The moduli space of germs of generic families of analytic diffeomorphisms unfolding a parabolic fixed point ⋆

Colin Christopher [a] Christiane Rousseau [b]

[a] *School of Mathematics, University of Plymouth*

[b] *DMS and CRM, Université de Montréal*

Received *****; accepted after revision +++++

Presented by

Abstract

In this note we give the moduli space of germs of generic families of analytic diffeomorphisms unfolding a parabolic fixed point of codimension 1. A complete modulus of analytic classification for such objects is given by an unfolding of the Écalle-Voronin modulus over a sector of opening greater than 2π of the universal covering of the space of the canonical parameter ϵ punctured at the origin. In the overlapping (Glutsyuk) sector of parameter space, which corresponds to the parameter values for which the two fixed points are hyperbolic and the diffeomorphism has orbits connecting the two fixed points, we have two representatives of the modulus. We identify a necessary compatibility condition satisfied by these representatives. The compatibility condition implies the existence of an adequate scaling for which the modulus is $\frac{1}{2}$-summable in ϵ with directions of non-summability coinciding with the directions of real multipliers at the fixed points. We show that the compatibility condition together with this summability property is also sufficient, thus allowing for a complete description of the space of moduli. *To cite this article: C. Christopher, C. Rousseau, C. R. Acad. Sci. Paris, Ser. I ?? (2007).*

Résumé

L'espace des modules des germes de familles génériques de difféomorphismes analytiques déployant un point fixe parabolique. Dans cette note on donne l'espace des modules des germes de familles génériques de difféomorphismes analytiques déployant un point fixe parabolique de codimension 1. Un module complet pour de tels objets est donnés par le déploiement du module d'Écalle-Voronin sur un secteur d'ouverture plus grande que 2π du recouvrement universel de l'espace du paramètre canonique percé à l'origine. Dans le sous-secteur recouvert deux fois de l'espace du paramètre (sous-secteur Glutsyuk), là où les deux points fixes sont hyperboliques et où le difféomorphisme a des orbites connectant les deux points fixes, on a deux représentants du module. On identifie une condition de compatibilité nécessaire satisfaite par les deux représentants. La condition de compatibilité implique l'existence d'une normalisation sous laquelle le module est $\frac{1}{2}$-sommable en ϵ avec directions de non-sommabilité coïncidant avec les directions de multiplicateurs réels aux points fixes. On montre que la condition de compatibilité jointe à cette propriété de sommabilité est aussi suffisante, permettant ainsi de décrire complètement l'espace des modules. *Pour citer cet article : C. Christopher, C. Rousseau, C. R. Acad. Sci. Paris, Ser. I ? ? (2007).*

1. Statement of the results

We consider here germs of generic analytic 1-parameter families diffeomorphisms of $(\mathbb{C},0)$ unfolding a parabolic point. Such families can be "prepared" so that the parameter becomes an analytic invariant ([2]). A prepared family with canonical parameter ϵ has the form

$$f_\epsilon(z) = z + (z^2-\epsilon)(1+b(\epsilon)+c(\epsilon)z+O(z^2-\epsilon)), \tag{1}$$

so that $\frac{1}{\sqrt{\epsilon}} = \frac{1}{\ln(f'_\epsilon(\sqrt{\epsilon}))} - \frac{1}{\ln(f'_\epsilon(-\sqrt{\epsilon}))}$. The formal invariant $a(\epsilon)$ is the analytic function in ϵ given by $a(\epsilon) = \frac{1}{\ln(f'_\epsilon(\sqrt{\epsilon}))} + \frac{1}{\ln(f'_\epsilon(-\sqrt{\epsilon}))}$.

Notation: we let T_B be the translation by B. We also take $\hat{\epsilon}$ to represent the parameter ϵ when lifted to the universal cover of the punctured disc.

In [2] it is shown that a modulus for these unfoldings can be obtained from the formal parameter, $a(\epsilon)$, and two analytic functions, $\Psi^0_{\hat{\epsilon}}$ and $\Psi^\infty_{\hat{\epsilon}}$, which describe the shift between two choices of Fatou coordinates on their region of overlap. These functions are always assumed to satisfy $T_1\circ\Psi = \Psi\circ T_1$. That is, they could also be considered as lifts of maps $\psi^{0,\infty}_{\hat{\epsilon}}$ between neighborhoods of the poles of two spheres. We demonstrate the following theorem, which identifies the extra conditions on the $\Psi^{0,\infty}_{\hat{\epsilon}}$ to form a moduli space for analytic unfoldings, and proves their sufficiency.

Theorem 1.1 *(i) We consider a prepared germ of the form (1), i.e. a germ of generic analytic 1-parameter family of diffeomorphisms of $(\mathbb{C},0)$ unfolding a parabolic point. Then for $\delta\in(0,\pi)$ there exists $\rho>0$ and a representative of the modulus defined for $\hat{\epsilon}\in V_{\delta,\rho} = \{\hat{\epsilon}; |\hat{\epsilon}|<\rho, \arg\hat{\epsilon}\in(-\delta, 2\pi+\delta)\}$ by $(\Psi^0_{\hat{\epsilon}}, \Psi^\infty_{\hat{\epsilon}})$ where*

- *there exists $Y_0>0$ such that $\Psi^0_{\hat{\epsilon}}$ (resp. $\Psi^{0,\infty}_{\hat{\epsilon}}$) is analytic on $ImW<-Y_0$ (resp. $ImW>Y_0$) for all $\hat{\epsilon}\in V_{\delta,\rho}$;*
- *$\Psi^{0,\infty}_{\hat{\epsilon}}$ are $\frac{1}{2}$-summable in ϵ with direction of non-summability given by $\mathbb{R}^+$;*
- *$\Psi^{0,\infty}_{\hat{\epsilon}}$ satisfy the following compatibility condition: we cover the overlap (Glutsyuk sector) by the two subsectors*

$$\begin{cases} \overline{V} = \{\hat{\epsilon}; 0<|\hat{\epsilon}|<\rho, \arg\hat{\epsilon}\in(-\delta,+\delta)\} \\ \widetilde{V} = \{\hat{\epsilon}; 0<|\hat{\epsilon}|<\rho, \arg\hat{\epsilon}\in(2\pi-\delta,2\pi+\delta)\}, \end{cases} \tag{2}$$

on which we denote $\hat{\epsilon}$ by $\overline{\epsilon}$ and $\tilde{\epsilon}$ respectively. We let

$$\alpha^0 = -\frac{2\pi i(1-a(\epsilon)\sqrt{\hat{\epsilon}})}{2\sqrt{\hat{\epsilon}}} \qquad \alpha^\infty = -\frac{2\pi i(1+a(\epsilon)\sqrt{\hat{\epsilon}})}{2\sqrt{\hat{\epsilon}}}. \tag{3}$$

On $\widetilde{V}$ (resp. $\overline{V}$), α^0 and α^∞ takes values $\tilde{\alpha}^{0,\infty}$ (resp. $\overline{\alpha}^{0,\infty}$). There exist maps $\overline{H}^{0,\infty}_{\overline{\epsilon}}$ and $\widetilde{H}^{0,\infty}_{\tilde{\epsilon}}$ commuting with T_1 and satisfying

$$\begin{cases} \widetilde{H}^0_{\tilde{\epsilon}}\circ T_{\tilde{\alpha}^0}\circ\widetilde{\Psi}^0_{\tilde{\epsilon}} = T_{\tilde{\alpha}^0}\circ\widetilde{H}^0_{\tilde{\epsilon}} \\ \widetilde{H}^\infty_{\tilde{\epsilon}}\circ T_{\tilde{\alpha}^0}\circ\widetilde{\Psi}^\infty_{\tilde{\epsilon}} = T_{\tilde{\alpha}^\infty}\circ\widetilde{H}^\infty_{\tilde{\epsilon}} \\ \overline{H}^0_{\overline{\epsilon}}\circ\overline{\Psi}^0_{\overline{\epsilon}}\circ T_{\overline{\alpha}^0} = T_{\overline{\alpha}^0}\circ\overline{H}^0_{\overline{\epsilon}} \\ \overline{H}^\infty_{\overline{\epsilon}}\circ\overline{\Psi}^\infty_{\overline{\epsilon}}\circ T_{\overline{\alpha}^0} = T_{\overline{\alpha}^\infty}\circ\overline{H}^\infty_{\overline{\epsilon}}. \end{cases} \tag{4}$$

⋆ This work is supported by NSERC in Canada.

Email addresses: C.Christopher@plymouth.ac.uk (Colin Christopher), rousseac@dms.umontreal.ca (Christiane Rousseau).

URL: www.dms.umontreal.ca/~rousseac (Christiane Rousseau).

The compatibility condition is then

$$\widetilde{H}^{\infty}_{\hat\epsilon}\circ(\widetilde{H}^{0}_{\hat\epsilon})^{-1}=T_{2\pi ia(\epsilon)}\circ\overline{H}^{0}_{\overline{\epsilon}}\circ(\overline{H}^{\infty}_{\overline{\epsilon}})^{-1}\circ T_{D(\epsilon)}\tag{5}$$

for some constant $D(\epsilon)=-2\pi ia(\epsilon)+O(\exp(-\frac{A}{|\sqrt{\epsilon}|}))$ *with* $A>0$.

(ii) Conversely we consider a germ of function $a(\epsilon)$ *analytic in* ϵ *and a germ of family*

$$\Psi=(\Psi^0_{\hat\epsilon},\Psi^\infty_{\hat\epsilon})_{\hat\epsilon\in V_{\delta,\rho}}\tag{6}$$

for some $\delta\in(0,\pi)$ *and* $\rho>0$, *where* $\Psi^0_{\hat\epsilon}$ *and* $\Psi^\infty_{\hat\epsilon}$ *are defined respectively on* $ImW<-Y_0$ *and* $ImW>Y_0$, *are* $\frac{1}{2}$*-summable in* ϵ *with direction of non-summability given by* $\mathbb{R}^+$ *and satisfy the compatibility condition* (5). *Then there exists a germ of family of analytic diffeomorphisms*

$$f_\epsilon=z+(z^2-\epsilon)(1+O(\epsilon)+O(z))\tag{7}$$

whose modulus over a fixed neighborhood in (z,ϵ)*-space is given by* (6) *together with* $a(\epsilon)$.

(iii) Furthermore, (i) and (ii) depend analytically on extra parameters.

Remark 1 In the unfolded modulus, the map $\Psi^0_{\hat\epsilon}$ *(resp.* $\Psi^\infty_{\hat\epsilon}$*) refers to the dynamics near* $-\sqrt{\hat\epsilon}$ *(resp.* $\Psi^0_{\hat\epsilon}$*). When* $\hat\epsilon$ *makes a full turn,* $-\sqrt{\hat\epsilon}$ *and* $\sqrt{\hat\epsilon}$ *are exchanged. We therefore have two different ways of describing the dynamics near each singular point: for instance one with* $\Psi^0_{\hat\epsilon}$ *and the other with* $\Psi^\infty_{\hat\epsilon e^{2\pi i}}$. *This is of course only possible in the region of parameter space where the two points are hyperbolic. The compatibility condition guarantees that the two dynamics are in fact the same.*

2. The proofs

The proof of Theorem 1.1 is composed of four parts:

(1) the construction of $(\Psi^{0,\infty}_{\hat\epsilon})_{\hat\epsilon\in V_{\delta,\rho}}$: this is done in [2];

(2) the derivation of the compatibility condition from which the $\frac{1}{2}$-summability of $\Psi^{0,\infty}_{\hat\epsilon}$ follows;

(3) the "local realization" over sectorial neighborhoods in $\hat\epsilon$ of small opening: this yields the realization by a family $g_{\hat\epsilon}$ ramified in ϵ which uniform limit g_0 when $\hat\epsilon\to 0$ along any ray $\arg\hat\epsilon=Cst$. The local realization does not use (2).

(4) the global realization: from $g_{\hat\epsilon}$, using the compatibility condition we construct a uniform family f_ϵ over an abstract 2-dimensional manifold. The $\frac{1}{2}$-summability of $\Psi^{0,\infty}_{\hat\epsilon}$ allows application of the Newlander-Nirenberg theorem to show that this manifold is an open set of $\mathbb{C}^2$.

In more detail:

(2) To decide when two diffeomorphisms f_ϵ unfolding a parabolic point are conjugate, we embed them in the flow of the vector field $v_\epsilon=\frac{z^2}{1+a(\epsilon)z}\frac{d}{dz}$ on adequate sectorial domains and we measure the obstruction to a global embedding. For this it is easier to work in a coordinate W which is the time of the vector field v_ϵ. The four maps $\widetilde{H}^{0,\infty}_{\hat\epsilon}$ and $\overline{H}^{0,\infty}_{\overline{\epsilon}}$ can be seen as a change of coordinate in W embedding the map in a flow near the hyperbolic fixed points $\pm\sqrt{\epsilon}$. In W-coordinate this flow is that of $\frac{\partial}{\partial W}$. The comparison of these embeddings is an invariant: this is exactly what is expressed by the compatibility condition, modulo a scaling given by the constant $D(\epsilon)$. To derive the $\frac{1}{2}$-summability of $\Psi^0_{\hat\epsilon}$ we explicitly calculate the maps $\widetilde{H}^{0,\infty}_{\tilde\epsilon}$ and $\overline{H}^{0,\infty}_{\overline{\epsilon}}$ in the region $ImW>Y_0$. There, both $\overline{H}^{0}_{\overline{\epsilon}}$ and $\widetilde{H}^{0}_{\hat\epsilon}$ are of the form $id+O(\exp(-\frac{A}{|\sqrt{\hat\epsilon}|}))$, while $(\overline{H}^{\infty}_{\overline{\epsilon}})^{-1}=\overline{\Psi}^{\infty}_{\overline{\epsilon}}+2\pi ia+O(\exp(-\frac{A}{|\sqrt{\hat\epsilon}|}))$ and $\widetilde{H}^{\infty}_{\hat\epsilon}=\widetilde{\Psi}^{\infty}_{\hat\epsilon}+2\pi ia+O(\exp(-\frac{A}{|\sqrt{\hat\epsilon}|}))$, so the compatibility condition yields the $\frac{1}{2}$-summability of $\Psi^\infty_{\hat\epsilon}$ by the theorem of Ramis-Sibuya. A similar study in the region $ImW<-Y_0$ allows to prove the $\frac{1}{2}$-summability of $\Psi^0_{\hat\epsilon}$.

(3) The local realization for $\hat{\epsilon}$ in a sector of small opening is first done for fixed $\hat{\epsilon}$ by gluing two sectors $U_{\hat{\epsilon}}^{\pm}$ as in Figure 1 with maps $\Xi_{\hat{\epsilon}}^{0,\infty}$ constructed from $\Psi_{\hat{\epsilon}}^{0,\infty}$ ($\Xi_{\hat{\epsilon}}^{0,\infty}$ is the conjugate of $\Psi_{\hat{\epsilon}}^{0,\infty}$ under the change $z \mapsto W$, where W is the time of the vector field $v_\epsilon = \frac{z^2}{1+a(\epsilon)z}\frac{d}{dz}$). This yields a complex manifold $M_{\hat{\epsilon}}$ which we endow with a diffeomorphism given on each $U_{\hat{\epsilon}}^{\pm}$ by the time-one map of the vector field v_ϵ. The theorem of Ahlfors-Bers allows to recognize in the abstract manifold $M_{\hat{\epsilon}}$ a neighborhood of $\pm\sqrt{\epsilon}$ in $\mathbb{C}$ and to fill the holes at the fixed points. It is then possible to see that the construction can be made so as to depend analytically on $\hat{\epsilon}$ with a continuous fixed limit M_0 as $\hat{\epsilon} \to 0$ along a ray. In this way we realize the modulus in a ramified family $g_{\hat{\epsilon}}$ for $\hat{\epsilon}$ in some $V_{\delta,\rho}$.

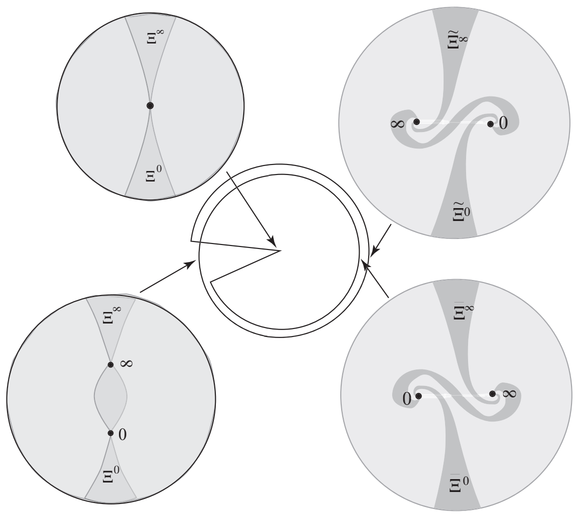

Figure 1. The sectors $U_{\hat{\epsilon}}^{\pm}$ and their intersection for the different values of $\hat{\epsilon} \in V_{\delta,\rho}$

(4) The compatibility condition ensures that $g_{\overline{\epsilon}}$ and $g_{\tilde{\epsilon}}$ are conjugate by some $J_{\hat{\epsilon}}$ for $\arg\overline{\epsilon} \in (-\delta,\delta)$ and $\tilde{\epsilon} = e^{2\pi i}\overline{\epsilon}$. Moreover $g_{\hat{\epsilon}}$ is $\frac{1}{2}$-summable in $\hat{\epsilon}$ with directions of non summaility given by $\arg\hat{\epsilon} \in \{0, 2\pi\}$. It follows that $|J_{\hat{\epsilon}}| = O(\exp(-\frac{A}{|\sqrt{\hat{\epsilon}}|}))$ for some positive A. We construct an abstract manifold by gluing $B(0,r)\times\{\arg\hat{\epsilon} \in (-\delta,\delta)\}$ with $B(0,r)\times\{\arg\hat{\epsilon} \in (2\pi-\delta, 2\pi+\delta)\}$, by means of $(z,\overline{\epsilon}) \mapsto (J_{\hat{\epsilon}}(z), \tilde{\epsilon})$. We paste in $B(0,r)\times\{\epsilon = 0\}$ to fill the hole and get a C^∞ manifold $\mathcal{M}$. Newlander-Nirenberg's theorem allows to recognize in $\mathcal{M}$ a neighborhood of the origin in $\mathbb{C}^2$ on which $g_{\hat{\epsilon}}$ induces a family of diffeomorphisms f_ϵ realizing the modulus. $\square$

Acknowledgements

We are grateful to Reinhard Schäfke and Loïc Teyssier for helpful discussions.

References

[1] A. A. Glutsyuk, Confluence of singular points and nonlinear Stokes phenomenon, *Trans. Moscow Math. Soc.* **62** (2001), 49–95.

[2] P. Mardešić, R. Roussarie and C. Rousseau, Modulus of analytic classification for unfoldings of generic parabolic diffeomorphisms, *Moscow Mathematical Journal* **4** (2004), 455–502.