assume now that λ is irrational. Then each leaf of the foliation defined by (3.1) is simply connected. This is easily seen as the whole foliation retracts to a torus $\mathbb{T}$ of the form $\mathbb{T} = \{(x,y) \in \mathbb{CP}^2 : |x| = r_1/2, |y| = r_2/2\}$, the leaves of the foliation giving the irrational linear flow on the torus.

There are however asymptotic cycles on the leaves, which we now define.

Definition 3.2. *Let $[\alpha_n, \beta_n] \subset \mathbb{R}$ be a sequence of intervals and let $\gamma_n : [\alpha_n, \beta_n] \to \mathbb{C}^2$ be a sequence of curves, all coinciding on the intersection of their domains. We say that γ_n, $n \in \mathbb{N}$, is an asymptotic cycle if*

$$\lim \gamma(\alpha_n) = \lim \gamma(\beta_n). \tag{3.8}$$

Proposition 3.3. *For each $\lambda \in \mathbb{R}^+$ and any (x_0, y_0) in a neighborhood of the origin in $\mathbb{C}^2$ there exists an asymptotic cycle passing through (x_0, y_0).*

Proof. If λ is rational the asymptotic cycles are all closed. They are exactly the cycles. If λ is irrational, let $\frac{p_n}{q_n} \in \mathbb{Q}$ be a sequence of rational numbers tending to λ. Then for any (x_0, y_0) an asymptotic cycle is given by the sequence $\gamma_n : [0, 2\pi q_n] \to \mathbb{C}^2$, $\gamma_n(\theta) = (x_0 e^{i\theta}, y_0 e^{-i\lambda\theta})$. $\square$

Remark. Our definition of asymptotic cycles is more general than the definition in [BL]. According to their definition of asymptotic cycles the foliation defined by (3.1) has no asymptotic cycles in the irrational case (Cf. Remark 3.10).

Generalizing Definition 3.1 we put:

Definition 3.4. *An integrable saddle of the form* (3.1)*, with h satisfying* (3.2)*, $\lambda \in \mathbb{R}$, is isochronous if*

$$\lim \int_{\gamma_n} \eta = 0, \tag{3.9}$$

for any asymptotic cycle γ_n belonging to a leaf of (3.1)*, where η is defined in* (3.5)*. (Relation* (3.9) *is to be understood in the classical sense of improper integrals.)*

Theorem E. *An integrable saddle* (3.1) *is linearizable if and only if it is isochronous.*

Of course if the saddle is linearizable then it is isochronous. This follows after linearizing the saddle as the form η in new coordinates is zero and the isochronicity condition on the form dt is independent of the chosen coordinates.

In order to prove the converse, we first introduce a definition. Let

$$\omega = P(x, y)dx + Q(x, y)dy \tag{3.10}$$

be a holomorphic one-form in $\mathbb{C}^2$.

Definition 3.5. *A holomorphic one-form η is relatively exact, with respect to ω, if there exist holomorphic functions g and m such that*

$$\eta = dg + m\omega. \tag{3.11}$$

Lemma 3.6. *Let ω be a one-form in $\mathbb{C}^2$ having isolated singularities. A holomorphic one-form η is relatively exact with respect to the form ω if and only if there exists a holomorphic function g such that*

$$\eta \wedge \omega = dg \wedge \omega. \tag{3.12}$$

Proof. Relation (3.11) implies immediately (3.12). Conversely, the assumption that ω has isolated singularities means that the components $P(x,y)$ and $Q(x,y)$ have no common factor vanishing at the origin. Hence putting $\eta - dg = R(x,y)dx + S(x,y)dy$, relation (3.12) means that P divides R and Q divides S, and moreover $\frac{R}{P} = \frac{S}{Q}$, which defines a holomorphic function m verifying (3.11). $\square$

Proposition 3.7. *An integrable saddle* (3.1) *is linearizable if and only if the form η, given by* (3.5)*, is relatively exact with respect to the linear form*

$$\omega = \lambda y dx + x dy. \tag{3.13}$$

Proof. Note that the solutions of (3.1) are leaves of the foliation defined by $\omega = 0$. The proof is essentially repeating the proof of Theorem A (cf. also [BC]). Let ω and η be given by (3.13) and (3.5). A linearizing transformation of an integrable saddle (3.1) exists, if and only if there exists a transformation of the form

$$(u,v) = (xf(x,y), yf^{-\lambda}(x,y)), \tag{3.14}$$

where

$$\frac{df}{f} \wedge \omega = \eta \wedge \omega, \quad f(0,0) = 1. \tag{3.15}$$

That is putting $g(x,y) = \log(f(x,y))$, the system is linearizable if and only if there exists a holomorphic function g verifying (3.12), which is equivalent to relative exactness of η by Lemma 3.6. $\square$

Remark 3.8. In the proof of Theorem A we proved that equation (3.12) can be solved for any one form η if and only if $\lambda \in \mathcal{C}^C$. For linearizability of a given system it is necessary and sufficient that equation (3.12) be solvable for the form η given by (3.5).

The proof of Theorem E is now reduced to the proof of

Theorem 3.9. *A holomorphic 1-form η is relatively exact with respect to the form ω given by* (3.13)*, if and only if*

$$\lim \int_{\gamma_n} \eta = 0 \tag{3.16}$$

for any asymptotic cycle γ_n belonging to a leaf of ω.

Proof. The direct implication is obvious. We prove only the converse. We distinguish and treat separately the cases λ rational (relatively easy) and λ irrational (more complicated).

Proof in the rational case.

For λ rational the claim follows from [BC], Lemma 2.1.1. We sketch the proof for completeness. Let $\lambda = \frac{p}{q} \in \mathbb{Q}$. Let γ_c, $c \in \mathbb{C}$, be a cycle belonging to $G^{-1}(c)$, where $G(x,y) = x^p y^q$.

We suppose that γ_c is of the form

$$\gamma_c(\theta) = (x_0 e^{iq\theta}, y_0 e^{-ip\theta}), \quad G(x_0,y_0) = c, \quad \theta \in [0, 2\pi]. \tag{3.17}$$

Let

$$\eta = \sum a_{ij} x^{i-1} y^j dx + \sum b_{ij} x^i y^{j-1} dy. \tag{3.18}$$

By substituting the parameterizations of γ_c into $\int_{\gamma_c} \eta$, one verifies that

$$\int_{\gamma_c} \eta = \sum_k \alpha_k c^k, \tag{3.19}$$

where

$$\alpha_k = 2i\pi(q a_{kp,kq} - p b_{kp,kq}), \quad k = 1, \dots . \tag{3.20}$$

Hence, the vanishing of (3.19) shows that

$$q a_{kp,kq} - p b_{kp,kq} = 0, \quad k = 1, \dots . \tag{3.21}$$

Substitute

$$g(x,y) = \sum \beta_{ij} x^i y^j, \tag{3.22}$$

in (3.12) and search for a formal solution of (3.12). We get the system of equations

$$(i - \lambda j)\beta_{ij} = a_{ij} - \lambda b_{ij}. \tag{3.23}$$

This defines β_{ij}, for $(i,j) \neq (kp,kq)$, $k \in \mathbb{Z}$. For the resonant terms β_{ij}, $(i,j) = (kp,kq)$, there could be a problem as the term $i - \lambda j$ vanishes. However, it is (3.21) that assures that a formal solution is obtained by putting

$$\beta_{kp,kq} = 0. \tag{3.24}$$

Now the function g of the form (3.22), with β_{ij} thus defined gives a formal solution of (3.12). This formal solution is easily verified to converge, as the coefficient $i - \lambda j$ in (3.23) is at least $\frac{1}{q}$, for $(i,j) \neq (kp,kq)$. This completes the proof of the theorem in the rational case.

Proof in the irrational case. In this case one can get a formal solution of (3.12) as in the rational case (without using the isochronicity condition). However, as shown in Theorem A, the formal solution does not converge in general. This is why we adopt a different approach in this case.

We have to define a holomorphic function g satisfying (3.12) in a neighborhood of the origin. By linear scaling we can assume that the polydisc $\Omega = \{(x,y) \in \mathbb{C}^2 : |x| \leq 1, |y| \leq 1\}$ belongs to the domain of convergence of the form η. Put $g(1,1) = 0$. Let γ_c be the curve lying in the leaf of the foliation (3.1), passing through the point $(1,c)$, given by

$$\gamma_c(\theta) = (e^{i\theta}, ce^{-i\lambda\theta}), \quad \theta \in \mathbb{R}. \tag{3.25}$$

Put

$$g(\gamma_1(\theta)) = \int_0^\theta \gamma_1^* \eta. \tag{3.26}$$

This defines the function g on a dense set of the torus $T = \{(x, y) \in \mathbb{C}^2 : |x| = 1, |y| = 1\}$. Next, by the hypothesis (3.8), the function g can be extended without ambiguity by continuity to the torus T.

Denote by $D = \{y \in \mathbb{C} : |y| < 1\}$, the unitary disc, $\bar{D}$ its closure and S its boundary. Note that g satisfies the condition

$$g(1, ce^{2\pi i\lambda}) = g(1, c) + \int_0^{2\pi} \gamma_c^* \eta, \tag{3.27}$$

for c on the circle S. We next want to extend g to a continuous function on the disc $\{1\} \times \bar{D}$,

$$g(1, c) = u(c), \tag{3.28}$$

with u holomorphic in D. Moreover, we want condition (3.27) to hold for all points $c \in \bar{D}$.

Initially, the function u is continuous complex valued defined on the circle S. Applying separately the existence theorem for solutions of Dirichlet's problem for the real and imaginary part of u, we extend u to a continuous function on $\bar{D}$, *harmonic* in D. Let g be given by (3.28), $c \in \bar{D}$. We claim that (3.27) holds for this extended function g. Indeed, consider the function

$$\psi(c) = u(ce^{2\pi i\lambda}) - u(c) - \int_0^{2\pi} \gamma_c^* \eta. \tag{3.29}$$

The function ψ is a harmonic function in D, as the last term in (3.29) is a holomorphic function in c. Moreover, $\psi(c)$ vanishes for $c \in S$. Now by the unicity of solutions of Dirichlet's problem it follows that ψ is identically zero on $\bar{D}$. This proves that(3.27) holds on $\bar{D}$. We claim next that u is in fact holomorphic in D. In order to prove it, introduce the differential operators

$$\partial = \frac{1}{2}(\frac{\partial}{\partial c'} + i\frac{\partial}{\partial c''}), \quad \bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial c'} - i\frac{\partial}{\partial c''}), \tag{3.30}$$

where c' and c'' are the real and imaginary part of c. As the last term in the definition of ψ is holomorphic, it follows, from the vanishing of ψ, that

$$\bar{\partial}(u(ce^{2\pi i\lambda})) = \bar{\partial}(u(c)). \tag{3.31}$$

Now, since λ is irrational, the numbers $ce^{2\pi ik\lambda}$, $k \in \mathbb{N}$, are dense on the circle of radius $|c|$, so (3.31) shows that the function $\bar{\partial}(u(c))$ depends only on $|c|$. Say $\bar{\partial}(u(c)) = \phi(|c|)$. However, since u is harmonic, then $\partial(\phi(|c|)) = 0$. This can only happen, if ϕ is a constant. We have so far proven that $u(c) = v(c) + k\bar{c}$, where v is a holomorphic function and $k \in \mathbb{C}$. To show that $k = 0$, we integrate relation(3.29) along $|c| = 1$. As ψ vanishes in D and moreover v and the last term in (3.29) are holomorphic, we get $2\pi ik(e^{-2\pi i\lambda}-1)=0$, so $k=0$ and we have shown that u is holomorphic on D.

We now extend g to a complement of the y-axis. Given any (x,y), denote $\mathcal{L}_{(x,y)}$ the leaf of the linear foliation $\omega=0$ passing through (x,y). For (x,y), $x\neq 0$, belonging to a neighborhood of the origin, the leaf $\mathcal{L}_{(x,y)}$ cuts the disc $\{1\}\times D$ infinitely many times. Let $\gamma_{(x,y)}$ be a curve in the leaf $\mathcal{L}_{(x,y)}$ starting at a point $(1,c)$, $c=x^\lambda y\in D$, and connecting it to (x,y). The leaf also cuts the disc $\{1\}\times D$ at points $(1, ce^{2\pi i\lambda k}$ with $k\in\mathbb{Z}$. We put

$$g(x,y)=g(1,c)+\int_{\gamma_{(x,y)}}\eta. \tag{3.32}$$

We claim that g is well defined and does not depend on the choice of $\gamma_{(x,y)}$. Indeed, let $\tilde{\gamma}_{(x,y)}$ be another choice of $\gamma_{(x,y)}$ starting at $(1,\tilde{c})\in D$ with $\tilde{c}=c\exp(2\pi ik\lambda)$. Note that the path obtained by taking $\gamma_{(x,y)}$ followed by $(\tilde{\gamma}_{(x,y)})^{-1}$ is homotopic to the path $\gamma_c:[0,2k\pi]\to\mathcal{L}_{(x,y)}$, given by $\gamma_c(\theta)=(e^{i\theta},ce^{-i\lambda\theta})$, for some $k\in\mathbb{Z}$. This follows from the simple conectedness of the leaf $\mathcal{L}_{(x,y)}$.

By induction (3.27) gives

$$g(1,ce^{2\pi ik\lambda})=g(1,c)+\int_0^{2k\pi}\gamma_c^*\eta,\quad k\in\mathbb{Z}. \tag{3.33}$$

This shows that

$$g(1,c)+\int_{\gamma_{(x,y)}}\eta=g(1,\tilde{c})+\int_{\tilde{\gamma}_{(x,y)}}\eta \tag{3.34}$$

and hence the function g is well defined on a neighborhood of the origin from which the y-axis has been deleted. Moreover, g is holomorphic, as the initial value $g(1,c)$ depends holomorphically on $c=x^\lambda y$ and g is extended by integration of the holomorphic form η.

In order to extend holomorphically g to the y-axis, note that a point (x,y) close to the y-axis can be linked to a point in the disc $\{1\}\times D$ by a path γ belonging to the leaf $\mathcal{L}_{(x,y)}$ whose length is uniformly bounded. This can be seen by taking the path obtained by following first

$$\gamma_1(\theta)=(xe^{-i\theta},ye^{i\lambda\theta}), \tag{3.35}$$

for θ varying from 0 to $arg(x)<2\pi$ and then following

$$\gamma_2(r)=(r,\frac{yx^\lambda}{r^\lambda}), \tag{3.36}$$

for r varying from $|x|$ to 1. The form η is bounded in the fixed neighborhood of the origin which we have chosen. As g is also holomorphic (hence bounded) on $\{1\}\times D$, it now follows from the definition (3.32) of g that it is bounded on a fixed neighborhood of the origin from which the y-axis has been deleted. By the removable singularity theorem, we can now extend g holomorphically to a full neighborhood of the origin.

Relation (3.12) follows from the definition of g. Indeed, it suffices to verify this relation locally in the complement of the origin, where ω is different from zero. By a local change of coordinates $(z,w)=(z(x,y),w(x,y))$, it can be assumed that

$\omega = dw$. Taking a section $z = k$ transverse to the leaves of the foliation and noting that g is obtained by integrating η along the leaves $w = const$, it follows that the dz coordinates of dg and η coincide. Hence $\eta - dg$ is colinear to ω. Now (3.12) is proved and the proof of Theorem 3.9 is completed. □

Remark 3.10. In [BC] the authors study the space

$$H^1_{top}(\omega) \tag{3.37}$$

given as the quotient of the space of germs of holomorphic relatively closed forms whose integral vanishes along any cycle tangent to leaves of a one-from ω by the space of relatively exact forms. A form η is relatively closed if

$$d\eta \wedge \omega = 0. \tag{3.38}$$

They study in particular the case when ω is a logarithmic form, i.e. has a Darboux first integral of the form

$$F = \prod F_i^{\lambda_i}. \tag{3.39}$$

They show that if F_i are irreducible and $\lambda_i \in \mathbb{N}$ are relatively prime then

$$H^1_{top}(\omega) = 0. \tag{3.40}$$

More generally, if λ_i in $\mathbb{C}^*$ satisfy a diophantine condition and ω is not dicritical, they prove that then (3.40) holds too. They observe also that $H^1_{top}(\omega)$ can be nonzero, if the exponents λ_i are arbitrary. As shown by our Theorem A in $\mathbb{C}^2$ this happens for the linear one-form ω given by (3.13) precisely if $\lambda \in \mathcal{C}$, since for λ irrational the leaves of the foliation defined by ω are simply connected so there are no nontrivial cycles γ on the leaves and condition

$$\int_\gamma \eta = 0 \tag{3.41}$$

is trivial.

Denote $\tilde{H}^1_{top}(\omega)$ the quotient space, whose numerator is formed of the space of germs of relatively closed one-forms such that (3.16) holds for any asymptotic cycle belonging to a leaf of the foliation defined by ω and the denominator is given by the space of relatively exact forms. Theorem 3.9 can be reformulated by saying that

$$\tilde{H}^1_{top}(\omega) = 0, \tag{3.42}$$

where ω denotes the one-form in $\mathbb{C}^2$ given by (3.13). It seems an interesting problem to determine the conditions on a logarithmic one-form ω under which (3.42) holds.

In [BL] the authors study the space $\tilde{H}^1_{top}(\omega)$ for ω given by

$$\omega = \frac{p}{q} y(1 + (a-1)u^k)dx + x(1 + au^k)dy. \tag{3.43}$$

They claim that the case of linear form ω (3.13) has been solved in [BC]. Their notion of asymptotic cycles coincides with ours for the form ω given by (3.43), but does not coincide for (3.13). There are no asymptotic cycles for linear system (3.13) in the sense of their definition. As a consequence (3.42) does not hold in general with their definition of asymptotic cycle. This justifies our study. Strangely enough, the study of the nonlinear case (3.43) is much easier than the study of the linear case (3.13) due to the simpler recurrence in the nonlinear case.

4. Normalizability and isochronous saddles

In the first part of this section, we give some classical results on orbital normalizability. Next we generalize Section 3 to normalizable systems. More precisely, we examine when an orbitally normalizable system can be conjugated to its orbital normal form by an analytic change of coordinates (without multiplication by a function). We introduce a generalization of the isochronicity condition which answers precisely to the above question.

When studying orbital normalizability, we work with analytic orbital equivalence. Recall that two systems are *orbitally analytically equivalent* if by an analytic change of coordinates one can be transformed to a (nonzero at the origin) multiple of the other. Hence, we can start with a system of the form

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -\frac{p}{q}y + yh(x,y). \end{aligned} \tag{4.1}$$

If the system (4.1) is normalizable but not integrable then it can be brought by an analytic change of coordinates $Y = y\phi(x,y)$, $\phi(0,0) \neq 0$, to the form

$$\begin{aligned} \dot{x} &= x \\ \dot{Y} &= -\frac{p}{q}Y(1+\psi(u)), \end{aligned} \tag{4.2}$$

where $\psi(u) = u^k + au^{2k} + o(u^{3k})$ is an analytic function in $u = x^pY^q$.

Proposition 4.1. *If a system* (4.1) *is normalizable and non integrable then there exists an analytic change of coordinates* $(\overline{X},\overline{Y}) = (x, \alpha y + o(x,y))$ *which transforms it into the normal form*

$$\begin{aligned} \dot{\overline{X}} &= \overline{X} \\ \dot{\overline{Y}} &= -\frac{p}{q}\overline{Y}(1 - \frac{U^k}{1+aU^k}). \end{aligned} \tag{4.3}$$

Proof. We start with a system in the form (4.2). There exists an analytic change of coordinates $U = u(1+f(u))$ which will transform $\dot{u} = -pu^{k+1}(1+au^k+o(u^{2k}))$ (corresponding to (4.2)) to $\dot{U} = p\frac{U^{k+1}}{1+aU^k}$ (corresponding to (4.3)) [K]. The corresponding analytic change of coordinates will be $\overline{Y} = \left(\frac{U}{x^p}\right)^{1/q}$. □

As a corollary, by multiplying (4.3) by $1+aU^k$, we get:

Corollary 4.2. *Given an orbitally normalizable system* (1.1) *which is non integrable, there exist $k \in \mathbb{N}$, $a \in \mathbb{C}$, an analytic function h, $h(0,0) \neq 0$, and an analytic change of coordinates transforming* (1.1) *to*

$$\begin{aligned} \dot{X} &= X(1 + aU^k)h(x,y), \\ \dot{Y} &= -\frac{p}{q}Y(1 + (a-1)U^k)h(x,y), \end{aligned} \tag{4.4}$$

where $U = X^pY^q$ is the resonant monomial (This corresponds to putting the 1-form into normal form).

The corollary justifies the equivalent definition 1.2 for orbital normalizability:

Theorem 4.3. *A non integrable system (1.1) with $\lambda = p/q$ is orbitally normalizable if and only if the holonomy of any separatrix is normalizable i.e. given by the composition of the time-one map of a vector field with vanishing linear part at the origin with a rotation of angle $-2\pi\frac{p}{q}$.*

Proof. Let $\lambda = \frac{p}{q}$. Suppose first that the system (1.1) is orbitally normalizable. Then it can be brought to the normal form (4.4). Take $X = e^{2\pi i\xi}$ and $Y = \overline{Y}e^{-2\pi i(p/q)\xi}$. To calculate the holonomy we must calculate

$$\begin{aligned} \frac{d\overline{Y}}{d\xi} &= \frac{\partial\overline{Y}}{\partial Y}\frac{dY}{dX}\frac{dX}{d\xi} + \frac{\partial\overline{Y}}{\partial\xi} \\ &= 2\pi i\frac{p}{q}\frac{\overline{Y}^{qk+1}}{1 + a\overline{Y}^{qk}}. \end{aligned} \tag{4.5}$$

As the holonomy is given by $e^{-2\pi i(p/q)}\overline{Y}(1)$ it is the time-one map of (4.5), composed with a rotation of angle $-2\pi\frac{p}{q}$.

Conversely, let us suppose that the holonomy of the x-separatrix of a system is given by the time-one flow of the vector field $\dot{y} = f(y) = y^{n+1}g(y)$, $g(0) \neq 0$ composed by a rotation of angle $-2\pi\frac{p}{q}$. By Kostov's theorem [K] or the proof of Proposition 4.1 above, we can find an analytic change of coordinates $\overline{Y} = \alpha y + o(y)$ transforming the vector field to $\dot{\overline{Y}} = 2\pi i\frac{p}{q}\frac{\overline{Y}^{n+1}}{1+a\overline{Y}^n}$. If we show that $n = kq$ for some k, then the holonomy is the same as that of (4.4) and since the holonomy characterizes the system up to orbital equivalence ([MM] and [MR]) the system is orbitally normalizable. To show that $n = qk$ we first remark that, if we perform analytic changes of coordinates tangent to the identity on (1.1), then the holonomies are conjugate, so we can decide to calculate the holonomy for a system in the prenormal form (2.25) (in which we take $\eta = 0$). The conclusion follows from (2.27). □

We now study when an orbitally normalizable system is normalizable to the resonant model (4.8) below. By Corollary 4.2, we assume the system of the form

$$\begin{aligned} \dot{x} &= x(1 + au^k)h(x,y) \\ \dot{y} &= -\frac{p}{q}y(1 + (a-1)u^k)h(x,y) \end{aligned} \tag{4.6}$$

with h analytic and $h(0) = 1$. Note that the system (4.6) has a first integral

$$F(x,y) = x^{pk(a-1)}y^{qka}e^{-1/x^{pk}y^{qk}} = u^k e^{-1/u^k}x^{-pk} = u^{k(a-1)}e^{-1/u^k}y^{kq}. \tag{4.7}$$

We examine when (4.6) can be put to the normal form

$$\begin{aligned} \dot X &= X(1+aU^k) \\ \dot Y &= -\frac{p}{q}Y(1+(a-1)U^k), \end{aligned} \tag{4.8}$$

where the formal invariants k and a are the same as in (4.6). Here $u = x^p y^q$ and $U = X^p Y^q$ are the resonant monomials.

Let

$$\omega = \frac{p}{q}y(1+(a-1)u^k)dx + x(1+au^k)dy, \tag{4.9}$$

$$\begin{aligned} dt &= \frac{dx}{x(1+au^k)h(x,y)}, \\ dt_{norm} &= \frac{dx}{x(1+au^k)} \\ \eta &= dt - dt_{norm}. \end{aligned} \tag{4.10}$$

Then η is of the form

$$\eta = \frac{O(x,y)dx}{x}. \tag{4.11}$$

Proposition 4.4. *The system* (4.6) *is analytically (resp. formally) normalizable to the form* (4.8) *if and only if there exists a germ of analytic function at the origin (resp. formal power series) g vanishing at the origin, such that*

$$(kp\eta - dg)\wedge\omega = 0, \tag{4.12}$$

for $xy \neq 0$.

Remark 4.5. By Lemma 3.6, condition (4.12) is equivalent to the form η being relatively exact, with respect to ω, i.e. to the existence of analytic functions (or formal power series) g and m such that

$$kp\eta = dg + m\omega. \tag{4.13}$$

Proof of Proposition 4.4. We give the proof in the analytic case. The proof in the formal case is identical except that g has to be a formal power series instead of being a germ of an analytic function.

Necessity. A change of coordinates which preserves the orbital normal form preserves the invariant coordinate axes, so must be of the form

$$
\begin{aligned}
X &= xm(x,y) = x(1+O(x,y)) \\
Y &= yn(x,y) = y(1+O(x,y)).
\end{aligned} \tag{4.14}
$$

We claim that the first integral F in (4.7) must be preserved, that is relation

$$F(x,y) = F(X(x,y), Y(x,y)) \tag{4.15}$$

must hold. Indeed, $F(X(x,y), Y(x,y))$ is also a first integral of (4.8), so the quotient (4.16) is also a first integral of (4.6). This first integral is of the form

$$\frac{F(X(x,y), Y(x,y))}{F(x,y)} = K(x,y)e^{1/u^k - 1/U^k}, \tag{4.16}$$

with

$$K = m^{pk(a-1)}n^{qka} = 1 + O(x,y). \tag{4.17}$$

Therefore taking the logarithm of (4.16), we obtain a meromorphic first integral of (4.6). However, (4.6) being non integrable (because of the presence of resonant term, any meromorphic first integral is trivial (i.e. constant). This follows directly from the results in the next section. We can of course suppose that $x = 0$ and $y = 0$ are invariant curves. Then the system has a rational integrating factor $V = x^{\alpha_1}y^{\beta_1}\prod_{i=1}^{m_1} F_i^{\gamma_i}$ and a first integral $H = x^{\alpha_2}y^{\beta_2}\prod_{j=1}^{m_2} G_j^{\delta_j}$ where $F_i(0,0), G_j(0,0) \neq 0$. By Theorem 5.10 we can show that the system is integrable (yielding a contradiction) as soon as we find an integrating factor which has only one of these factors: such an integrating factor is given by $V_1 = VH^{\frac{a_1}{a_2}}$ if $a_2 \neq 0$ and V otherwise. Putting

$$g(x,y) = \frac{1}{u^k} - \frac{1}{U^k} = \frac{1}{u^k}\frac{m^{pk}n^{qk} - 1}{m^{pk}n^{qk}}, \tag{4.18}$$

it follows that Ke^g is constant i.e. $g(x,y) = -\log(K) + C$. Moreover $C = 0$ as $g(0,0) = \log(K)(0,0) = 0$. Hence $K = e^{-g}$. Relation (4.18) gives

$$m^{pk}n^{qk} = (1 - u^k g(x,y))^{-1}. \tag{4.19}$$

Hence $n = m^{-\frac{p}{q}}(1-u^k g(x,y))^{-\frac{1}{qk}}$. Putting this relation in (4.17) allows to calculate m, namely $m = (1-u^k g)^{-\frac{a}{pk}}e^{\frac{g}{pk}}$, so that the change of coordinates (4.14) must be of the form

$$
\begin{aligned}
X &= x(1-u^k g(x,y))^{-a/(pk)}e^{g/(pk)}, \\
Y &= y(1-u^k g(x,y))^{(a-1)/(qk)}e^{-g/(qk)}.
\end{aligned} \tag{4.20}
$$

Note that (4.6) gives

$$\dot{u} = pu^{k+1}h \tag{4.21}$$

and (4.8) gives

$$\dot{U} = pU^{k+1}. \tag{4.22}$$

From (4.18)

$$\dot{g} = -k\frac{\dot{u}}{u^{k+1}} + k\frac{\dot{U}}{U^{k+1}} = kp(1-h). \tag{4.23}$$

Here $\dot{g} = \frac{\partial g}{\partial x}x(1+au^k)h - \frac{p}{q}\frac{\partial g}{\partial y}y(1+(a-1)u^k)h$. Calculating the terms in (4.12) we get $\eta = \frac{dx}{1+au^k}\frac{1-h}{h}$, $kp\eta \wedge \omega = kp\frac{1-h}{h}$ and $dg \wedge \omega = \frac{\dot{g}}{h}$. Hence g satisfies (4.12).

Sufficiency a straight-forward verification shows that if (4.12) holds, then (4.20) defines an analytic normalizing change of coordinates. This concludes the proof of the Proposition. □

The problem of relative exactness with respect to the form ω given by (4.9) has been studied by Berthier and Loray [BL]. They prove:

Lemma 4.6 [BL]**.** *Let*

$$\eta = \sum_{\substack{m,n\geq 0\\ m+n>0}} a_{m,n}x^m y^n \frac{dx}{x} \tag{4.24}$$

be a formal 1*-form. Then there exists a formal power series g vanishing at the origin such that* (4.12) *holds if and only if*

$$a_{p,q} = \cdots = a_{kp,kq} = 0. \tag{4.25}$$

This gives

Corollary 4.7.

(i) For any one-form η of the form (4.24) *there exists a formal power series g vanishing at the origin such that*

$$(pk\eta - \sum_{i=1}^{k} a_{ip,iq}u^i\frac{dx}{x} - dg) \wedge \omega = 0, \tag{4.26}$$

where ω is given by (4.9) *and $u = x^p y^q$ is the resonant monomial.*

(ii) There are exactly k obstructions to the existence of a formal change of variables transforming the system (4.6) *to the form* (4.8)*. They are given by the nonvanishing of the coefficients $a_{p,q}, \dots, a_{kp,kq}$ in* (4.24)*.*

(iii) For η in (4.10) *coming from* (4.6) *a necessary condition for the existence of g satisfying* (4.12) *is that there be no terms in u^l, $1 \leq l \leq k$, in $h(x,y)$.*

In the spirit of Section 3 we want to find an *isochronicity condition* under which an orbitally normalizable saddle will be analytically normalizable to its orbital normal form.

Note that the trajectories of the system (4.6) are described by the holonomy of its normal form. The dynamics of the holonomy is of *flower type* [Ca]. The leaves of the foliation are simply connected, so there are no cycles. However, there are plenty of *asymptotic cycles* (cf. Definition 3.2). Take a transversal Σ to the x-axis in the domain of convergence of orbital normalization. Then any point of

Σ sufficiently close to the origin belongs either to a stable or unstable *petal* (or to the intersection of a stable and an unstable petal). Loosely speaking this means that the orbit spins indefinitely around the y-axis in positive time, negative time, or both, and its intersection with Σ converges to the x-axis.

Definition 4.8. *An orbitally normalizable system* (4.6) *is isochronous if*

(i) there exists $A > 0$ bounding uniformly the integral $\int_\gamma \eta$, where η is given by (4.10) *and γ is any curve tangent to the foliation defined by the form ω, which belongs to the domain of validity of* (4.12)*;*

(ii)

$$\int_\gamma \eta = 0, \tag{4.27}$$

for any curve γ as in (i) which is an asymptotic cycle.

It is proved in [BL] that conditions (i) and (ii) in Definition 4.8 are equivalent to the relative exactness of the form η, which in turn is equivalent to the normalizability of (4.6) (Remark 4.5 and Proposition 4.4).

We thus obtain

Theorem F. *An orbitally normalizable system* (4.6) *is normalizable by an analytic change of coordinates (without multiplication by a function) to the form* (4.8) *if and only if it is isochronous.*

Note that the form η is essentially coordinate independent. More precisely the form dt is coordinate independent and for a change of coordinates of the form (4.14),

$$dt_{norm} = dx/x - (a/p)du/u = d(X)/(X) - (a/p)d(U)/(U) + d\phi, \tag{4.28}$$

for some analytic ϕ.

Proposition 4.9. *Any normalizable non integrable system can (without multiplication by a function) be put to the normal form*

$$\begin{aligned} \dot X &= X(1 + aU^k)(1 + a_1U + \cdots a_kU^k) \\ \dot Y &= -\frac{p}{q}Y(1 + (a-1)U^k)(1 + a_1U + \cdots a_kU^k), \end{aligned} \tag{4.29}$$

for some $a_1, \ldots, a_k \in \mathbb{C}$.

Proof. Given a normalizable system, we can assume that it is of the form

$$\begin{aligned} \dot x &= xk_1(u) \\ \dot y &= yk_2(u) \end{aligned} \tag{4.30}$$

with k_1 and k_2 analytic. As the system is in particular orbitally normalizable and nonintegrable, we assume without loss of generality that it is of the form

$$\begin{aligned} \dot x &= x(1 + au^k)h(u) \\ \dot y &= -(p/q)y(1 + (a-1)u^k)h(u), \end{aligned} \tag{4.31}$$

with $h(0) = 1$. Indeed, we observe first that we can put (4.30) in the form (4.31), but with $h = h(x, y)$. Next, h depends only on u, as k_1 and k_2 depend only on u.

Relation (4.31) gives (4.21), but with h depending only on u and (4.29) gives

$$\dot{U} = pU^{k+1}(1 + a_1U + \cdots a_kU^k) = pU^{k+1}H(U). \tag{4.32}$$

As in the proof of Proposition 4.1, the change of coordinates has to be of the form (4.20), with g given by (4.18), but depending only on u. Moreover now $g = g(u)$ has to verify

$$\dot{g} = \frac{dg}{du}pu^{k+1}h(u) = -kp(H(U) - h(u)) \tag{4.33}$$

instead of (4.23). Substituting

$$U = \frac{u}{(1 + gu^k)^{1/k}} \tag{4.34}$$

in (4.33) gives

$$\frac{dg}{du}u^{k+1} = k(1 - \frac{H(\frac{u}{(1+gu^k)^{1/k}})}{h(u)}). \tag{4.35}$$

In order to be able to solve the equation (4.35) it is necessary that

$$1 - H(u)/h(u) = O(u^{k+1}). \tag{4.36}$$

This is easily achieved by choosing $H(u)$ as the polynomial given by the k-jet of $h(u)$. This determines the coefficients a_i, $i = 1, \dots, k$, in the statement of the proposition. Now the right hand side of (4.35) is divisible by u^{k+1} and after this division, the function g verifying (4.35) is obtained by the standard existence theorem for solutions of differential equations.

A direct verification now shows that (4.20), for this choice of g, transforms the system (4.31) to (4.29). □

5. Darboux mechanism for normalizability, integrability and linearizability

In this section we introduce a generalization of Darboux methods, including the use of integrating factors vanishing at the origin and non analytic first integrals expanded in convergent series with log terms to yield normalizability. Together with a second mechanism introduced in Section 6, namely a blow-down transforming the saddle into a node, these mechanisms are sufficient to explain all quadratic integrable and linearizable systems

$$\begin{aligned} \dot{x} &= x + c_{20}x^2 + c_{11}xy + c_{02}y^2 \\ \dot{y} &= -\lambda y + d_{20}x^2 + d_{11}xy + d_{02}y^2 \end{aligned} \tag{5.1}$$

for $\lambda = 1$ ([CR]) and $\lambda = 2$ (Section 7 and [FZ]). They also allow to construct several classes of linearizable systems for arbitrary $\lambda \in \mathbb{R}^+$. Moreover they allow to identify several normalizable quadratic systems. We do not know if they are sufficient for describing all normalizable, integrable and linearizable quadratic systems for $\lambda = \frac{p}{q} \in \mathbb{Q}^+$.

We recall and generalize known results (Theorems 5.1 and 5.3) and definitions (Definitions 5.2 and 5.4) on Darboux integrability and linearizability (see [D], [S] or [MMR]). To make the paper self-contained we give 1-line proofs of the two theorems.

Theorem 5.1. *A polynomial system* (1.1) *of degree n is integrable if there exist analytic functions $F_1(x,y),\dots,F_m(x,y), K_1(x,y),\dots,K_m(x,y)$ and numbers $\alpha_1,\dots\alpha_m \in \mathbb{C}$ satisfying*

i) *for $i=1,\dots,m$*
$$\frac{\partial F_i}{\partial x}\dot{x} + \frac{\partial F_i}{\partial y}\dot{y} = F_i(x,y)K_i(x,y); \tag{5.2}$$

ii) $\sum_{i=1}^m \alpha_i K_i = 0$*;*

iii) *There are at most two values of i such that $F_i(0,0)=0$. In such a case the function F_i should have one of the forms $F_i(x,y) = x + o(x,y)$ (resp. $F_i(x,y) = y + o(x,y)$) and the corresponding α_i should be $\alpha_i = 1$ (resp. $\alpha_i = \lambda$).*

The first integral is given by $H(x,y) = \prod_{i=1}^m F_i^{\alpha_i}$.

Proof. It is easily verified that i) and ii) ensure that $\dot{H} = 0$. Condition iii) yields that the integral has the right regularity in the neighborhood of the origin. (This theorem goes back to Darboux when the $K_i(x,y)$ are polynomials. It is obvious here that this hypothesis is not necessary.) □

Definition 5.2.

(1) *An analytic function $F_i(x,y)$ satisfying* (5.2) *is called a generalized Darboux factor and the corresponding analytic function $K_i(x,y)$ is called the cofactor of $F_i(x,y)$. We note $K_i = \mathrm{cof}(F_i)$.*

(2) *A function $F(x,y)$ of the form $F(x,y) = \prod_{i=1}^m F_i^{\alpha_i}$, where the $F_i(x)$ are generalized Darboux factors is a generalized Darboux function. Although it may not be analytic such a function has a cofactor $K(x,y) = \sum_{i=1}^m \alpha_i K_i$.*

(3) *A system for which a first integral is found by means of the mechanism described in Theorem 5.1 is called generalized Darboux integrable.*

(4) *A (reciprocal) integrating factor is a Darboux function $M(x,y)$ such that its cofactor $K(x,y)$ satisfies $K(x,y) = div$, where div is the divergence of the vector field* (1.1).

Note that dividing a vector field by a reciprocal integrating factor yields a vector field of divergence zero.

Theorem 5.3. *A polynomial system* (1.1) *of degree n is linearizable if one of the three following situations occur:*

Case I. *There exist analytic functions $F_1(x,y),\dots,F_m(x,y), K_1(x,y),\dots,K_m(x,y)$ defined in a neighborhood of $(0,0)\in\mathbb{C}$ and numbers $\alpha_1,\dots,\alpha_{m-1},\beta_2,\dots,\beta_m \in \mathbb{C}$ satisfying* (5.2) *and*

i) *$F_1(x,y) = x + o(x,y)$, $F_m(x,y) = y + o(x,y)$, $F_i(0,0) \neq 0$ for $i = 2,\dots,m-1$;*

ii) $\sum_{i=1}^{m-1} \alpha_i K_i = 1$;
iii) $\sum_{i=2}^{m} \beta_i K_i = -\lambda$.

The linearizing change of coordinates is given by

$$(X,Y) = \left(\prod_{i=1}^{m-1} F_i^{\alpha_i}, \prod_{i=2}^{m} F_i^{\beta_i}\right) \tag{5.3}$$

and the system is integrable with first integral $X^\lambda Y$.

Case II. *The system is integrable with first integral $H(x,y) \sim x^\lambda y$ and there exist analytic functions $F_1(x,y), \dots, F_m(x,y), K_1(x,y), \dots, K_m(x,y)$ defined in a neighborhood of $(0,0) \in \mathbb{C}$ and numbers $\alpha_1, \dots \alpha_m \in \mathbb{C}$ satisfying* (5.2) *and*

i) $F_1(x,y) = x + o(x,y)$, $F_i(0,0) \neq 0$ *for* $i = 2, \dots, m$;
ii) $\sum_{i=1}^{m-1} \alpha_i K_i = 1$.

The linearizing change of coordinates is given by

$$(X,Y) = \left(\prod_{i=1}^{m} F_i^{\alpha_i}, \frac{H(x,y)}{X^\lambda}\right). \tag{5.4}$$

Case III. *An analogous case follows from case II under the change*

$$(x,y,t,\lambda) \mapsto (y,x,-\lambda t, \frac{1}{\lambda}). \tag{5.5}$$

Proof. The proof is an obvious generalization of [MMR] and [CR]. A simple calculation in Case I yields $\dot{X} = X, \dot{Y} = -\lambda Y$. In Case II we get $\dot{X} = X$ and $X^\lambda Y = x^\lambda y$ from which $\dot{Y} = -\lambda Y$ follows. (The previous versions of this theorem required the $K_i(x,y)$ to be polynomials. It is obvious that this hypothesis is not necessary.) □

Definition 5.4. *A system is called generalized Darboux linearizable if we can construct a linearizing change of coordinates by one of the three mechanisms described in Theorem 5.2.*

By Corollary 4.2 an orbitally normalizable system can be put to the form (4.6) by an analytic change of coordinates. It then has a first integral F given by (4.7). It will here be more convenient to work with the first integral $-\frac{1}{\ln F}$, which is of the form

$$H(x,y) = \frac{u^k}{1 + Au^k \ln x + Bu^k \ln y}. \tag{5.6}$$

It occurs sometimes in examples (as in Proposition 5.12) that we can directly find a generalized first integral of a system in the form of a convergent series with monomials of the form $x^i y^j \ln^k x \ln^l y$. We want to find conditions allowing to conclude to orbital normalizability. A first step in this direction is the following theorem.

Theorem 5.5. *Suppose a system of the form* (1.1), *with* $\lambda = \frac{p}{q}$ *rational has a first integral of the form*

$$H(x,y) = \frac{u^k F(x,y)}{1 + Au^k F(x,y) \ln x + Bu^k F(x,y) \ln y + G(x,y)} \tag{5.7}$$

where $u = x^p y^q$ *and* $G(x,y) = O(x,y)$ *and* $F(x,y) = 1 + O(x,y)$ *are analytic functions. Then the system is orbitally normalizable.*

Proof. We write $H(x,y)$ as

$$H_2(x,y) = \frac{u^k}{\phi(x,y) + Au^k \ln x + Bu^k \ln y}, \tag{5.8}$$

where $\phi(x,y) = \frac{1+G(x,y)}{F(x,y)} = 1 + O(x,y)$. Assume that $A \neq 0$. We seek a transformation $y = Y\ell(x,Y) = Y(1 + o(x,Y))$ to normalize this first integral, i.e. to bring it to the form

$$H(x,Y) = \frac{U^k}{1 + AU^k \ln x + BU^k \ln Y},$$

where $U = x^p Y^q$ and $H(x,Y)$ is the first integral of (4.3) (in coordinates (x,Y)). That is we want to solve

$$M(x,Y,\ell) = \phi(x, Y\ell(x,Y)) + Bx^{kp} Y^{kq} \ell^{kq} \ln \ell - \ell^{kq} = 0. \tag{5.9}$$

A solution of the form $\ell(x,Y) = 1 + \cdots$ follows directly from the implicit function theorem. $\square$

The two last theorems can be extended to the case of a saddle-node.

Corollary 5.6. *If a system of the form* (1.1) *with a saddle-node of finite order at the origin (i.e.* $\lambda = 0$*) is orbitally normalizable then there is an analytic change of coordinates bringing it (after possible division by a nonzero function) to the following polynomial normal form:*

$$\begin{aligned} \dot{X} &= \pm X(1 - aY^k) \\ \dot{Y} &= Y^{k+1} \end{aligned} \tag{5.10}$$

Proof. The proof is exactly the same as that of Proposition 4.1, with u (resp. U) replaced by y (resp. Y). $\square$

Corollary 5.7. *If a system of the form* (1.1) *with a saddle-node of finite order at the origin has a first integral of the form*

$$H(x,y) = \frac{y^k f(x,y)}{1 + ay^k f(x,y) \ln x - by^k f(x,y) \ln y + g(x,y)} \tag{5.11}$$

where $g(x,y) = O(x,y)$ *and* $f(x,y) = 1 + O(x,y)$ *are analytic, then the system is orbitally normalizable.*

Proof. The proof goes as in Theorem 5.5. $\square$

The Darboux method allows the construction of first integrals as in Theorem 5.5 and Corollary 5.7, allowing to prove the normalizability of some systems. In order to minimize the work to put, for each particular case, the first integrals into one of the forms described in Theorem 5.5 we introduce a general method involving integrating factors.

Lemma 5.8. *Let the system*

$$D_H = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \tag{5.12}$$

be Hamiltonian i.e. $P = H_y$, $Q = -H_x$ and let $(X,Y) = \phi(x,y) = (x + o(x,y), y + o(x,y))$ be a change of coordinates. Then the transformed system

$$D = \frac{1}{J}[(PX_x + QX_y)\frac{\partial}{\partial X} + (PY_x + QY_y)\frac{\partial}{\partial Y}], \tag{5.13}$$

where

$$J = \partial(X,Y)/\partial(x,y)], \tag{5.14}$$

is the Hamiltonian system for the Hamiltonian function $G = H \circ \phi^{-1}$, i.e. $D = D_G$.

Proof. Indeed,

$$\begin{aligned} D_G &= G_Y\frac{\partial}{\partial X} + -G_X\frac{\partial}{\partial Y} \\ &= (H_x x_Y + H_y y_Y)\frac{\partial}{\partial X} - (H_x x_X + H_y y_X)\frac{\partial}{\partial Y}. \end{aligned} \tag{5.15}$$

Now $D_G = D$ follows from

$$\begin{pmatrix} x_X & x_Y \\ y_X & y_Y \end{pmatrix} = \frac{1}{J}\begin{pmatrix} Y_y & -X_y \\ -Y_x & X_x \end{pmatrix}. \tag{5.16}$$

□

Corollary 5.9. *If the system* (1.1) *has a (reciprocal) integrating factor, then we can find a reciprocal integrating factor for any transformation of the system. The new integrating factor can be chosen to be a (locally non-vanishing) multiple of the pullback of the integrating factor in the original coordinates.*

Theorem 5.10. *If the system* (1.1) *has a local (reciprocal) integrating factor of the form*

$$H = \prod_{i=1}^{m} F_i^{\alpha_i}, \tag{5.17}$$

with F_i analytic in x and y and nonzero α_i, then the system is

- *integrable if λ is irrational;*
- *integrable or orbitally normalizable if λ is a nonzero rational.*

More precisely:

(i) *if all $F_i(0,0) \neq 0$, then the system is integrable;*
(ii) *if at most one $F_i(0,0)$ vanishes and the corresponding Darboux factor has one of the two forms $F_i(x,y) = x + o(x,y)$ or $F_i(x,y) = y + o(x,y)$, then the system is integrable;*
(iii) *if exactly two factors $F_1(x,y) = x + o(x,y)$ and $F_2(x,y) = y + o(x,y)$ vanish at the origin then the system is integrable, except when the two coefficients α_1 and α_2 are both integers greater than 1, in which case it is orbitally normalizable;*
iv) *if iii) is satisfied and there exists a Darboux linearization of one of the coordinates as in Case II or Case III of Theorem 5.3 then the system is normalizable.*

Proof. We straighten the axes of the system (1.1) first. The system will still have an integrating factor of the same form by Corollary 5.9. This yields an analytic system

$$\begin{aligned}\dot{x} &= x \sum_{i,j\geq 0} a_{i,j} x^i y^j \\ \dot{y} &= y \sum_{i,j\geq 0} b_{i,j} x^i y^j,\end{aligned} \tag{5.18}$$

with $a_{0,0} = 1$ and $b_{0,0} = -\lambda$.

Any F_i which vanishes at the origin must be of the form $x^k y^l h_i(x,y)$, with $h_i(x,y)$ analytic and non-vanishing at the origin. Hence, we can choose our integrating factor to be of the form $x^\alpha y^\beta h(x,y)$, with $h(x,y)$ analytic and non-vanishing at the origin. Since we are only interested in an orbital classification we can absorb the factor $h(x,y)$ into the system and relabel the $a_{i,j}$'s and $b_{i,j}$'s accordingly.

Since $x^\alpha y^\beta$ is a (reciprocal) integrating factor of (5.18), i.e. $div = \alpha\,\mathrm{cof}(x) + \beta\,\mathrm{cof}(y)$ we must have

$$(1+i-\alpha)a_{i,j} + (1+j-\beta)b_{i,j} = 0 \tag{5.19}$$

for all i and j.

This gives us a convergent first integral

$$\begin{aligned}\phi = &\sum_{1+j-\beta\neq 0} \frac{a_{i,j}}{1+j-\beta} x^{1+i-\alpha} y^{1+j-\beta} - \sum_{\substack{1+j-\beta=0\\ 1+i-\alpha\neq 0}} \frac{b_{i,j}}{1+i-\alpha} x^{1+i-\alpha} \\ &+ a_{\alpha-1,\beta-1}\ln(y) - b_{\alpha-1,\beta-1}\ln(x).\end{aligned} \tag{5.20}$$

If either α or β is not a positive integer, then we clearly have no logarithmic terms and so we have a first integral of the form $x^{1-\alpha}y^{1-\beta}m(x,y)$ with $m(x,y)$ analytic and non-zero at the origin. This covers (i) and (ii).

We now treat the case when α and β are integers.

The first case is $\alpha = \beta = 1$. Then $\phi(x,y) = a_{0,0}\ln(y) - b_{0,0}\ln(x) + n(x,y)$ for some analytic function n. Then $\exp(\phi) = x^{-b_{0,0}} y^{a_{0,0}} \exp(n(x,y))$ is a first integral for (5.18) yielding integrability.

This completes the case λ irrational as we must have either $\alpha = \beta = 1$ or one of α and β is irrational, since by (5.19)

$$(1-\alpha) = \lambda(1-\beta). \tag{5.21}$$

Finally we are left with the remaining case where $\alpha > 1$ and $\beta > 1$ are both integers. We therefore have a first integral

$$\phi(x,y) = x^{1-\alpha}y^{1-\beta}h(x,y) + a_{\alpha-1,\beta-1}\ln(y) - b_{\alpha-1,\beta-1}\ln(x), \tag{5.22}$$

for some analytic function $h(x,y)$ with $h(0,0)\neq 0$. The function ϕ^{-1} is of the form treated in Theorem 5.5 from which normalizability follows.

In the particular case where $a_{\alpha-1,\beta-1}$ and $b_{\alpha-1,\beta-1}$ vanish the system is integrable.

In case iv) we first linearize one equation of the system. The new system is again orbitally normalizable, which is in this case equivalent to being normalizable. $\square$

Corollary 5.11. *If the system* (1.1) *with* $\lambda = 0$ *has a local (reciprocal) integrating factor of the form* (5.17) *with* F_i *analytic in* x *and* y*, then the system is integrable or orbitally normalizable and the center manifold is analytic. More precisely:*

(i) *if* $F_1(0,0) = x + o(x,y)$ *and* $F_i(0,0) \neq 0$ *for* $i > 1$*, then the system is integrable;*

(ii) *if* $F_1(0,0) = x+o(x,y)$*,* $F_2(x,y) = y+o(x,y)$ *and* $F_i(0,0) \neq 0$ *for* $i > 1$*, then the system is integrable if* $\alpha_2 = 1$*. It is orbitally normalizable otherwise. It is normalizable if there exist* α_i *with* $i > 1$ *such that* $K_1 + \sum_{i>1} \alpha_i K_i = 1$*.*

Proof. The proof is very similar to that of Theorem 5.10 and we use the same notations and reductions. Indeed we do not have (5.18) a priori but $F_1(x,y) = 0$ gives an analytic separatrix (center manifold) and we can bring the system to the form (5.18). Because of (5.21) we need have $\alpha = 1$. In case $\beta = 0$ then the first integral (5.20) is analytic. In case $\beta = 1$ the first integral has the form $\phi(x,y) = a_{0,0}\ln y + n(x,y) = \ln Y$ for $Y = y\exp(n(x,y))$. When $\beta > 1$ we have a first integral of the form $\phi(x,y) = y^{1-\beta}h(x,y) + a_{0,\beta-1}\ln(y) - b_{0,\beta-1}\ln(x)$ for some analytic function $h(x,y)$ with $h(0,0) \neq 0$. The function ϕ^{-1} is of the form treated in Corollary 5.7 from which normalizability follows. □

As an example we treat the following family which shows that the union of integrable and normalizable systems seem to form nice algebraic sets in the space of all parameters. This example is partially treated in [Z].

Proposition 5.12. *The system*

$$\begin{aligned} \dot{x} &= x + c_{20}x^2 \\ \dot{y} &= -\lambda y + d_{20}x^2 + d_{11}xy + d_{02}y^2 \end{aligned} \tag{5.23}$$

for $\lambda > 0$ *is always linearizable or normalizable. More precisely it is*

i) *linearizable if* $\lambda \notin \mathbb{N}$*;*

ii) *linearizable if* $\lambda \in \mathbb{N}$ *and* $d_{02} = 0$*;*

iii) *linearizable if* $\lambda \in \mathbb{N}$ *and*

$$[d_{11} + (k-1)c_{20}][d_{11} + (\lambda - k)c_{20}] + \frac{(\lambda + 1 - 2k)^2}{k(\lambda + 1 - k)} d_{20}d_{02} = 0 \tag{5.24}$$

with $k = 1, \dots, [(\lambda+1)/2]$*;*

iv) *normalizable in the other cases where* $\lambda \in \mathbb{N}$*.*

Proof. The proof contains several subcases. We will treat completely the first ones and sketch the other ones. As the first equation can be linearized it suffices to consider integrability and normalizability. Apart from the invariant lines the other generalized Darboux factors come from the fact that the equation can be transformed to a Riccati equation. Details in [Z].

1. $c_{20}d_{02} \neq 0$**.** We scale $c_{20} = -1$, $d_{02} = 1$

$$\begin{aligned} \dot{x} &= x(1-x) \\ \dot{y} &= -\lambda y + d_{20}x^2 + d_{11}xy + y^2. \end{aligned} \tag{5.25}$$

The system has the following Darboux factors:

$$
\begin{array}{cc}
G_1(x,y) = x & K_1(x,y) = 1-x \\
G_2(x,y) = 1-x & K_2(x,y) = -x \\
G_3(x,y) = (y-\alpha x)F_1(x) + x(1-x)F_1'(x) & K_3(x,y) = -\lambda + x(\alpha + d_{11}) + y \\
G_4(x,y) = (\lambda - y + (\alpha-\lambda)x)F_2(x) - x(1-x)F_2'(x) & K_4(x,y) = (\alpha + d_{11} - \lambda)x + y
\end{array}
\tag{5.26}
$$

where $F_1(x) = F(a,b;c;x)$ and $F_2(x) = F(a-c+1, b-c+1; 2-c; x)$ with $F(a,b;c;x)$ is the Gauss hypergeometric function

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n \tag{5.27}$$

with

$$(a)_n = \begin{cases} a(a+1)\dots(a+n-1) & n \geq 1 \\ (a)_0 = 1 \end{cases} \tag{5.28}$$

and similarly for $(b)_n$ and $(c)_n$. Here we define α as a root of $\alpha^2 - (\lambda - d_{11})\alpha + d_{20} = 0$, $c = \lambda + 1$ and a and b are the roots of $A^2 - (1 + 2\alpha + d_{11})A + \alpha(\lambda+1) = 0$. When $\lambda \in \mathbb{N} \setminus \{0\}$ then $0 \geq 2 - c \in \mathbb{Z}$. We let $a_1 = a - c + 1$, $b_1 = b - c + 1$ and $c_1 = 2 - c$. Then $(c_1)_{c-1} = 0$ and $F_2(x)$ has terms in $\ln x$ (is not analytic) unless $(a_1)_{c-1} = 0$ or $(b_1)_{c-1} = 0$. Since $(c_1)_n = 0$ as soon as $n \geq \lambda$ we have $a = \lambda - i$ or $b = \lambda - i$ for $i = 0, \dots \lambda - 1$. This is case iii). In this case $F_2(z)$ is a polynomial of degree i.

Note that if we fix our choice of α then a is one root of the above equation if and only if $a' = \lambda + 1 - a$ is a root of the equation with the other choice of α. Thus the conditions for an integer root can be paired to give the expression (5.24).

In case iv) then the system has an integrating factor. Theorem 5.10 allows to conclude that the system is orbitally normalizable.

2. $c_{20} = 0$**,** $d_{02} \neq 0$**.** We scale $d_{02} = 1$. If $2\alpha - d_{11} \neq 0$ where $\alpha^2 - d_{11}\alpha + d_{20} = 0$ we can scale $2\alpha - d_{11} = -1$. The system has the following Darboux factors:

$$
\begin{array}{cc}
G_1(x,y) = x & K_1(x,y) = 1 \\
G_2(x,y) = e^x & K_2(x,y) = x \\
G_3(x,y) = (y+\alpha x)\overline{F}_1(x) + x\overline{F}_1'(x) & K_3(x,y) = -\lambda + x(\alpha+1) + y \\
G_4(x,y) = (\lambda - y - \alpha x)\overline{F}_2(x) - x\overline{F}_2'(x) & K_4(x,y) = (\alpha+1)x + y
\end{array}
\tag{5.29}
$$

where the function $\overline{F}_1(x) = {}_1F_1(a,b;x)$ is the confluent hypergeometric function

$${}_1F_1(a;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!(c)_n} x^n, \tag{5.30}$$

and $\overline{F}_2(x) = {}_1F_1(a-c+1; 2-c; x)$ for $a = -\alpha(1+\lambda)$, $c = \lambda+1$ and $2\alpha - d_{11} = -1$. The different cases are treated as above.

If $2\alpha - d_{11} = 0$, i.e $d_{20} = \frac{1}{4}d_{11}^2$, then we can suppose $d_{11} \neq 0$ (otherwise the system is obviously linearized) and we scale $d_{11} = -2$. The system has the same Darboux factors $G_1(x)$ and $G_2(x)$ as in (5.29). An additional Darboux factor G_3 with cofactor K_3 is given by

$$G_3(x) = (y-x)\tilde{F}(x) + x\tilde{F}'(x) \qquad K_3(x,y) = -\lambda - x + y \tag{5.31}$$

where $\tilde{F}(x) = \sum_{n=0}^{\infty} \frac{(1+\lambda)^n x^n}{n!(1+\lambda)_n}$ is analytic. We have a relation $div = 2G_3 + (1+\lambda)G_1$, which yields that the system is integrable if $\lambda \notin \mathbb{N}$ and normalizable otherwise.

3. $d_{02} = 0$**.** In that case we have the two invariant lines with respective cofactors:

$$\begin{array}{rc} F_1(x) = x & K_1(x) = 1 + c_{20}x \\ F_2(x) = 1 + c_{20}x & K_2(x) = c_{20}x \\ & div = 1 - \lambda + (2c_{20} + d_{11})x \end{array} \tag{5.32}$$

which yield an integrating factor. Using the integrating factor we can even find generically a third invariant algebraic curve as λ is a positive integer. Indeed the system is reduced to the linear equation

$$\frac{dy}{dx} = \left(-\frac{\lambda}{x} + \frac{\lambda c_{20} + d_{11}}{1 + c_{20}x}\right) y + \frac{d_{20}x}{1 + c_{20}x}. \tag{5.33}$$

For $c_{20} \neq 0$ (allowing to scale $c_{20} = 1$) the system has the general solution

$$y = x^{-\lambda}(1+x)^{\lambda + d_{11}}(h(x) + C), \tag{5.34}$$

where

$$h'(x) = d_{20}x^{1+\lambda}(1+x)^{-1-\lambda-d_{11}}. \tag{5.35}$$

Hence generically $h(x) = P(x)(1+x)^{-\lambda-d_{11}}$, where $P(x)$ is the following polynomial of degree $\lambda + 1$:

$$P(x) = -d_{20} \sum_{i=0}^{\lambda+1} \frac{\binom{1+\lambda}{i}}{\lambda + d_{11} - i} (-1)^i (1+x)^i. \tag{5.36}$$

For $C = 0$ this yields an invariant algebraic curve $yx^\lambda - P(x) = 0$, except when $\lambda + d_{11} = i$ for some i in which case a term in $\log(x+1)$ occurs in the function $P(x)$.

For $c_{20} = 0$ the linear system can be integrated as

$$y = x^{-\lambda} \exp(d_{11}x)(h(x) + C), \tag{5.37}$$

where $h(x) = d_{20}P(x)\exp(-d_{11}x)$, with $P(x)$ a polynomial of degree $\lambda+1$, yielding again for $C = 0$ an invariant algebraic curve $yx^\lambda - d_{20}P(x) = 0$. $\square$

6. Normalizability or linearizability via a blow-down bringing a saddle to a node or a saddle-node

This mechanism (see Theorem below) makes use of the result of Poincaré that a system is linearizable by means of an analytic change of coordinates in a neighborhood of a non-resonant node. In the case of a resonant node there is at most one resonant monomial and the system is linearizable if and only if that monomial has a zero coefficient. In all cases the system is normalizable.

Theorem 6.1. *A polynomial system*

$$\begin{aligned}\dot{x} &= x + \sum_{2\le i+j\le n} a_{ij}x^i y^j\\ \dot{y} &= -\lambda y + \sum_{2\le i+j\le n} b_{ij}x^i y^j\end{aligned} \tag{6.1}$$

is linearizable if there exists $r = \frac{p}{q} \in \mathbb{Q}^+$ such that the map

$$(X, Y) = (xy^r, y^r). \tag{6.2}$$

transforms it into a polynomial system with a linearizable node at the origin.

Such an r exists if the following conditions (i)–(iv) are satisfied:

i) $b_{ij} = 0$ *if*

$$j < r(i-1) + 1 \quad \text{or} \quad j \not\equiv 1 \ (mod\, p). \tag{6.3}$$

In particular $y = 0$ is invariant, yielding that the transformations are well-defined.

ii) *For $i = 0$,*

$$a_{0,j} = 0 \quad \text{if} \quad j \not\equiv 0 \quad (mod\, p); \tag{6.4}$$

for $i > 0$,

$$a_{ij} + rb_{i-1,j+1} = 0 \quad \text{if} \quad (j < r(i-1) \quad \text{or} \quad j \not\equiv 0 (mod\, p)\ ). \tag{6.5}$$

iii) *We have*

$$\lambda r - 1 > 0. \tag{6.6}$$

iv) *One of the following two conditions is satisfied: either*

$$\frac{\lambda r}{\lambda r - 1} \notin \mathbb{N} \tag{6.7}$$

or

$$\frac{\lambda r}{\lambda r - 1} \in \mathbb{N} \tag{6.8}$$

and the system is linearizable in the neighborhood of the node of the transformed system under the change of coordinates (6.2)*.*

When $\frac{\lambda r}{\lambda r - 1} \in \mathbb{N}$, and the condition iv) is violated, i.e. the node of the transformed system is not linearizable, then the origin of (6.1) *is normalizable.*

The system (6.1) *has a node at $(\frac{1}{rb_{11}}, 0)$ if $b_{11} \neq 0$. The node is linearizable if and only if the node at the origin of the transformed system by* (6.2) *is linearizable.*

Proof. The idea of the proof is to construct the linearizing change of coordinates $(x_1, y_1) = (x + o(x, y), y + o(x, y))$ of (6.1) from the linearizing change of coordinate of the node of the system in (X, Y)-coordinates. Indeed we transform (6.1) by the change of coordinates (6.2). Conditions i) and ii) ensure that the transformed system is polynomial. Condition iii) ensures that the origin is sent to a node and condition iv) ensures that this node is linearizable by an analytic change of coordinates (Poincaré theorem)

$$\begin{aligned}(X_1, Y_1) &= (X + o(X, Y), AX + Y + o(X, Y)) =\\ &= (y^r(x + o(x, y), y^r(1 + O(x, y)).\end{aligned} \tag{6.9}$$

Indeed the expression in x and y in (6.9) is well-defined since, in the system in X and Y, only monomials X^lY^m exist where $l+m\equiv 1 \mod q$. Then it is clear that this property is also true for X_1 and Y_1 from which the second part of (6.9) is well-defined. The original system is linearized by means of a transformation

$$(x_1,y_1)=\left(\frac{X_1}{Y_1},Y_1^{1/r}\right). \tag{6.10}$$

When $\frac{\lambda r}{\lambda r-1}=n\in\mathbb{N}$ and the node is not linearizable, i.e there is a resonant monomial X_1^n, then $n\equiv 1 \mod q$, which implies that $(n-1)r\in\mathbb{N}$. We can find (X_1,Y_1) as in (6.9) bringing the system to the form:

$$\begin{aligned}\dot X_1&=X_1\\ \dot Y_1&=nY_1+aX_1^n,\end{aligned} \tag{6.11}$$

with $a\neq 0$, yielding that (6.1) is transformed by means of (6.10) into

$$\begin{aligned}\dot x_1&=(1-n)x_1-ax_1^{n+1}y_1^{(n-1)r}\\ \dot y_1&=\frac{n}{r}y_1+\frac{a}{r}x_1^ny_1^{(n-1)r+1}.\end{aligned} \tag{6.12}$$

This is the normal form of the system provided $(n-1)r\in\mathbb{N}$.

For the last part of the proof we first remark that $y=0$ is an invariant curve of (6.1) and that (6.1) is invariant under $y\mapsto\omega y$ with $\omega^p=1$. The node at $(\frac{1}{rb_{11}},0)$ has an analytic invariant curve of the form $x=\frac{1}{rb_{11}}+f(y^p)$, which corresponds to a curve $X=\frac{1}{rb_{11}}Y+Yf(Y^q)$, i.e. an invariant curve through the origin in (6.13). Both are analytic exactly when the function f is analytic. □

The previous theorem can be generalized to the case where the blow-down of the system has a saddle-node at the origin. In particular it provides a useful criterion to prove that a saddle point is not normalizable.

Theorem 6.2. *We consider a system* (6.1) *satisfying* (6.3)-(6.5) *with* $\lambda=\frac{1}{r}=\frac{q}{p}\in\mathbb{Q}$. *The transformation* $(X,Y)=(xy^r,y^r)$ *brings it to a system of the form*

$$\dot X=o(X,Y),\qquad \dot Y=-Y+bX+o(X,Y). \tag{6.13}$$

Then the origin of (6.1) *is orbitally normalizable (linearizable) if and only if the origin of* (6.13) *is orbitally normalizable (linearizable). Moreover the system* (6.1) *has a semi-hyperbolic point at* $(\frac{1}{rb_{11}},0)$ *(exactly one zero eigenvalue). This point has an analytic center manifold if and only if the origin of* (6.13) *has an analytic center manifold. In particular if the origin of* (6.13) *does not have an analytic center manifold, then the origin of* (6.1) *is not orbitally normalizable.*

Proof. The proof runs much like Theorem 6.1 with some extra checking.

1. In (6.13) we have $b=rb_{1,1}$. Conditions (6.3)-(6.5) yield that the system (6.1) has $y=0$ as an invariant line, and a semi-hyperbolic point at $(1/b,0)$. (This point is at infinity when $b = 0$.) Furthermore, the system (6.1) is invariant under the substitution $y \mapsto \omega y$ where $\omega^p = 1$.

2. Consider now the transformed system (6.13). For each monomial in this system X^mY^n we must have $m + n \equiv 1 \bmod q$. Thus the system is invariant under the transformation $(X, Y) \mapsto (\eta X, \eta Y)$ where $\eta^q = 1$. Hence, if $Y - g(X) = 0$ is a formal expansion for the center manifold, then $g(X) = Xf(X^q)$. Similarly, if $X - h(Y) = 0$ is the expression for the stable manifold, then $h(Y) = Y\ell(Y^q)$. We must have $f(0) = b$ and $\ell(0) = 0$ from consideration of equation (6.13). Also the cofactors of these two expressions must have monomials X^mY^n with $m + n \equiv 0 \bmod q$.

3. Suppose that (6.13) is orbitally normalizable, with resonant monomials $X^{k'+1}$ and $YX^{k'}$, then we must have $k' = kq$ for some k. Thus (6.13) has a (reciprocal) integrating factor

$$[Y + Xf(X^q)][X - Y\ell(Y^q)]^{kq+1}e^{\phi(X,Y)}, \tag{6.14}$$

for some analytic function ϕ. The equation satisfied by ϕ is of the form

$$\dot{\phi} = H(X, Y), \tag{6.15}$$

where H is a linear combination of the divergence of (6.13) and the cofactors above. Hence, the terms of ϕ must be of the form X^mY^n with $m + n \equiv 0 \bmod q$. Thus $\phi(X, Y) = \tilde{\phi}(x, y)$ for some analytic function $\tilde{\phi}$.

4. Then, if (6.13) is orbitally normalizable we have (using Lemma 5.8) the following integrating factor for (6.1):

$$\frac{\partial(x, y)}{\partial(X, Y)} y^r[1 + xf(x^qy^p)]y^{kp+r}[x - \ell(y^p)]^{kq+1}e^{\phi(xy^r,y^r)}, \tag{6.16}$$

which simplifies to

$$\frac{1}{r}[1 + xf(x^qy^p)]y^{kp+1}[x - \ell(y^p)]^{kq+1}e^{\tilde{\phi}(x,y)}. \tag{6.17}$$

Thus (6.1) is orbitally normalizable [with resonant monomials xu^k and yu^k where $u = x^qy^p$].

5. Conversely, let us assume that (6.1) is orbitally normalizable, and that the invariant separatrix tangent to the y axis is $x - \ell(y^p) = 0$, with $\ell(0) = 0$. It follows that there is an integrating factor of the form

$$y^{kp+1}[x - \ell(y^p)]^{kq+1}e^{\tilde{\phi}(x,y)}, \tag{6.18}$$

for some analytic function $\tilde{\phi}$. Let S (resp. S') be the vector space generated by all monomials x^iy^j with $j \equiv 0 \bmod p$ and $j \geq r(i - 1)$ (resp. $j \equiv 0 \bmod p$ and $j \geq ri$). It is clear from the conditions of §1 that the cofactor of y has all monomials in S and also the divergence of (6.1). Furthermore, it is clear that the cofactor of $y^r(x - \ell(y^p))$ also has all monomials in S (this follows, because this expression is of the form $X - Y\ell(Y^q)$ which defines an invariant curve of (6.13) and therefore has a cofactor whose monomials X^mY^n have $m+n \equiv 0 \mod q$. Whence, $(x-\ell(y^p))$ has its cofactor in S also. Finally $\tilde{\phi}(x,y)$ must satisfy the equation

$$\dot{\tilde{\phi}} = \tilde{H}(x,y), \tag{6.19}$$

where $\tilde{H}$ has only monomials in S. Let us show that $\tilde{\phi}$ has only monomials in S'.

The first case is when the system (6.1) has only monomials xu (resp. yu) where $u = x^iy^j$ with $j \geq ri$. Then y, $x-\ell(y^p)$ have cofactors with monomials in S' and $div \in S'$, yielding that H is in S'. Let us suppose that $\tilde{\phi}$ has a monomial not in S' and let $v = x^{i'}y^{j'}$ with $j' < ri'$ be such a monomial of least degree. Then $\dot{\tilde{\phi}}$ contains the monomial v whose coefficient has been multiplied by $(i' - j'/r) \neq 0$, yielding a contradiction.

Otherwise the function $\tilde{\phi}$ can be decomposed as $\tilde{\phi} = \tilde{\phi}_1 + \tilde{\phi}_2$ where $\tilde{\phi}_1$ (resp. $\tilde{\phi}_2$) is composed with monomials in S' (resp. not in S'). We want to show that $\dot{\tilde{\phi}}_1$ has only monomials in S, while ϕ_2 is identically zero.

Let $u = x^iy^j$ a monomial in S' and $\dot{u} = u_x\dot{x} + u_y\dot{y}$. We have $xu_x, yu_y \in S'$ and $\frac{\dot{x}}{x}, \frac{\dot{y}}{y} \in S$ yielding $\dot{u} \in S$.

Suppose now $u = x^iy^j$ be a monomial of lower degree in $\tilde{\phi}_2$ and let $v = x^{i'}y^{j'}$ be a monomial of lower degree in $\dot{x}$ such that $\frac{v}{x} \in S \setminus S'$ is a nonzero monomial of least degree of (6.1) in $\dot{x}$. (If there are several monomials of same least degree we choose the ones with minimal i and i'.) Then yv appears with a nonzero coefficient $b_{i',j'+1}$ in $\dot{y}$. We have $\dot{u} = u_x\dot{x} + u_y\dot{y}$. In $\dot{\tilde{\phi}}_2$ appears the monomial $w = x^{i+i'}y^{j+j'} \notin S$ with coefficient $b_{i',j'+1}(-ir+j) \neq 0$ yielding a contradiction. Hence $\tilde{\phi}$ has all its monomials in S'.

6. Thus, if (6.1) is orbitally normalizable we must have an integrating factor

$$\frac{\partial(X,Y)}{\partial(x,y)} y^{kp+1}[x-\ell(y^p)]^{kq+1}e^{\tilde{\phi}(x,y)}, \tag{6.20}$$

which simplifies to

$$[xy^r - y^r\ell(y^p)]^{kq+1}y^re^{\tilde{\phi}(x,y)}. \tag{6.21}$$

The final step is to note that $y^re^{\tilde{\phi}}$ is analytic in X and Y since $\tilde{\phi}$ is in S. The reciprocal integrating factor must therefore be of the form

$$[X - Y\ell(Y^q)]^{kq+1}B(X,Y), \tag{6.22}$$

where $B(X,Y)$ is analytic in X and Y and must therefore represent (via $B=0$) the center manifold of (6.13).

The last part of the proof follows by remarking that the center manifold of $(\frac{1}{b}, 0)$ for (6.1) has the form $x = \frac{1}{b} + f(y^p)$, i.e. corresponds to a curve $X = \frac{1}{b}Y + Yf(Y^q)$, i.e. precisely to the center manifold of the origin in (6.13). Both are analytic exactly when the function f is analytic. $\square$

Corollary 6.3. *Similar criteria as in Theorem 6.1 and Theorem 6.2 obviously exist if we interchange the roles of x and y by means of* (5.5) *which transforms* (6.1) *in a system of the same form. The new coefficients A_{ij} and B_{ij} satisfy $A_{ij} = \frac{b_{ji}}{\lambda}$, $B_{ij} = \frac{a_{ji}}{\lambda}$.*

As an example we treat the following family:

Proposition 6.4. *The system*

$$\begin{aligned}\dot{x} &= x - \frac{2}{q}x^2 + y^2\\ \dot{y} &= -\lambda y + xy\end{aligned} \tag{6.23}$$

with $q \in \mathbb{N}$

i) *is linearizable when* $\lambda \notin \mathbb{Q}$ *and* $\lambda > \frac{q}{2}$*;*
ii) *is normalizable but not integrable when* $\lambda = \frac{kq+1}{2k}$ *for* $k \in \mathbb{N}$*;*
iii) *is linearizable when* $\lambda \in \mathbb{Q}$*,* $\lambda > \frac{q}{2}$ *and* $\forall k \in \mathbb{N}, \lambda \neq \frac{kq+1}{2k}$*;*
iv) *is not orbitally normalizable for* $\lambda = \frac{q}{2}$*;*

Proof. We use the change of coordinates $(X, Y) = (xy^{2/q}, y^{2/q})$ which transforms (6.23) into

$$\begin{aligned}\dot{X} &= (1 - \frac{2\lambda}{q})X + Y^{1+q}\\ \dot{Y} &= \frac{2}{q}X - \frac{2\lambda}{q}Y.\end{aligned} \tag{6.24}$$

The origin is a node for $\lambda > \frac{q}{2}$ and a semi-hyperbolic point if $\lambda = \frac{2}{q}$. For $\lambda > \frac{q}{2}$ the system is then linearizable as soon as the node is not resonant i.e. $\frac{2\lambda}{2\lambda - q}$ is an integer or the resonant monomial has a vanishing coefficient. In the latter case let $n = \frac{2\lambda}{2\lambda - q}$. Since the system (6.24) is symmetric under $(X, Y) \mapsto (\eta Z, \eta Y)$ with $\eta^q = 1$ the resonant monomial is zero unless $n = kq + 1$ for some $k \in \mathbb{N}$, which implies $\lambda = \frac{kq+1}{2k}$. Hence, unless $\lambda = \frac{kq+1}{2k}$ for some $k \in \mathbb{N}$ the normal form for the node is linear and the normalizing change of coordinates is analytic.

Let us now consider the case where $\lambda = \frac{kq+1}{2k}$ and check that the normal form for the resonant node is not linear. Indeed, if the node is linearizable then it has an analytic invariant curve of the form

$$Y = \sum_{i=1}^{\infty} a_i X^i. \tag{6.25}$$

The a_i are found inductively by

$$\begin{cases} a_1 = \frac{2}{q} \\ \left((iq)\frac{2\lambda}{q} - iq + 1\right) a_{iq+1} = \sum_{i_1+\cdots+i_{q+2}=iq+1} (i_1+1)a_{i_1+1}a_{i_2}\dots a_{i_{q+2}}, \end{cases} \tag{6.26}$$

($a_0 = 0$ in (6.25)). When $\lambda = \frac{kq+1}{2k}$ this yields

$$\left(\frac{i}{k}(kq+1) - (iq+1)\right) a_{iq+1} = \sum_{i_1+\cdots+i_{q+2}=iq+1} (i_1+1)a_{i_1+1}a_{i_2}\dots a_{i_{q+2}}. \tag{6.27}$$

The a_{iq+1} have alternating signs yielding a fixed sign for all terms in the right hand side of (6.26), which hence does not vanish. So (6.27) has no solution for $i = k$.

In the case $\lambda = \frac{2}{q}$ the origin of (6.24) is a semi-hyperbolic point. We show that it has no analytic center manifold. Indeed such a manifold would be of the form (6.25). The conditions (6.26) yield in that case

$$-a_{iq+1} = \sum_{i_1+\cdots+i_{q+2}=iq+1} (i_1+1)a_{i_1+1}a_{i_2}\dots a_{i_{q+2}}, \tag{6.28}$$

which yields divergence of the series because of the factor i_1+1 and as all terms in the right hand side of (6.28) have a fixed sign. □

Question. *What happens with system* (6.23) *for the other values of* λ*? Could it happen that the system is not linearizable (resp. integrable) as soon as* $\lambda < \frac{q}{2}$*, except maybe for a few values (indeed when* $q = 1$ *the system is integrable but not linearizable for* $\lambda = \frac{1}{4}$ *and not integrable for* $\lambda = \frac{1}{3}$*)?*

7. Linearizable quadratic systems

This section is more technical. It contains the "experimental" material that led to the questions studied in the paper. Initially, the idea was to classify quadratic systems with integrable saddles with $\lambda = 2$. However many of the strata extend to continuous values of λ. These are given in Theorem 7.2. It also shows that the techniques developed here seem to be the right tools to use when we examine polynomial (rather than analytic) systems. In [Z], Żołądek gives a list of strata of integrable quadratic systems for rational values of λ. We have enlarged this list and have also studied the irrational values of λ. Although the process for finding integrable systems looks to be discontinuous—depending on whether λ is rational, or diophantian irrational, or Liouvillian—we were surprised by what seemed to be a "continuous" skeleton of integrable (resp. linearizable) systems in the complete 7-parameter space of

$$\begin{aligned} \dot{x} &= x + c_{20}x^2 + c_{11}xy + c_{02}y^2 \\ \dot{y} &= -\lambda y + d_{20}x^2 + d_{11}xy + d_{02}y^2. \end{aligned} \tag{7.1}$$

Żołądek only gives the first integrals from which the Darboux factors could be recovered. We present all the Darboux factors so we are in the position of applying the theorems of Sections 5 and 6.

In [FSZ] we find the complete list of the 20 integrability conditions of (7.1) when $\lambda = 2$. We determine precisely which of these systems are linearizable. Whenever we can we generalize the results to families with arbitrary values of $\lambda \in \mathbb{R}_{>0}$, but this is in no sense meant to be an exhaustive list of possibilities.

Theorem 7.1. *A system*

$$\begin{aligned} \dot{x} &= x + c_{20}x^2 + c_{11}xy + c_{02}y^2 \\ \dot{y} &= -2y + d_{20}x^2 + d_{11}xy + d_{02}y^2 \end{aligned} \tag{7.2}$$

is linearizable if and only if one of the conditions I-VIII, X-XII listed below in Theorem 7.2 is satisfied for the special case $\lambda = 2$*, or if one of the two additional conditions XIII, XIV is satisfied.*

XIII. $c_{20} + 2d_{11} = 2c_{11}d_{20} - d_{20}d_{02} + 2d_{11}^2 = 2c_{11}d_{11} + c_{02}d_{20} = 0$;
XIV. $4c_{20} - 19d_{11} = c_{11} - d_{02} = 4c_{11}d_{20} + 35d_{11}^2 = 16c_{02}d_{20}^2 + 125d_{11}^3$.

Theorem 7.2. *A system* (7.1) *is linearizable if one of the conditions I-XI listed below is satisfied.*

I. $d_{20} = d_{11} = 0$ *for* $\lambda \neq 1/n$ *for all* $n \in \mathbb{N}$. *When* $\lambda = 1/n$ *the stratum is smaller and splits into the two parts* $c_{20} = d_{20} = d_{11} = 0$ *and IX.*

II. $d_{11} = c_{02} = \lambda c_{11} - (\lambda - 1)d_{02} = 0$.

III. $d_{11} + (\lambda - 1)c_{20} = c_{11} = d_{20} = 0$.

IV. $c_{11} = c_{02} = 0$ *for* $\lambda \notin \mathbb{N}$. *For* $\lambda \in \mathbb{N}$ *the stratum is smaller and splits into two parts, namely* $c_{11} = c_{02} = d_{02}$ *and VIII.*

V. $c_{20} - d_{11} = c_{11} - d_{02} = c_{02} = d_{20} = 0$.

VI. $c_{20} - 2d_{11} = c_{11} = d_{20} = d_{02} = 0$.

VII. $c_{20} + \frac{2}{q}d_{11} = c_{11} = d_{20} = d_{02} = 0$ *for* $q \in \mathbb{N}$ *if* $\lambda > \frac{q}{2}$ *and* $\lambda \neq \frac{nq}{2(n-1)}$ *for all* $n \in \mathbb{N} \setminus \{1\}$.

VIII. $c_{11} = c_{02} = [d_{11} + (k-1)c_{20}][d_{11} + (\lambda - k)c_{20}] + \frac{(\lambda+1-2k)^2}{k(\lambda+1-k)}d_{20}d_{02} = 0$, *for* $\lambda \in \mathbb{N}$ *and* $k = 1, \dots [(\lambda+1)/2]$ *(Solutions of the last equation come in pairs* $(k, \lambda + 1 - k)$*).*

IX. $d_{11} = d_{20} = [c_{11} + (k-1)d_{02}][c_{11} + (\frac{1}{\lambda} - k)d_{02}] + (\frac{1}{\lambda} + 1 - 2k)^2(k(\frac{1}{\lambda} + 1 - k))^{-1}c_{20}c_{02} = 0$, *for* $\frac{1}{\lambda} \in \mathbb{N}$ *and* $k = 1, \dots [(1/\lambda + 1)/2]$.

X. $2(2+\lambda)d_{02} - (3+4\lambda)c_{11} = 2(1+2\lambda)c_{20} - (4+3\lambda)d_{11} = 4(1+2\lambda)^2 d_{20}d_{02} - \lambda(2+\lambda)(3+4\lambda)d_{11}^2 = \lambda(3+4\lambda)^2 d_{11}c_{02} - 2(1+2\lambda)^2 d_{02}^2 = (2+\lambda)d_{11}d_{02} - 2(3+4\lambda)c_{02}d_{20} = 0$.

XI. $c_{02} = c_{20} - (2\lambda - 1)d_{11} = \lambda c_{11} - (2\lambda - 1)d_{02} = d_{20}d_{02} + \lambda(\lambda - 1)d_{11}^2 = 0$.

XII. $2\lambda c_{20} - (3 + 2\lambda)d_{11} = (2 + 3\lambda)c_{11} - 2d_{02} = (\lambda + 2)d_{11}^2 + 2\lambda^2 c_{11}d_{20} = 8\lambda^3 c_{02}d_{20}^2 + (1+2\lambda)(2+\lambda)^2 d_{11}^3 = 0$.

Proof of Theorems 7.1 and 7.2. The necessity of the conditions for $\lambda = 2$ has been proved in two different independent ways: the first one has been to annihilate the coefficients of the normal form which have been computed on Reduce. The second one has been to consider the 20 strata of integrable points given in [FSZ] and to list the systems for which the first four coefficients of the normal form (calculated in Maple) vanish.

We prove the sufficiency of the conditions by proving that all the systems appearing in cases I-XIV are indeed linearizable. Whenever we can, we make the proof directly for a general λ, the case $\lambda = 2$ being a particular case.

We only consider the generic cases. The non-generic cases are computable in a similar way from Darboux factors coming from coalescence of algebraic curves and algebraic curves coalescing with the line at infinity [C]. Most calculations were performed by Reduce or Maple, and we do not include them, as they would add very little (apart from bulk) to the exposition.

I. This case is the dual of Proposition 5.12 under (5.5).

II. The system can be brought to

$$\begin{aligned} \dot{x} &= x + c_{20}x^2 + (\lambda - 1)hxy \\ \dot{y} &= -\lambda y + d_{20}x^2 + \lambda hy^2, \end{aligned} \tag{7.3}$$

with $h \in \mathbb{C}$. It has the following three invariant lines:

$$\begin{aligned} F_1(x,y) &= x & K_1(x,y) &= 1 + c_{20}x + (\lambda - 1)hy \\ F_{2,3}(x,y) &= 1 + A_{2,3}x - hy & K_{2,3}(x,y) &= A_{2,3}x + \lambda hy \\ & & div &= 1 - \lambda + 2c_{20}x + (3\lambda - 1)hy \end{aligned} \tag{7.4}$$

where $A_{2,3}$ are the two roots of $A^2 - Ac_{20} + hd_{20} = 0$. This allows to construct an integrating factor, yielding integrability by Theorem 5.10 and linearizability by Theorem 5.3.

III. This case is dual of II.

IV. This case is treated in Proposition 5.12.

V. The system can be brought to the form

$$\begin{aligned} \dot{x} &= x(1 + c_{20}x + c_{11}y) \\ \dot{y} &= y(-\lambda x + c_{20}x + c_{11}y. \end{aligned} \tag{7.5}$$

This system has the invariant lines with respective cofactors:

$$\begin{array}{cc} F_1(x,y) = x & K_1(x,y) = 1 + c_{20}x + c_{11}y \\ F_2(x,y) = y & K_2(x,y) = -\lambda + c_{20}x + c_{11}y \\ F_3(x,y) = 1 + c_{20}x - \frac{c_{11}}{\lambda}y & K_3(x,y) = c_{20}x + c_{11}y \end{array} \tag{7.6}$$

from which we can construct a linearizing change of coordinates.

VI. The system

$$\begin{aligned} \dot{x} &= x + 2d_{11}x^2 + c_{02}y^2 \\ \dot{y} &= -\lambda y + d_{11}xy \end{aligned} \tag{7.7}$$

is Darboux integrable with the following invariant algebraic curves and respective cofactors:

$$\begin{array}{cc} F_1(x,y) = (1+2\lambda)x + c_{02}y^2 & K_1(x,y) = 1 + 2d_{11}x \\ F_2(x,y) = y & K_2(x,y) = -\lambda + d_{11}x \\ F_3(x,y) = 1 + 2d_{11}x + \frac{c_{02}d_{11}}{\lambda}y^2 & K_3(x,y) = 2d_{11}x. \end{array} \tag{7.8}$$

VII. This system has been treated in Proposition 6.4.

VIII. This system has been treated in Proposition 5.12.

IX. This system is dual of VIII.

X. The system can be brought to the form

$$\begin{aligned} \dot{x} &= x + (3\lambda+4)x^2 - 2\lambda(2+\lambda)xy + \lambda(1+2\lambda)y^2 \\ \dot{y} &= -\lambda y - (2+\lambda)x^2 + 2(1+2\lambda)xy - \lambda(3+4\lambda)y^2 \end{aligned} \tag{7.9}$$

and has the invariant algebraic curves with respective cofactors:

$$\begin{array}{cc} F_1(x,y) = x + \lambda(x+y)^2 = 0 & K_1(x,y) = 1 + 4(1+\lambda)x - 4\lambda(1+\lambda)y \\ F_2(x,y) = y + (x+y)^2 = 0 & K_2(x,y) = -\lambda + 4(1+\lambda)x - 4\lambda(1+\lambda)y \\ F_3(x,y) = 1 + 2(\lambda+1)(x+y) = 0 & K_3(x,y) = 2(\lambda+1)(x - \lambda y). \end{array} \tag{7.10}$$

XI. We only consider the generic case $c_{20}c_{11} \neq 0$ allowing to scale the system to

$$\begin{aligned} \dot{x} &= x + (2\lambda - 1)x^2 - (2\lambda - 1)xy \\ \dot{y} &= -\lambda y + (\lambda - 1)x^2 + xy - \lambda y^2. \end{aligned} \tag{7.11}$$

The system has two invariant curves yielding an integrating factor. Also the first separatrix can be linearized, yielding that the system is linearizable:

$$\begin{aligned} F_1(x,y) &= x & K_1(x,y) &= 1 + (2\lambda - 1)(x - y) \\ F_2(x,y) &= 1 + 2\lambda x + 2y + (x-y)^2 & K_2(x,y) &= 2\lambda(x-y) \\ & & div &= 1 - \lambda + (4\lambda - 1)(x - y). \end{aligned} \tag{7.12}$$

When $\lambda = 2$ there is a cubic invariant. In general we have a DSC first integral.

XII. The system in the generic case $c_{11}d_{11} \neq 0$ can be brought to the form

$$\begin{aligned} \dot{x} &= x + \frac{3+2\lambda}{2}x^2 + xy - \frac{1+2\lambda}{2}y^2 \\ \dot{y} &= -\lambda y - \frac{\lambda+2}{2}x^2 + \lambda xy + \frac{2+3\lambda}{2}y^2. \end{aligned} \tag{7.13}$$

It has the following invariant curves yielding a linearization:

$$\begin{aligned} F_1(x,y) &= x - \frac{1}{2}(x+y)^2 \\ & \qquad K_1(x,y) = 1 + (\lambda+1)(x+y) \\ F_2(x,y) &= y + \frac{1}{2}(x+y)^2 \\ & \qquad K_2(x,y) = -\lambda + (\lambda+1)(x+y) \\ F_3(x,y) &= \lambda + \lambda(1+\lambda)x - (1+\lambda)y - \frac{1}{2}(1+\lambda)^2(x+y)^2 \\ & \qquad K_3(x,y) = (1+\lambda)(x+y). \end{aligned} \tag{7.14}$$

XIII. The system can be brought to the form:

$$\begin{aligned} \dot{x} &= x - 2x^2 + c_{11}xy - 2c_{11}y^2 \\ \dot{y} &= -2y + x^2 + xy + (2c_{11}+2)y^2 \end{aligned} \tag{7.15}$$

and has the following invariant algebraic curves with respective cofactors:

$$\begin{aligned} F_1(x,y) &= -4y + (x+2y)^2 = 0 & K_1(x,y) &= -2 + 2(c_{11}+2)y \\ F_{2,3}(x,y) &= 1 + A_{2,3}x - (A_{2,3} + 2 + c_{11})y = 0 & K_{2,3}(x,y) &= A_{2,3}x + 2(A_{2,3} + 2 + c_{11})y \\ & & div &= -1 - 3x + (5c_{11}+4)y, \end{aligned} \tag{7.16}$$

where $A_{2,3}$ are the two roots of $A^2 + 3A + c_{11} + 2 = 0$. From the invariant curves one can find an integrating factor yielding a first integral, and also a linearization of the y-separatrix. The integral is of the DHE (Darboux-hyperelliptic) type and can be found in [FSZ].

XIV. The system can be brought to the form

$$\begin{aligned}\dot{x} &= x - 19x^2 - 14xy + 5y^2 \\ \dot{y} &= -2y + 10x^2 - 4xy - 14y^2\end{aligned} \tag{7.17}$$

and has the following invariant algebraic curves with respective cofactors:

$$\begin{array}{cc} F_1(x,y) = x + (x+y)^2 = 0 & K_1(x,y) = 1 - 18(x+y) \\ F_2(x,y) = 2y - 5x^2 - 64xy + 22y^2 & K_2(x,y) = -2 - 36(x+y) \\ +18(7x-2y)(x+y)^2 + 81(x+y)^4 = 0 & \\ F_3(x,y) = 1 - 24x + 12y = 0 & K_3(x,y) = -24(x+y) \end{array} \tag{7.18}$$

from which a Darboux linearization can be found. □

For the sake of completeness we also consider the case $\lambda = 0$ (next theorem) and what is the type of points appearing at the limits of the strata of Theorem 7.2 when $\lambda \to 0$ (Theorem 7.4 below) .

Theorem 7.3. *For $\lambda = 0$ the system* (5.1) *is integrable (i.e. has an analytic first integral) if and only if one of the two following conditions is satisfied*

(A) $d_{20} = d_{11} = d_{02} = 0$;
(B) $c_{02} = d_{02} = 0$.

It is linearizable if and only if one of the two following conditions is satisfied:

(C) $c_{20} = c_{11} = d_{20} = d_{11} = d_{02} = 0$.
(D) $c_{11} = c_{02} = d_{02} = 0$.

Proof. For the system to be integrable it is necessary that it has a curve of singular points which is either the line $x = 0$ (case (B)) or a conic (case (A)). In the first case we divide the system by x. The flow-box theorem allows to find a local first integral. In the second case y is simply a first integral.

For linearizability the necessity comes from a direct calculation of the conditions for the existence of a linearizing change of coordinates. We now study the sufficiency
Case (C): The linearizing change of coordinates is simply given by $(X, Y) = (x + c_{02}y^2, y)$.
Case (D): We consider the case $c_{20} \neq 0$ and we scale $c_{20} = 1$. We first make the change $X = x(1 + x)$ which linearizes the first equation yielding a system:

$$\begin{aligned}\dot{X} &= X \\ \dot{y} &= d_{20}\frac{X^2}{(1-X)^2} + d_{11}\frac{Xy}{1-X}\end{aligned} \tag{7.19}$$

We look for a change of coordinates: $Y = h(X) + yk(X)$, with $k(0) = 1$ and $h(0) = 0$. Then $\dot{Y} = 0$ is guaranteed as soon as $k'(X) + d_{11}\frac{k(X)}{1-X} = 0$, i.e. $k(X) = (1-X)^{d_{11}}$ and $h'(X) + d_{20}\frac{Xk(X)}{(1-X)^2} = 0$, i.e. $h(x) = -\frac{d_{20}}{d_{11}(d_{11}-1)} + \frac{d_{20}}{d_{11}-1}(1-X)^{d_{11}-1} - \frac{d_{20}}{d_{11}}(1-X)^{d_{11}}$. The cases $c_{20} = 0$ is done similarly. □

We explore what kind of points appear in the closure of the strata when $\lambda \to 0$.

Theorem 7.4. *The closure for $\lambda = 0$ of the strata appearing in Theorem 7.2 consists of normalizable or integrable systems except for the strata I, III and IX which contain systems for which the normalizing transformation is divergent and for strata VII and VIII which are far from $\lambda = 0$.*

Proof.

I For $d_{02} = 0$ the system is integrable by Theorem 7.3. If $d_{02} \neq 0$ then generically we can consider a system of the form

$$\begin{aligned} \dot{x} &= x - x^2 + c_{11}xy + c_{02}y^2 \\ \dot{y} &= y^2 \end{aligned} \tag{7.20}$$

We show that the system is not normalizable for $c_{11} < 2$ and $c_{02} < 0$. Indeed it suffices to show that the system has no analytic center manifold. A center manifold has the form $x = f(y) = \sum_{i=2}^k a_k y^k + o(y^k)$, with $a_2 = -c_{02} > 0$ and

$$a_{i+1} = (i - c_{11})a_i + \sum_{j=2}^{i-1} a_j a_{i+1-j}. \tag{7.21}$$

The series $\sum a_i y^i$ is clearly divergent.

II The case $\lambda = 0$ corresponds to the stratum (B) of Theorem 7.3. i.e. to integrable systems.

III In the limit we get $d_{11} - c_{20} = c_{11} = d_{20}$. In the particular case $d_{11} = c_{20} = 0$ we can show as in case I that the center manifold of

$$\begin{aligned} \dot{x} &= x + c_{02}y^2 \\ \dot{y} &= d_{02}y^2 \end{aligned} \tag{7.22}$$

is non analytic if $c_{02}d_{02} \neq 0$ yielding the divergence of the normalizing series in these cases.

IV The system generically has the form

$$\begin{aligned} \dot{x} &= x - x^2 \\ \dot{y} &= d_{20}x^2 + d_{11}xy + d_{02}y^2 \end{aligned} \tag{7.23}$$

The first three Darboux factors appearing in (5.23) are still valid for $\lambda = 0$ yielding normalizability by Corollary 5.11. The non generic cases are done similarly.

V The two invariant lines $x = 0$ and $y = 0$ give an integrating factor and normalizability by Corollary 5.11.

VI The invariant curves $F_1(x, y) = 0$ and $y = 0$, with F_1 given in (7.6) yield normalizability by Corollary 5.11. The closure of VI is a component of that of X.

IX Limit points are of two kinds: either $d_{20} = d_{11} = d_{02} = 0$ which is stratum (A) of Theorem 7.3 or $d_{11} = d_{20} = k[c_{11} + (k-1)d_{02}]d_{02} + c_{20}c_{02} = 0$ with $k \in \mathbb{N}$. As in case I we can show that the center manifold can be non analytic if we take $c_{02} = -1$, $d_{02} = 1$ and take $c_{11} < 2$ so that $c_{20} = k(k-1+c_{11}) < 0$.

X splits in two cases: X_1 and X_2. X_1 corresponds to $c_{11} = c_{02} = d_{02} = c_{20} - 2d_{11} = 0$ which is a subcase of IV. X_2 corresponds to $c_{11} = d_{20} = d_{02} = c_{20} - 2d_{11}$ which is the same as VI.

XI The systems are contained in stratum (B) of Theorem 7.3 and are integrable.

XII After scaling of x and y the stratum XII in Theorem 7.2 is zero dimensional. The closure of the stratum is found by putting $\lambda = 0$ in (7.13). The two curves $F_1(x,y) = 0$ and $F_2(x,y) = 0$ in (7.14) still exist and give an integrating factor yielding normalizability by Corollary 5.11. $\square$

Remark 7.5. All limit points are at least half-normalizable (the cochain not governing the analycity of the center manifold vanishes) as soon as the saddle-node is approached by integrable saddles for $\lambda = \frac{1}{n}$. A general theory explaining this kind of phenomena will be developed in a further work. We discuss briefly without explanation the three cases to appear here, namely the substratum $c_{20} = d_{20} = d_{11} = 0$ of I, III and IX.

i) Substratum $c_{20} = d_{20} = d_{11} = 0$ of I: we scale $d_{02} = 1$. In that case the system is reduced to a linear equation of Euler type:

$$y^2 \frac{dx}{dy} = x(1 + c_{11}y) + c_{02}y^2. \tag{7.24}$$

As the equation can be explicitly integrated this yields us a first integral of the form

$$H(x,y) = xy^{-c_{11}} e^{\frac{1}{y}} - c_{02} \int_0^y e^{\frac{1}{\zeta}} \zeta^{-c_{11}} d\zeta. \tag{7.25}$$

The non analytic center manifold is given by

$$x - y^{c_{11}} e^{-\frac{1}{y}} \int_0^y e^{\frac{1}{\zeta}} \zeta^{-c_{11}} d\zeta = 0. \tag{7.26}$$

which is Borel-summable in y except in the direction $\mathbb{R}^+$. However the first integral has no pathology in the direction $\mathbb{R}^-$. We say that the system is half-normalizable. We do not go further in that direction. In a forthcoming paper we will explain how this half-normalizability is forced by the fact that the saddle-node is approached by integrable saddles when $\lambda = \frac{1}{n}$.

ii) Let us show however that the limit points of III are half-normalizable at least in the generic case $d_{02} \neq 0$. Scaling $d_{02} = 1$ we work with a system

$$\begin{aligned} \dot{x} &= x + c_{20}x^2 + c_{02}y^2 \\ \dot{y} &= c_{20}xy + y^2. \end{aligned} \tag{7.27}$$

As for the limit of the substratum of I we can find an inverse integrating factor. A first integral can be deduced whose only direction of non Borel-summability is $\mathbb{R}^+$, yielding half-normalizability. The integrating factor is constructed from the following factors (one of them is not analytic so we do not use the term generalized Darboux factor)

$$\begin{array}{cc} F_1(x,y) = y & K_1(x,y) = c_{20}x + y \\ F_2(x,y) = 1 + c_{20}x - c_{20}c_{02}y & K_2(x,y) = c_{20}x \\ F_3(x,y) = \exp\left(-\frac{1+c_{20}x}{y}\right) & K_3(x,y) = 1 + c_{20}x - c_{20}c_{02}y \\ div = 1 + 3c_{20}x + 2y & \end{array} \tag{7.28}$$

From $div = (2 + c_{20}c_{02})K_1 - c_{20}c_{02}K_2 + K_3$ we deduce that

$$y^{2+c_{20}c_{02}} F_2^{-c_{20}c_{02}} \exp\left(-\frac{1 + c_{20}x}{y}\right) \tag{7.29}$$

is an inverse integrating factor.

iii) We show as before that the systems of IX are half-normalizable. Indeed only the generic case $d_{02} \neq 0$ needs to be considered. We first treat $c_{02} \neq 0$ and scale $c_{02} = -1$. Scaling $d_{02} = 1$ we consider a system with $c_{20} = k(c_{11} + k - 1)$, i.e. of the form

$$\begin{aligned} \dot{x} &= x + k(c_{11} + k - 1)x^2 + c_{11}xy - y^2 \\ \dot{y} &= y^2. \end{aligned} \tag{7.30}$$

The system has the following factors:

$$\begin{array}{cc} F_1(x,y) = y & K_1(x,y) = y \\ F_2(x,y) = e^{-\frac{1}{y}} & K_2(x,y) = 1 \\ F_3(x,y) = \sum_{i=0}^{k} b_i y^i + x \sum_{i=0}^{k} i b_i y^{i-1} & K_3(x,y) = k(c_{11} + k - 1)x \\ div = 1 + 2k(c_{11} + k - 1)x + (2 + c_{11})y, & \end{array} \tag{7.31}$$

where $b_0 = 1$,

$$(i + 1)b_{i+1} = b_i(k - i)[c_{11} + k + i - 1]. \tag{7.32}$$

Hence $div = (2 + c_{11})K_1 + K_2 + 2K_3$. We conclude as in cases I and III. When $c_{02} = 0$ we have $c_{11} = -k + 1$ (with $d_{02} = 1$). As before F_1 and F_2 are Darboux factors. We have the additional one

$$\begin{array}{cc} F_4(x,y) = x & K_1(x,y) = 1 + c_{20}x + (1 - k)y \\ div = 1 + 2c_{20}x + (3 - k)y & \end{array} \tag{7.33}$$

yielding the first integral

$$H(x,y) = e^{-\frac{1}{y}} y^{1-k} x^{-1} + c_{20} \int_0^y e^{-\frac{1}{\zeta}} \zeta^{-1-k} d\zeta \tag{7.34}$$

The system is normalizable since the integral part is a rational function in y for $k \in \mathbb{N} \setminus \{0\}$.

8. Lotka-Volterra systems

The Lotka-Volterra subfamily is a rich subfamily of the quadratic systems, which may exhibit the same kind of pattern for the global organization of normalizable, integrable and linearizable systems. Indeed the examples of quadratic systems constructed in Section 2 were Lotka-Volterra systems. Computer calculations for small values of $p + q$ show an interesting pattern of linearizability and integrability. As a first step in investigating this pattern further we include the following results.

Theorem 8.1. *For $\lambda \in \mathbb{N} \setminus \{1\}$ the Lotka-Volterra system*

$$\begin{aligned} \dot{x} &= x(1 + c_{20}x + c_{11}y) \\ \dot{y} &= y(-\lambda + d_{11}x + d_{02}y) \end{aligned} \tag{8.1}$$

has a linearizable saddle at the origin if and only of one of the following conditions is satisfied:

$$\begin{array}{lll} (A_m) & mc_{20} + d_{11} = 0, & m = 0, \dots, \lambda - 2; \\ (B) & c_{11} = d_{02} = 0; & \\ (C) & c_{20} - d_{11} = c_{11} - d_{02} = 0; & \\ (D) & c_{11} = (\lambda - 1)c_{20} + d_{11} = 0. & \end{array} \tag{8.2}$$

Proof. The proof uses the necessary and sufficient conditions for integrability of (8.1) appearing in [FSZ]. These are the following (we use the same letter as in (8.2)) when the condition is identical:

$$\begin{array}{lll} (A_m) & mc_{20} + d_{11} = 0, & m = 0, \dots, \lambda - 2; \\ (E) & \lambda c_{20}c_{11} - (\lambda - 1)c_{20}d_{02} - d_{11}d_{02} = 0. & \end{array} \tag{8.3}$$

From Theorem 7.2 we know that elements of strata (A_0), (B), (C) and (D) are linearizable: indeed stratum (A_0) is included in case I, stratum (B) is included in case IV, stratum (D) in case III and stratum (C) coincides with case V.

The rest of the proof consists in two parts: showing that elements of (A_m) for $m > 0$ are linearizable and showing that elements of stratum (E) which do not belong to strata (A_0) to (D) are not linearizable. The first part is shown in Proposition 8.2 below.

For the second part we look for a linearization of the $\dot{y}$ equation as a series

$$Y = \sum_{n=1}^{\infty} f_n(x)y^n, \quad f_1(0) = 1, \tag{8.4}$$

and we find an obstruction.

We now consider a system in stratum (E) and we must show that any linearizable system in this stratum is an element of (A_m) or (C) or (D). (Note that condition (E) is equivalent to the existence of a third invariant line allowing, together with the coordinate axes, the construction of a Darboux first integral). We first consider the case $c_{20}d_{02} = 0 \neq 0$. Scaling them both to 1 yields the system

$$\begin{aligned} \dot{x} &= x(1 + x + c_{11}y) \\ \dot{y} &= y(-\lambda + (\lambda c_{11} - \lambda + 1)x + y). \end{aligned} \tag{8.5}$$

If $c_{11} = 0$ we are in case (D). Hence we consider $c_{11} \neq 0$. Looking for a linearization $Y = \sum_{n \geq 1} f_n(x)y^n$, $f_1(0) = 1$, of the second equation, yields the following system of differential equations:

$$x(1+x)f_n'(x) + (-(n-1)\lambda + n(\lambda c_{11} - \lambda + 1)x)f_n(x) + c_{11}xf_{n-1}'(x) + (n-1)f_{n-1}(x). \tag{8.6}$$

Then $f_1(x) = (1+x)^{-\lambda c_{11}+\lambda-1}$ and $f_2(x) = x^\lambda(1+x)^{-2\lambda c_{11}+\lambda-2}h_2(x)$ with

$$h_2'(x) = x^{-\lambda-1}(1+x)^{\lambda c_{11}-1}P_1(x), \tag{8.7}$$

with $P_1(x)$ not divisible by $x+1$ and of degree exactly 1 if $c_{11}(\lambda c_{11}-\lambda+1) = (c_{11}-1)(-\lambda c_{11}+1) \neq 0$. If $P_1(x)$ has degree 1 (resp. 0) this equation is solvable without logarithmic terms if and only if $\lambda c_{11} - 1 = i$ is an integer satisfying $0 \leq i \leq \lambda - 2$ (resp. $0 \leq i \leq \lambda - 1$). Hence $d_{11} = 2 - \lambda + i = m$ and we fall into one of the (A_m) when $i \leq \lambda - 2$. If $P_1(x)$ has degree 0 we need only treat the missing case $c_{11} = 1$. Then (8.7) is solvable without logarithmic terms and we are in case (C).

If $c_{20} = 0$ and $d_{11} \neq 0$ (to exclude (A)) then $d_{02} = 0$. If $d_{02} = 0$ and $c_{11} \neq 0$ (to exclude (B)) then $d_{02} = 0$. So we only need to consider the case $c_{20} = d_{02} = 0$, and scale so that $c_{11} = d_{11} = 1$ which yields the system of differential equations for a linearization:

$$xf_n'(x) + f_n(x)(-(n-1)\lambda + nx) + xf_{n-1}'(x) = 0.$$

Hence $f_1(x) = e^{-x}$ and $f_2(x) = x^\lambda e^{-2x}h_2(x)$ with $h_2'(x) = -x^{-\lambda}e^x$. Hence $f_2(x)$ is not an analytic function, yielding the non-linearizability of the system. □

Proposition 8.2. *Let m be a positive integer. The system*

$$\begin{aligned} \dot{x} &= x - \frac{d_{11}}{m}x^2 + c_{11}xy \\ \dot{y} &= -\lambda y + d_{11}xy + d_{02}y^2 \end{aligned} \tag{8.8}$$

is linearizable as soon as $\lambda > m$ unless $\lambda = \frac{jm+1}{j}$ for j an integer, in which case the system is normalizable and an additional condition is required for linearizability. The system is generically not orbitally normalizable for $\lambda = m$.

Proof. This system is of the form studied in Theorem 6.1. Under the change of variables $(X,Y) = (xy^{1/m}, y^{1/m})$ the system becomes:

$$\begin{aligned} \dot{X} &= \left(1 - \frac{\lambda}{m}\right)X + \left(c_{11} + \frac{d_{02}}{m}\right)XY^m \\ \dot{Y} &= -\frac{\lambda}{m}Y + \frac{d_{11}}{m}X + d_{02}Y^{m+1}. \end{aligned} \tag{8.9}$$

The origin is a node as soon as $\lambda > m$. The node is in $1:n$ resonance when $\lambda = \frac{nm}{n-1}$. However the system has a special form: linear plus homogeneous of degree $m+1$. Hence all terms in the normal form are of degree $jm+1$ for j an integer, yielding that all nodes are linearizable unless $n = jm+1$ for j an integer, i.e. $\lambda = m + \frac{1}{j}$.

When $\lambda = m$, $d_{02} = 0$ and $c_{11}d_{11} \neq 0$ the origin in (8.9) has a non analytic center manifold. Scaling $c_{11} = -1$ and $d_{11} = m$ it has divergent expansion $Y = \sum_{i=1}^\infty a_iX^i$ with

$$\begin{cases} a_1 = 1 \\ a_i = \sum_{i_1+\dots i_{m+1}=i} i_1 a_{i_1} \dots a_{i_{m+1}}. \end{cases} \tag{8.10}$$

□

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