The moduli space of germs of generic families of analytic diffeomorphisms unfolding a parabolic fixed point*

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Abstract

In this paper we describe the moduli space of germs of generic families of analytic diffeomorphisms which unfold a parabolic fixed point of codimension 1.

In [11] (and also [15]), it was shown that the Ecalle-Voronin modulus can be unfolded to give a complete modulus for such germs. The modulus is defined on a ramified sector in the canonical perturbation parameter ϵ. As in the case of the Ecalle-Voronin modulus, the modulus is defined up to a linear scaling depending only on ϵ.

Here, we characterize the moduli space for such unfoldings by finding the compatibility conditions on the modulus which are necessary and sufficient for realization as the modulus of an unfolding.

The compatibility condition is obtained by considering the region of sectorial overlap in ϵ-space. This lies in the Glutsyuk sector where the two fixed points are hyperbolic and connected by the orbits of the diffeomorphism. In this region we have two representatives of the modulus which describe the same dynamics. We identify the necessary compatibility condition between these two representatives by comparing them both with their common Glutsyuk modulus.

The compatibility condition implies the existence of a linear scaling for which the modulus is 1/2-summable in ϵ, whose direction of non-summability coincides with the direction of real multipliers at the fixed points. Conversely, we show that the compatibility condition (which implies the summability property) is sufficient to realize the modulus as coming from an analytic unfolding, thus giving a complete description of the space of moduli. The terminology "space" of moduli is justified by the fact that the moduli depend analytically on extra parameters.

1 Introduction

The analytic classification of germs of analytic diffeomorphisms with a parabolic fixed point of codimension 1 was given by Ecalle [3] and Voronin [22]. A complete modulus for a germ of diffeomorphism $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ of the form

$$f(z) = z + z^2 + o(z^2)$$

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is given by a formal invariant $a \in \mathbb{C}$ and an equivalence class of a pair of germs (ψ^0, ψ^∞) where $\psi^0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $\psi^\infty : (\mathbb{C}, \infty) \to (\mathbb{C}, \infty)$, and where the equivalence relation is defined as follows:

$$(\psi^0, \psi^\infty) \sim (\check{\psi}^0, \check{\psi}^\infty) \Longleftrightarrow \exists C, C' \in \mathbb{C}^* \begin{cases} \check{\psi}^0 = L_C \circ \psi^0 \circ L_{C'} \\ \check{\psi}^\infty = L_C \circ \psi^\infty \circ L_{C'} \end{cases}$$

where L_C (resp. $L_{C'}$) is the linear map $w \mapsto Cw$ (resp. $w \mapsto C'w$). Moreover all tuples $(a, [\psi^0, \psi^\infty])$ are realizable, where $[\psi^0, \psi^\infty]$ represents the equivalence class of (ψ^0, ψ^∞).

The paper [11] addresses the similar question for the analytic classification of generic 1-parameter families of analytic diffeomorphisms unfolding a parabolic fixed point. It was shown that it is possible to prepare the family so that the parameter becomes an analytic invariant. Then a conjugacy between two germs of prepared families must preserve the canonical parameter. The main result of [11] is that the unfolding of $(a, [\psi^0, \psi^\infty])$ is a complete modulus of analytic classification for a prepared germ $f_\epsilon : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ of the form

$$f_\epsilon(z) = z + (z^2 - \epsilon)(1 + b(\epsilon) + c(\epsilon)z + O(z^2 - \epsilon)),$$

such that $\frac{\partial f_\epsilon}{\partial \epsilon} \neq 0$ and f_0 has formal invariant a. The paper [11] also allows an explanation of the meaning of the coefficients which form the Ecalle-Voronin modulus. Indeed the formal invariant a indicates a shift between the multipliers of the two fixed points in the limit $\epsilon = 0$. To interpret the coefficients of $\psi^{0,\infty}$ it is better to split the parameter space in two regions: in the Glutsyuk region where the two fixed points are hyperbolic and there is an orbit connecting them, then the coefficients of the unfolded $\psi^{0,\infty}_{\hat{\epsilon}}$ measure the non compatibility of the two "models" at the fixed points. In the Lavaurs region, they control the complicated dynamics of the fixed points. In particular the "parametric resurgence" phenomenon allows one to predict from the coefficients of ψ^0 (resp. ψ^∞) some discrete sequences $\{\epsilon_n\}$ converging to the origin for which the fixed point $-\sqrt{\epsilon_n}$ (resp. $\sqrt{\epsilon_n}$) of f_{ϵ_n} is resonant and nonlinearizable. Moreover it was shown in [11] that it is possible to take a representative of the equivalence class $[\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}}]$ depending analytically on $\hat{\epsilon}$, for $\hat{\epsilon}$ in a sector V of opening less than 4π of the universal covering of ϵ space punctured at 0.

While it is easily shown that a function $a(\epsilon)$ is realizable as the formal modulus of the family if and only if it is analytic, the other part of the necessary and sufficient conditions for realizability of a modulus and the determination of the moduli space was completely open. The difficulty comes from the fact that the construction leading to the modulus $[\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}}]$ of a family cannot be extended to make a full turn in $\sqrt{\epsilon}$. This is because the unfolded $\psi^0_{\hat{\epsilon}}$ (resp. $\psi^\infty_{\hat{\epsilon}}$) is attached to $-\sqrt{\hat{\epsilon}}$ (resp. $\sqrt{\hat{\epsilon}}$), which gives two completely different descriptions of the same dynamics of f_ϵ when $\hat{\epsilon}$ makes a full turn. This fact is precisely what we need to exploit to identify the sufficient condition for realizability. Indeed, in the Glutsyuk region, i.e. the region where the fixed points are hyperbolic, the renormalized return map near $-\sqrt{\hat{\epsilon}}$ (resp. $\sqrt{\hat{\epsilon}}$) is the composition of $\psi^0_{\hat{\epsilon}}$ (resp. $\psi^\infty_{\hat{\epsilon}}$) with a linear map. Since the fixed points are hyperbolic, these renormalized return maps are linearizable. The comparison of the linearizing maps is an analytic invariant, thus allowing one to derive a compatibility condition between $(\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}})$ and $(\psi^0_{\hat{\epsilon}e^{2\pi i}}, \psi^\infty_{\hat{\epsilon}e^{2\pi i}})$ so that they describe the same dynamics.

One important consequence of the compatibility condition is that it is possible to choose a representative of the equivalence class $[\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}}]$ such that $\psi^0_{\hat{\epsilon}}$ and $\psi^\infty_{\hat{\epsilon}}$ are both 1/2-summable in ϵ, with $\mathbb{R}^+$ as direction of non-summability. This property, together with the compatibility condition, is sufficient for a germ of family $(a(\epsilon), [\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}}])$ to be realizable.

The realization is done in two steps. We first realize locally by a family $f_{\hat{\epsilon}}$ ramified in $\hat{\epsilon}$. We do this by first giving the realization for a fixed $\hat{\epsilon}$: we construct the realization on an abstract manifold and use the Ahlfors-Bers theorem to show that this manifold is indeed an open set of $\mathbb{C}$. We then show that the construction can be performed so as to depend analytically on $\hat{\epsilon}$. We call this part the local realization. The second step is to correct the ramification. Indeed using the local realization and the compatibility condition, this allows us to construct a realization on an abstract 2-dimensional manifold. The Newlander-Nirenberg theorem can be applied to show that this manifold is indeed an open set of $\mathbb{C}^2$ containing a product of a neighborhood of the origin in ϵ-space with an open set of $\mathbb{C}$.

The compatibility condition puts very strong constraints on the families $(a(\epsilon), [\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}}])$ that can be realized. Indeed, we have already mentioned that this forces the existence of a representative $\psi^{0,\infty}_{\hat{\epsilon}}$ which is 1/2-summable in ϵ. But this is far from being sufficient. For instance, we analyze in detail the case of the Riccati equation and prove that the compatibility condition implies in that case that there exists representatives of the modulus $\psi^{0,\infty}_{\hat{\epsilon}}$ which are analytic in ϵ. This allows us to completely characterize the modulus space in this special case. We also exhibit an example of family $(a(\epsilon), [\psi^0_\epsilon, \psi^\infty_\epsilon])$ depending analytically on ϵ which cannot be realized as a modulus.

The identification of the moduli space opens great possibilities. Indeed, while the knowledge of the Ecalle-Voronin modulus of f_0 allows one to deduce the nonlinearizability of the fixed points of f_ϵ when special kinds of resonance occurs (the "parametric resurgence" phenomenon mentioned earlier), the dependence in ϵ is crucial to be able to draw similar conclusions in the case of fixed points whose multipliers are irrational rotations, or, in the case of resonance, when we consider the more complex question of the convergence of the change of coordinate to normal form. For instance, it is known that the quadratic map $f_\epsilon(z) = z(1+\epsilon) + z^2$ is never linearizable when $1+\epsilon = e^{2\pi i \alpha}$ with $\alpha \notin \mathbb{Q}$ not a Brjuno number. The system is also never normalizable when $1+\epsilon$ is a root of unity. But what can be said of a map $g_\epsilon(z) = f_\epsilon(z) + h_\epsilon(z)$ with $h_\epsilon(z) = o(z^2)$? We hope that our results will give tools to answer such questions.

Another potential application is in the spirit of Hilbert's 16th problem. This problem deals with the maximum number $H(n)$ and relative positions of limit cycles of polynomial vector fields of degree $\leq n$. The finiteness subproblem deals with the existence of a uniform upper bound for the number of limit cycles of polynomial vector fields of degree at most n for each integer n: $H(n) < \infty$. In the paper [2] it is shown how the finiteness part for $n = 2$ can be reduced to 121 local problems, namely showing that 121 graphics have finite cyclicity: let us call this the DRR-program. A significant step in the DRR-program was performed in the paper [1] where it is shown how the use of the Martinet-Ramis invariant of a saddle-node allows to prove the finite cyclicity of several generic graphics of this program. The most difficult graphics of the DRR-program are graphics surrounding centers. An efficient method to prove their cyclicity is to divide the displacement map in the Bautin ideal. This method requires a deep understanding and a fine control of the dependence on the parameters. The compatibility condition is a natural candidate for obtaining further results in this direction.

The paper is organized as follows. In Section 2 we recall the definition of the Ecalle-Voronin modulus, the preparation of the family and the results of [11]. In Section 3 we prove the local realization theorem. In Section 4 we derive the compatibility condition and we prove the 1/2-summability of $\psi^{0,\infty}_{\hat{\epsilon}}$ in ϵ. In Section 5 we prove the global realization theorem. Finally in Section 6 we study examples including the unfolding of a Riccati equation with a saddle-node, and give a complete analytic classification of its local unfoldings.

2 Preliminaries

2.1 Notations

The notations collected here are often referred to in the paper.

- L_C: the linear map
$$L_C(w) = Cw; \tag{2.1}$$
- m_A: the Möbius transformation
$$m_A(w) = \frac{w}{1+Aw}; \tag{2.2}$$
- T_B: the translation
$$T_B(W) = W + B; \tag{2.3}$$
- E: the map
$$E(W) = \exp(-2\pi i W) \tag{2.4}$$
with inverse $E^{-1}(w) = -\frac{1}{2\pi i}\ln(w)$;
- R^0 and R^∞ are the domains of $\mathbb{C}$ defined by:
$$\begin{cases} R^0 = \{W | \operatorname{Im} W < -Y_0\}, \\ R^\infty = \{W | \operatorname{Im} W > Y_0\}, \end{cases} \tag{2.5}$$
where Y_0 is some sufficiently large constant.
- We will be dealing with fixed points $\pm\sqrt{\epsilon}$ of a diffeomorphism f_ϵ. In order to make this well-defined, we work on the universal covering of ϵ-space punctured at 0 parameterized by $\hat{\epsilon}$. The function $\sqrt{\hat{\epsilon}}$ is defined by $\arg\sqrt{\hat{\epsilon}} = \frac{\arg\hat{\epsilon}}{2}$. In particular $\sqrt{\hat{\epsilon}} \in \mathbb{R}^+$, when $\arg\hat{\epsilon} = 0$.
- Upper indices 0 and ∞ will be associated to the two parts of the modulus and to other objects. In all cases, 0 (resp. ∞) will be associated with $-\sqrt{\hat{\epsilon}}$ (resp. $\sqrt{\hat{\epsilon}}$).

2.2 The Ecalle-Voronin modulus of a diffeomorphism and its unfolding

We briefly summarize some results of [11] on the unfoldings of the Ecalle-Voronin invariants of a generic parabolic point of a diffeomorphism
$$f(z) = z + z^2 + o(z^2). \tag{2.6}$$

Since the paper [11] only deals with 1-parameter families, we start by proving a "preparation theorem" for generic unfoldings with several parameters. The preparation makes clear the role of the "canonical parameter".

The perspective of [11] is to compare a generic 1-parameter family f_ϵ with a "model" family, namely the time-one map for the family of vector fields

$$v_\epsilon(z) = \frac{z^2-\epsilon}{1+a(\epsilon)z}\frac{\partial}{\partial z}. \tag{2.7}$$

If $\mu^0_{\hat\epsilon}$ and $\mu^\infty_{\hat\epsilon}$ are the eigenvalues at the singular points $-\sqrt{\hat\epsilon}$ and $\sqrt{\hat\epsilon}$ of (2.7), then we can remark that

$$a(\epsilon) = \frac{1}{\mu^\infty_{\hat\epsilon}} + \frac{1}{\mu^0_{\hat\epsilon}}, \qquad \frac{1}{\sqrt{\hat\epsilon}} = \frac{1}{\mu^\infty_{\hat\epsilon}} - \frac{1}{\mu^0_{\hat\epsilon}}, \tag{2.8}$$

i.e. ϵ and $a(\epsilon)$ are analytic invariants of the system (2.7). Moreover $a(\epsilon)$ depends analytically on ϵ. We wish to prepare our family of diffeomorphisms so that the multipliers at the fixed points, $\lambda^{0,\infty}_{\hat\epsilon}$, correspond to those of the time-1 map of (2.7), and hence $\lambda^{0,\infty}_{\hat\epsilon} = \exp(\mu^{0,\infty}_{\hat\epsilon})$.

Theorem 2.1 *We consider a germ of a* k*-parameter analytic family of diffeomorphisms* $f_\eta : (\mathbb{C},0) \to (\mathbb{C},0)$ *depending on a multi-parameter* $\eta = (\eta_1,\dots,\eta_k)$ *with a double fixed point at the origin for* $\eta = 0$*, such that* $\frac{\partial f}{\partial \eta_j} \neq 0$ *for some* $j \in \{1,\dots,k\}$*. There exists a germ of analytic change of coordinates and parameters* $(z,\eta) \mapsto (\overline{z},\epsilon,\nu_1,\dots,\nu_{k-1})$ *transforming the family to the prepared form*

$$\overline{f}_{\epsilon,\nu}(\overline{z}) = \overline{z} + (\overline{z}^2-\epsilon)\left[1+\overline{\beta}(\epsilon,\nu)+\overline{A}(\epsilon,\nu)\overline{z}+(\overline{z}^2-\epsilon)\overline{Q}(\overline{z},\epsilon,\nu)\right], \tag{2.9}$$

with the additional property that $\overline{\beta}(0,0)=0$ *and*

$$\frac{1}{\sqrt{\epsilon}} = \frac{1}{\ln(\overline{f}'_{\epsilon,\nu}(\sqrt{\epsilon}))} - \frac{1}{\ln(\overline{f}'_{\epsilon,\nu}(-\sqrt{\epsilon}))}.$$

The parameter ϵ *is unique and called the canonical parameter. With this choice of canonical parameter, the function*

$$a(\epsilon) = \frac{1}{\ln(\overline{f}'_{\epsilon,\nu}(\sqrt{\epsilon}))} + \frac{1}{\ln(\overline{f}'_{\epsilon,\nu}(-\sqrt{\epsilon}))},$$

is a formal invariant of the system which depends analytically on ϵ*.*

PROOF. Since $\frac{\partial^2(f_\eta - id)}{\partial z^2}(0) \neq 0$, the Weierstrass preparation theorem allows to write $f_\eta(z) - z = P_\eta(z)U_\eta(z)$ where $P_\eta(z)$ is a Weierstrass polynomial of degree 2 and $U_\eta(z) \neq 0$ for small (z,η). A translation in z allows to bring $P_\eta(z)$ to the form (we do not change the name of the variable) $z^2 - D(\eta)$. Moreover the genericity implies that $\frac{\partial D(\eta)}{\partial \eta_j} \neq 0$, allowing to replace the parameter η_j by $\overline{\epsilon} = D(\eta)$. Let $\nu = (\eta_1,\dots,\eta_{j-1},\eta_{j+1},\dots,\eta_k)$ Using a dilatation in z and $\overline{\epsilon}$ (without changing their names) we can suppose that the initial family has the two fixed points located at $z = \pm\sqrt{\overline{\epsilon}}$ and that $U(0,0)=1$, i.e. that we start with a family:

$$f_\eta(z) = z + (z^2-\overline{\epsilon})h(z,\overline{\epsilon},\nu),$$

where $h(z,\overline{\epsilon},\nu) = 1 + O(z) + O(|\overline{\epsilon},\nu|)$. By the Weierstrass division theorem we have

$$h(z,\overline{\epsilon},\nu) = Q(z,\overline{\epsilon},\nu)(z^2-\overline{\epsilon}) + (a_0 + B(\overline{\epsilon},\nu))z + 1 + C(\overline{\epsilon},\nu),$$

where $B(0,0)=C(0,0)=0$. The multipliers at $\pm\sqrt{\overline{\epsilon}}$ are given by

$$\begin{cases}\lambda^0 = f'_\eta(-\sqrt{\overline{\epsilon}}) = 1-2\sqrt{\overline{\epsilon}}\left[1+C(\overline{\epsilon},\nu)-(a_0+B(\overline{\epsilon},\nu))\sqrt{\overline{\epsilon}}\right], \\ \lambda^\infty = f'_\eta(\sqrt{\overline{\epsilon}}) = 1+2\sqrt{\overline{\epsilon}}\left[1+C(\overline{\epsilon},\nu)+(a_0+B(\overline{\epsilon},\nu))\sqrt{\overline{\epsilon}}\right].\end{cases} \tag{2.10}$$

An additional scaling in z and $\overline{\epsilon}$ is necessary of the form

$$(\overline{z},\epsilon) = (z(1+b(\overline{\epsilon},\nu)), \overline{\epsilon}(1+b(\overline{\epsilon},\nu))^2),$$

with $b(\overline{\epsilon},\overline{\eta}) = O(|\overline{\epsilon},\nu|)$ to be determined. It changes the family to the form

$$\overline{f}_{\epsilon,\nu}(\overline{z}) = \overline{z} + (\overline{z}^2-\epsilon^2)\left(\frac{1+C(\epsilon,\nu)}{1+b(\overline{\epsilon},\nu)} + \frac{a_0+B(\epsilon,\nu)}{(1+b(\overline{\epsilon},\nu))^2}\overline{z} + (\overline{z}^2-\epsilon)\overline{Q}(\overline{z},\epsilon,\nu)\right).$$

We ask that the new multipliers at $\pm\sqrt{\epsilon}$ satisfy

$$\frac{1}{\ln(\lambda^\infty)} - \frac{1}{\ln(\lambda^0)} = \frac{1}{\sqrt{\epsilon}}.$$

This equation is solvable since

$$\left.\frac{\partial\sqrt{\epsilon}\left(\frac{1}{\ln(\lambda^\infty)} - \frac{1}{\ln(\lambda^0)}\right)}{\partial b}\right|_{\epsilon=0} \neq 0$$

and yields $b(\overline{\epsilon},\nu) = O(|\overline{\epsilon},\nu|)$. The other formal invariant is given by

$$a(\overline{\epsilon},\nu) = \frac{1}{\ln(\lambda^0)} + \frac{1}{\ln(\lambda^\infty)}, \tag{2.11}$$

which is clearly analytic in ϵ and ν. Thus, we obtain the required form,

$$\begin{cases}\lambda^0 = \exp\left(-\frac{2\sqrt{\overline{\epsilon}}}{1-\overline{a}(\overline{\epsilon},\nu)\sqrt{\overline{\epsilon}}}\right), \\ \lambda^\infty = \exp\left(\frac{2\sqrt{\overline{\epsilon}}}{1+\overline{a}(\overline{\epsilon},\nu)\sqrt{\overline{\epsilon}}}\right).\end{cases} \tag{2.12}$$

□

The paper [11] describes a complete modulus of analytic classification for one-parameter prepared families of the form (2.9) for values of ϵ in a small neighborhood of the origin. This modulus is given by an unfolding of the Ecalle-Voronin modulus of f_0. Since ϵ is an analytic invariant for a prepared family, it is given by a family of moduli for each fixed value of ϵ. However no family of moduli analytic in ϵ exists in general, so the modulus must be defined in a ramified way. Furthermore [11] does not identify a sufficient condition for such a family to be realizable as the modulus of an unfolding.

Description of the Ecalle-Voronin modulus ($\epsilon = 0$). This modulus is effectively given by the *orbit space* of f_ϵ. We consider two fundamental domains $C^\pm$ of crescent shapes as in Figure 1, which are given by two curves $l_\pm$ and their images by f_0.

Each orbit is represented by at most one point in each crescent, but some orbits can have representatives in the two crescents. Hence the orbit space is the union of the two crescents

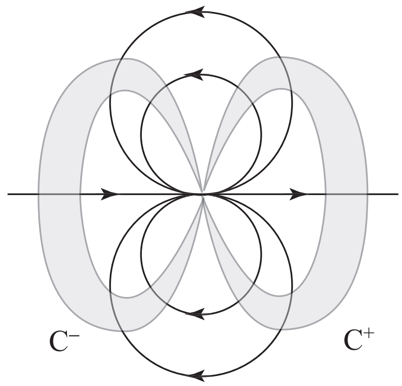


Figure 1: The Ecalle-Voronin modulus

modulo the identification of points of the same orbit. To give this identification in an intrinsic way, one remarks that the two crescents in which we identify the curves $l_\pm$ and $f(l_\pm)$ have the conformal structure of spheres $\mathbb{S}^\pm$, with the points 0 and ∞ identified. The coordinates on the spheres are unique up to linear changes of coordinates. Then the Ecalle-Voronin modulus is the equivalence class of pairs of germs (ψ^0, ψ^∞) of analytic diffeomorphisms, where $\psi^0 : (\mathbb{S}^+, 0) \to (\mathbb{S}^-, 0)$ and $\psi^\infty : (\mathbb{S}^+, \infty) \to (\mathbb{S}^-, \infty)$ are defined respectively in the neighborhoods of 0 and ∞, under conjugation by linear changes of coordinates in the source and target space. Let us define a map f_0 to be *iterable* or *embedable* if f_0 is the time-one map of an analytic vector field. The map f_0 is iterable if and only if both of the germs ψ^0 and ψ^∞ are linear.

The unfolded Ecalle-Voronin modulus. In [11] it is proved that for any sufficiently small neighborhood U of the origin in z-space and for any $\delta \in (0, \pi)$ (later we will restrict to $\delta \in (0, \frac{\pi}{2})$), there exists $\rho > 0$, which is the radius of a small sectorial neighborhood

$$V_{\rho,\delta} = \{\hat{\epsilon} : |\hat{\epsilon}| < \rho, \arg(\hat{\epsilon}) \in (-\delta, 2\pi + \delta)\} \cup \{\epsilon = 0\}, \tag{2.13}$$

of the origin in the universal covering of the parameter space punctured at 0 such that for each $\hat{\epsilon} \in V_{\rho,\delta}$ the orbit space is described as follows

- There exists two crescents $C^\pm_{\hat{\epsilon}}$ with endpoints at the two singular points bounded by curves $l_{\pm,\hat{\epsilon}}$ and their images $f_\epsilon(l_{\pm,\hat{\epsilon}})$ (Figure 2).
- The crescents $C^\pm_{\hat{\epsilon}}$ in which we identify the curves $l_{\pm,\hat{\epsilon}}$ and their images $f_\epsilon(l_{\pm,\hat{\epsilon}})$ have the conformal structure of spheres $\mathbb{S}^\pm_{\hat{\epsilon}}$ with the singular point $\sqrt{\hat{\epsilon}}$ (resp. $-\sqrt{\hat{\epsilon}}$) located at ∞ (resp. 0).
- Points in the two neighborhoods of 0 and ∞ on the spheres $\mathbb{S}^\pm_{\hat{\epsilon}}$ are identified modulo analytic maps, $\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}} : \mathbb{S}^+_{\hat{\epsilon}} \to \mathbb{S}^-_{\hat{\epsilon}}$, defined in the neighborhoods of 0 and ∞ respectively. These maps are obviously uniquely defined up to the choice of coordinates on the spheres. Hence it is natural to consider the equivalence classes of pairs $(\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}})$ under the equivalence relation:

$$(\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}}) \sim (\breve{\psi}^0_{\hat{\epsilon}}, \breve{\psi}^\infty_{\hat{\epsilon}}) \Longleftrightarrow \exists c(\hat{\epsilon}), c'(\hat{\epsilon}) \in \mathbb{C}^* \begin{cases} \breve{\psi}^0_{\hat{\epsilon}}(w) = c'(\hat{\epsilon})\psi^0_{\hat{\epsilon}}(c(\hat{\epsilon})w) \\ \breve{\psi}^\infty_{\hat{\epsilon}}(w) = c'(\hat{\epsilon})\psi^\infty_{\hat{\epsilon}}(c(\hat{\epsilon})w) \end{cases} \tag{2.14}$$

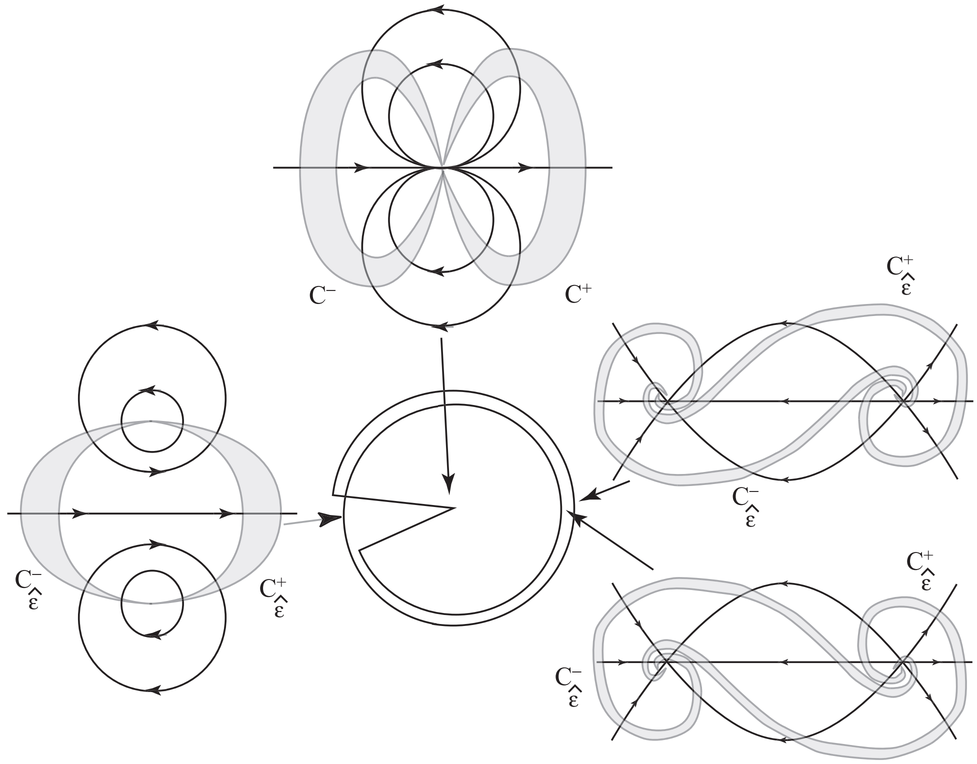


Figure 2: The modulus for the family

where $c(\hat\epsilon), c'(\hat\epsilon)$ are analytic in $V_{\rho,\delta}\setminus\{0\}$ with continuous non-zero limit at 0. Let us denote the equivalence class of the family $(\psi^0_{\hat\epsilon}, \psi^\infty_{\hat\epsilon})$ under $\sim$ by $[\psi^0_{\hat\epsilon}, \psi^\infty_{\hat\epsilon}]$.

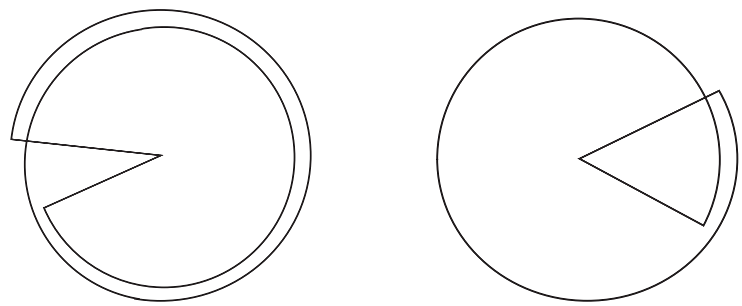
Theorem 2.2 1. *[11] The family* $(a(\epsilon), [\psi^0_{\hat\epsilon}, \psi^\infty_{\hat\epsilon}])$ *for some choice of* $V_{\rho,\delta}$ *is a complete modulus of analytic classification for the one-parameter prepared family* (2.9)*, called the modulus of the family* (2.9)*.*

2. *In the case of a* k*-parameter prepared family, the modulus* $(a(\epsilon,\nu), [\psi^0_{\hat\epsilon,\nu}, \psi^\infty_{\hat\epsilon,\nu}])$ *has representatives which depend analytically on the additional parameters* ν*.*

In this paper we will always use one degree of freedom in the equivalence relation $\sim$ to manage that $(\psi^0_{\hat\epsilon})'(0) = 1$. To preserve this property we will limit ourselves to $c' \equiv c$ in (2.14). It follows from [11] that we then have $(\psi^\infty_{\hat\epsilon})'(\infty) = \exp(4\pi^2 a(\epsilon))$.

In practice, we will prefer to work with other presentations of the modulus, $(\Psi^0_{\hat\epsilon}, \Psi^\infty_{\hat\epsilon})$, where $\Psi^{0,\infty}_{\hat\epsilon} = E^{-1}\circ\psi^{0,\infty}_{\hat\epsilon}\circ E$, with E defined in (2.4). The functions $\Psi^{0,\infty}_{\hat\epsilon}$ will have a direct construction from the Fatou coordinates defined in Section 2.3 below.

Remark 2.3 1. δ is characterized by the property that for $\arg(\hat\epsilon) \in (-\delta,\delta)$ (resp. $\arg(\hat\epsilon) \in (2\pi-\delta, 2\pi+\delta)$) there exists an orbit with α-limit (resp. ω-limit) in $\sqrt{\hat\epsilon}$ (resp. $-\sqrt{\hat\epsilon}$) and ω-limit (resp. α-limit) in $-\sqrt{\hat\epsilon}$ (resp. $\sqrt{\hat\epsilon}$). Moreover for $\arg(\hat\epsilon) \in (-\delta,\delta)\cup(2\pi-\delta, 2\pi+\delta)$, three cases are possible for orbits:



(a) δ large (b) δ small

Figure 3: $V_{\rho,\delta}$ for different sizes of δ

- they have α-limit at the repellor and escape the neighborhood;
- they have ω-limit at the attractor and the backwards orbits escape the neighborhood;
- they have α-limit at the repellor and ω-limit at the attractor.

2. While in [11] it was shown that we could take δ as close as π as wanted provided ρ be sufficiently small, we can remark that even with very small δ we cover a whole neighborhood of the origin in ϵ-space. The first point of view, namely taking δ close to π, is linked with the $1/2$-summability properties in ϵ which will be shown below. However, there will be no need to work with δ large when we will study the compatibility condition and indeed the Figures and estimates will be simpler if we work with $\delta \in (0, \frac{\pi}{2})$. Figure 3 describes the extreme situations for δ.

3. In fact, it would be natural here to re-express all our results in terms of germs of functions with respect to the family of sectors $V_{\rho,\delta}$. We have not used this language here, though it is implicit in what we do, as we wished to make clear at each point the dependence on ρ and δ. However, we will make use of arbitrary restrictions of ρ or δ in what follows without further comment.

The dependence of the modulus on ϵ. As stated above, it is not possible in general to define the modulus so that its definition depends continuously on ϵ in a neighborhood V of the origin. However, given $\delta \in (0, \pi)$, we can choose V sufficiently small that the sectorial neighborhood $V_{\rho,\delta}$ projects onto V. There exist representatives of the modulus $\psi_{\hat{\epsilon}}^{0,\infty}$ which depend analytically on $\hat{\epsilon} \neq 0$ and continuously on $\hat{\epsilon}$ at $\hat{\epsilon} = 0$.

In this way we obtain two presentations of the modulus for $\arg \epsilon \in (-\delta, \delta)$. We compare them via the Glutsyuk modulus defined below.

From the unfolded modulus we can deduce the dynamics near each of the fixed points by means of a renormalized return map when the multiplier is on the unit circle. Otherwise the renormalized return maps at the fixed points are linearizable.

The renormalized return maps. These maps are defined on one sphere, for instance $\mathbb{S}_{\hat{\epsilon}}^+$. In the neighborhood of $\sqrt{\hat{\epsilon}}$ (resp. $-\sqrt{\hat{\epsilon}}$) which we identify to ∞ (resp. 0) on $\mathbb{S}_{\hat{\epsilon}}^+$ we define return maps by iterating f_ϵ until the image is contained in $\mathbb{S}_{\hat{\epsilon}}^+$: given $z \in C_{\hat{\epsilon}}^+$ in the neighborhood

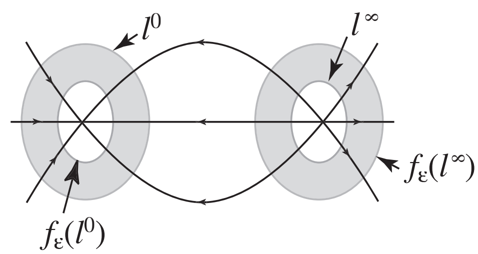


Figure 4: The fundamental domains in the Glutsyuk modulus

of $\sqrt{\hat\epsilon}$ (resp. $-\sqrt{\hat\epsilon}$) and w its coordinate on $\mathbb{S}^+_{\hat\epsilon}$, let $n \in \mathbb{N}$ be minimum such that $f^n_\epsilon(z) \in C^+_{\hat\epsilon}$ and let $k^\infty_{\hat\epsilon}(w)$ (resp. $k^0_{\hat\epsilon}(w)$) be its coordinate on $\mathbb{S}^+_{\hat\epsilon}$. Then $k^\infty_{\hat\epsilon}$ (resp. $k^0_{\hat\epsilon}$) is the renormalized return map in the neighborhood of $\sqrt{\hat\epsilon}$ (resp. $-\sqrt{\hat\epsilon}$). These return maps are given by the composition of the maps $\psi^0_{\hat\epsilon}$ and $\psi^\infty_{\hat\epsilon}$ with a global transition map $L_{\hat\epsilon} : \mathbb{S}^-_{\hat\epsilon} \to \mathbb{S}^+_{\hat\epsilon}$, the Lavaurs map. The Lavaurs map is an analytic map from $\mathbb{CP}^1$ to $\mathbb{CP}^1$ fixing 0 and ∞. Hence it is linear, yielding that the nonlinear part of the return map comes from the unfolding of the two components of the Ecalle-Voronin modulus. Let us call these two return maps $k^0_{\hat\epsilon} = L_\epsilon \circ \psi^0_{\hat\epsilon}$ and $k^\infty_{\hat\epsilon} = L_\epsilon \circ \psi^\infty_{\hat\epsilon}$. From [11], they have multipliers

$$\begin{cases} (k^0_{\hat\epsilon})'(0) = \exp\left(\frac{4\pi^2}{\mu^0_{\hat\epsilon}}\right), \\ (k^\infty_{\hat\epsilon})'(\infty) = \exp\left(\frac{4\pi^2}{\mu^\infty_{\hat\epsilon}}\right). \end{cases}$$

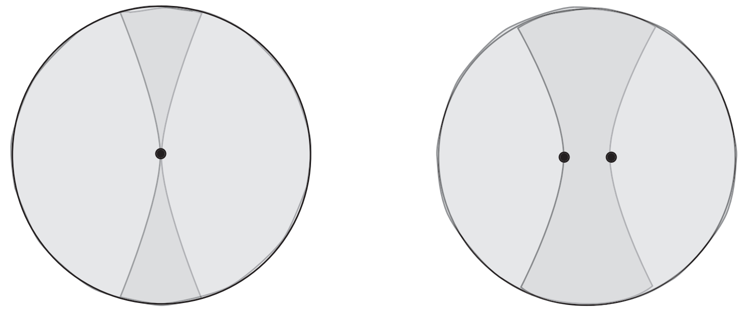
In the Glutsyuk domain, namely $\arg\hat\epsilon \in (-\delta,\delta) \cup (2\pi-\delta, 2\pi+\delta)$, $(k^0_{\hat\epsilon})'(0)$ and $(k^\infty_{\hat\epsilon})'(\infty)$ are exponentially small or large in $\sqrt{\hat\epsilon}$ ($\sim \exp(\pm\frac{C}{|\sqrt{\hat\epsilon}|})$)

The Glutsyuk modulus. The Glutsyuk modulus is defined for small values of ϵ with $\arg\epsilon \in (-\delta,\delta)$ and we will decide to work with $\delta \in (0, \frac{\pi}{2})$. For such ϵ, the fixed points $\sqrt{\epsilon}$ and $-\sqrt{\epsilon}$ are respectively hyperbolic repeller and attractor. Moreover, as stated in Remark 2.3, there are orbits of f_ϵ in U which have $\sqrt{\epsilon}$ (resp. $-\sqrt{\epsilon}$) as α- (resp. ω-) limit set.

We take two closed curves l^0 and l^∞ surrounding $-\sqrt{\epsilon}$ and $\sqrt{\epsilon}$. Since the fixed points are hyperbolic, we can choose $l^{0,\infty}$ so that the region $C^{0,\infty}_\epsilon$ between the curves $l^{0,\infty}$ and their images $f_\epsilon(l^{0,\infty})$ are homeomorphic to annuli (see Figure 4). We identify $l^{0,\infty}$ and $f_\epsilon(l^{0,\infty})$ to get two tori $\mathbb{T}^{0,\infty}_\epsilon$ which represent the local orbit space of the hyperbolic fixed points. Since f_ϵ has connecting orbits, we can iterate f_ϵ in such a way as to identify a collar of $\mathbb{T}^\infty_\epsilon$ with a collar in $\mathbb{T}^0_\epsilon$. In the limit $\epsilon = 0$, the tori become pinched and the map between the collars splits into two maps between the respective ends of the pinched tori. The moduli of the tori depend on $a(\epsilon)$ and ϵ and can be derived directly from the multipliers of the fixed points.

This map is one presentation of the Glutsyuk modulus. A more usual but less geometric presentation is equivalent to the covering map of the above construction. That is, we describe the Glutsyuk modulus in the following way. Since the two points are hyperbolic, there exists in the neighborhood of each fixed point $\pm\sqrt{\epsilon}$ a diffeomorphism $\varphi^\pm_\epsilon$ conjugating f_ϵ to the model, i.e. the time one map of (2.7). For a sufficiently small choice of $V_{\rho,\delta}$ the domains of $\varphi^\pm_\epsilon$ overlap allowing to define the map

$$\varphi^G_\epsilon = \varphi^-_\epsilon \circ (\varphi^+_\epsilon)^{-1}.$$



(a) $\epsilon = 0$ (b) $\arg\hat{\epsilon} \in (-\delta, \delta)$

Figure 5: The domain of the Glutsyuk modulus in the original coordinate z

If we call

$$V_G(\rho) = \{\epsilon; |\epsilon| < \rho, \arg\epsilon \in (-\delta, \delta)\},$$

then it is easy to verify that, for sufficiently small ρ, $(\varphi_\epsilon^G)_{\epsilon\in V_G(\rho)}$ is an analytic invariant of the family f_ϵ under analytic families of change of coordinates preserving the canonical parameter. The Glutsyuk modulus is unique up to composition on the left and on the right by time t maps v_ϵ^t of the vector field (2.7). The family $(\varphi_\epsilon^G)_{\epsilon\in V_\eta(\rho)}$ gives the presentation of the Glutsyuk modulus. The domain for φ_ϵ^G appears in Figure 5.

In practice we will also need to work with other presentations obtained with the use of Fatou coordinates described now.

2.3 Fatou coordinates and other presentations of the modulus

On U we make the change of coordinate $Z = p_\epsilon^{-1}(z)$ defined by

$$Z = p_\epsilon^{-1}(z) = \begin{cases} \frac{1}{2\sqrt{\epsilon}} \ln \frac{z-\sqrt{\epsilon}}{z+\sqrt{\epsilon}}, & \epsilon \neq 0, \\ -\frac{1}{z}, & \epsilon = 0. \end{cases} \tag{2.15}$$

In the Z-coordinate, the map f_ϵ is transformed to F_ϵ which is very close to the translation T_1. *Fatou coordinates* are changes of coordinates $Z \mapsto W$ defined on simply connected domains in Z-space called *translation domains* and conjugating F_ϵ to T_1.

A translation domain is constructed by choosing an admissible line ℓ in the image of $p_\epsilon^{-1}(U)$ in Z-space, i.e. a line such that ℓ and $F_\epsilon(\ell)$ are disjoint and bound a strip in $p_\epsilon^{-1}(U)$, and by saturating this strip under the action of F_ϵ,

Given an admissible line ℓ in Z-space, the associated Fatou coordinate is uniquely defined up to left composition with a translation.

The corresponding presentation of the modulus is a comparison of two Fatou coordinates.

In the Lavaurs point of view, we compare two Fatou coordinates $\Phi_{\hat{\epsilon}}^{\pm}$ defined on translation domains constructed with slanted lines $\ell_{\hat{\epsilon}}^{\pm}$ passing between two holes as in Figure 6, while in the Glutsyuk point of view we compare two Fatou coordinates $\Phi_{\hat{\epsilon}}^{0,\infty}$ defined on translation domains constructed with lines $\ell_{\hat{\epsilon}}^{0,\infty}$ parallel to the line of holes as in Figure 7.

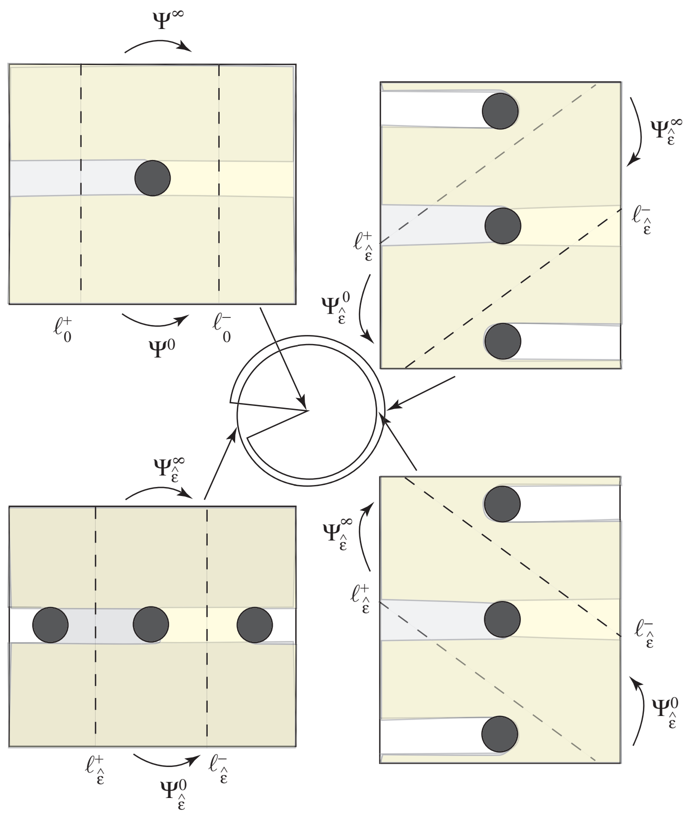


Figure 6: Fatou coordinates in Lavaurs point of view

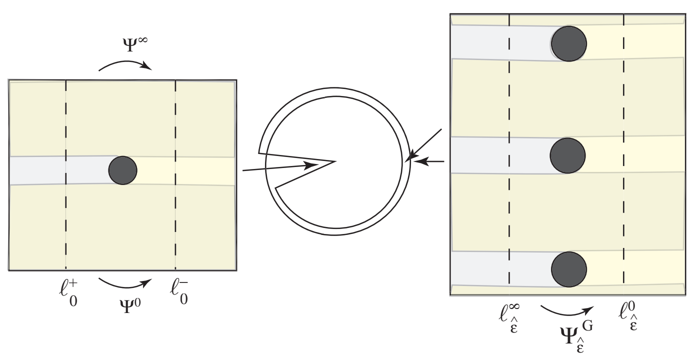


Figure 7: Fatou coordinates in Glutsyuk point of view

Definition 2.4 (1) The modulus in the Lavaurs point of view is given by

$$\Psi_{\hat\varepsilon} = \Phi^-_{\hat\varepsilon} \circ (\Phi^+_{\hat\varepsilon})^{-1}, \tag{2.16}$$

up to composition with a translation on the left and a translation on the right. Since the domain is disconnected, this map is indeed described by the two maps $\Psi^0_{\hat\varepsilon}$ (resp. $\Psi^\infty_{\hat\varepsilon}$)

defined for $\mathrm{Im}(W) < -Y_0$ (resp. $\mathrm{Im}(W) > Y_0$). We also use the alternative presentation

$$\psi_{\hat\epsilon} = E \circ \Psi_{\hat\epsilon} \circ E^{-1}. \tag{2.17}$$

Here the domain of $\psi_{\hat\epsilon}$ is the union of a neighborhood of 0 and a neighborhood of ∞ on $\mathbb{CP}^1$. The respective restrictions of $\psi_{\hat\epsilon}$ to these neighborhoods are noted $\psi^0_{\hat\epsilon}$ and $\psi^\infty_{\hat\epsilon}$. These clearly coincide with the definitions $\psi^{0,\infty}_{\hat\epsilon}$ given previously when considering the spheres $\mathbb{S}^\pm$.

(2) When $\arg\hat\epsilon \in (-\eta,\eta)$, there exist Fatou coordinates $\Phi^{0,\infty}_{\hat\epsilon}$ associated to translation domains defined with lines parallel to the holes as in Figure 7. (We call these Fatou coordinates the Fatou Glutsyuk coordinates.) The modulus in the Glutsyuk point of view is then given by

$$\Psi^G_{\hat\epsilon} = \Phi^0_{\hat\epsilon} \circ (\Phi^\infty_{\hat\epsilon})^{-1}, \tag{2.18}$$

up to composition with a translation on the left and a translation on the right. The maps φ^G (resp. $\varphi^\pm$) mentioned previously are just the push forward of Ψ^G (resp. $\Phi^{0,\infty}$) via p_ϵ.

Remark 2.5 From the uniqueness of the Fatou Glutsyuk coordinates, when $\arg\hat\epsilon \in (2\pi - \eta, 2\pi + \eta)$ the Glutsyuk modulus is defined by $\Psi^G_{\hat\epsilon} = \Phi^\infty_{\hat\epsilon} \circ (\Phi^0_{\hat\epsilon})^{-1}$.

3 The local realization

We will work with parameter values $\hat\epsilon$ in some $V_{\rho,\delta}$, as in (2.13). Unless specified, we will always suppose that the sectors $V_{\rho,\delta}$ contain $\epsilon = 0$. It is clear that we can extend our definition of the modulus of a family f_ϵ, to cover the case of a ramified prepared family $f_{\hat\epsilon}$ defined for $\hat\epsilon \in V_{\rho,\delta}$, where $f_{\hat\epsilon}$ is analytic in $V_{\rho,\delta}$ and locally of the form

$$f_{\hat\epsilon}(z) = z + (z^2 - \epsilon)(1 + h_{\hat\epsilon}(z)),$$

with $h_{\hat\epsilon}(z) = O(|\hat\epsilon, z|)$.

We denote

$$\begin{cases} \mu^0(\hat\epsilon) = -\frac{2\sqrt{\hat\epsilon}}{1-a(\epsilon)\sqrt{\hat\epsilon}} \\ \mu^\infty(\hat\epsilon) = \frac{2\sqrt{\hat\epsilon}}{1+a(\epsilon)\sqrt{\hat\epsilon}}, \end{cases}$$

and hence $\mu^{0,\infty}(e^{2\pi i}\hat\epsilon) = \mu^{\infty,0}(\hat\epsilon)$ and

$$a(\epsilon) = \frac{1}{\mu^0(\hat\epsilon)} + \frac{1}{\mu^\infty(\hat\epsilon)}$$

(which is not ramified in ϵ!).

For such ramified families we prove the following theorem:

Theorem 3.1 *Let $\delta \in (0, \frac{\pi}{2})$, and consider a germ of analytic function $a(\epsilon)$ at the origin. Let $V_{\rho,\delta}$ be a sectorial neighborhood of the origin in the universal covering of ϵ-space punctured at the origin of the form* (2.13), *such that $a(\epsilon)$ has a representative on $V_{\rho,\delta}$.*

Let $\Psi^0_{\hat\epsilon}(W)$ (resp. $\Psi^\infty_{\hat\epsilon}(W)$) be families of germs of analytic diffeomorphisms at $\mathrm{Im}(W) = -\infty$ (resp. $\mathrm{Im}(W) = +\infty$) having representatives $\Psi^0_{\hat\epsilon} : R^0 \to \mathbb{C}$ (resp. $\Psi^\infty_{\hat\epsilon} : R^\infty \to \mathbb{C}$) defined for $\hat\epsilon \in V_{\rho,\delta}$ in domains $R^0 = \{\mathrm{Im}(W) < -Y_0\}$ (resp. $R^\infty = \{\mathrm{Im}(W) > Y_0\}$) for some $Y_0 > 0$ and such that *(i)* $\Psi^{0,\infty}_{\hat\epsilon}$ *depend analytically on* $\hat\epsilon \in V_{\rho,\delta}\setminus\{0\}$ *and have continuous limits when* $\hat\epsilon \to 0$.

(ii) $\Psi^{0,\infty}_{\hat\epsilon}$ *commute with* T_1.

(iii) We have

$$\begin{cases}\Psi^0_{\hat\epsilon}(W) = W + O(\exp(2\pi i W)), & \mathrm{Im}(W) << 0,\\ \Psi^\infty_{\hat\epsilon}(W) = W - 2\pi i a(\epsilon) + O(\exp(2\pi i W)), & \mathrm{Im}(W) >> 0.\end{cases} \tag{3.1}$$

Then for any $\delta' \in (0,\delta)$ *there exists* $\rho' \in (0,\rho]$, *a neighborhood* U *of the origin in* $\mathbb{C}$ *containing the two points* $\pm\sqrt{\hat\epsilon}$ *and a family of analytic diffeomorphisms* $f_{\hat\epsilon}(z) : U \to \mathbb{C}$ *depending on* $\hat\epsilon \in V_{\rho',\delta'}$, *such that:*

- *For all* $\hat\epsilon \in V_{\rho',\delta'}$, $f_{\hat\epsilon}(z)$ *has exactly two fixed points located at* $\pm\sqrt{\hat\epsilon}$ *and is of the form*

$$f_{\hat\epsilon}(z) = z + (z^2 - \epsilon)(1 + h_{\hat\epsilon}(z)),$$

  with $h_{\hat\epsilon}(z) = O(|\hat\epsilon, z|)$.

- $f'_{\hat\epsilon}(\sqrt{\hat\epsilon}) = \exp(\mu^\infty)$ *and* $f'_{\hat\epsilon}(-\sqrt{\hat\epsilon}) = \exp(\mu^0)$. *(So* $f_{\hat\epsilon}$ *is prepared.)*

- $f_{\hat\epsilon}(z)$ *depends analytically of* $\hat\epsilon \in V_{\rho',\delta'}\setminus\{0\}$ *and has a continuous limit when* $\hat\epsilon \to 0$.

- *The modulus of* $f_{\hat\epsilon}$ *is given by* $[\Psi^0_{\hat\epsilon}, \Psi^\infty_{\hat\epsilon}]$.

If the functions $a(\epsilon,\nu)$ *and* $\Psi^{0,\infty}_{\hat\epsilon,\nu}$ *depend analytically on a multi-parameter* ν, *then the function* $f_{\hat\epsilon,\nu}$ *depends analytically on* ν.

For the proof of the theorem we will concentrate on the one-parameter case. It will be obvious that all steps will be analytic in extra parameters.

The following lemma will be used in the proof and elsewhere in the paper.

Lemma 3.2 *(i) We consider families of germs of analytic diffeomorphisms* $\Psi^0_{\hat\epsilon}(W)$ *(resp.* $\Psi^\infty_{\hat\epsilon}(W)$*) at* $\mathrm{Im}(W) = -\infty$ *(resp.* $\mathrm{Im}(W) = +\infty$*) commuting with* T_1, *having representatives* $\Psi^0_{\hat\epsilon} : R^0 \to \mathbb{C}$ *(resp.* $\Psi^\infty_{\hat\epsilon} : R^\infty \to \mathbb{C}$*) defined for* $\hat\epsilon \in V_{\rho,\delta}$ *in domains* $R^0 = \{\mathrm{Im}(W) < -Y_0\}$ *(resp.* $R^\infty = \{\mathrm{Im}(W) > Y_0\}$*) for some* $Y_0 > 0$ *and such that* $\Psi^{0,\infty}_{\hat\epsilon}$ *depend analytically on* $\hat\epsilon \in V_{\rho,\delta}$ *and have continuous limits when* $\hat\epsilon \to 0$. *Let*

$$\begin{cases}\Psi^0_{\hat\epsilon}(W) = W + \sum_{n\le -1} b_n(\hat\epsilon)\exp(2\pi i n W),\\ \Psi^\infty_{\hat\epsilon}(W) = W - 2\pi i a(\epsilon) + \sum_{n\ge 1} c_n(\hat\epsilon)\exp(2\pi i n W),\end{cases} \tag{3.2}$$

let $\beta > 0$ *be small and let*

$$\begin{cases}M^0 = \max_{\mathrm{Im}(W)\le -Y_0-\beta}\left|\Psi^0_{\hat\epsilon}(W) - W\right|,\\ M^\infty = \max_{\mathrm{Im}(W)\ge Y_0+\beta}\left|\Psi^\infty_{\hat\epsilon}(W) - W + 2\pi i a(\epsilon)\right|.\end{cases}$$

Then

$$\begin{cases}|b_n(\hat\epsilon)| < M^0\exp(-2\pi n(Y_0+\beta)), & n \le -1,\\ |c_n(\hat\epsilon)| < M^\infty\exp(2\pi n(Y_0+\beta)), & n \ge 1.\end{cases}$$

The series $\Psi^0_{\hat\epsilon}$ (resp. $\Psi^\infty_{\hat\epsilon}$) in (3.2) *is absolutely convergent for* $\mathrm{Im}(W) \le -Y_0 - \beta$ *(resp.* $\mathrm{Im}(W) \ge Y_0 + \beta$*). Moreover there exists a constant* $N = N(\beta)$ *depending only on* β *such that*

$$\begin{cases} \left|\Psi^0_{\hat\epsilon}(W) - W\right| < M^0 N(\beta)\exp(2\pi(Y_0+\beta+\mathrm{Im}(W))), & \mathrm{Im}(W) < -Y_0 - 2\beta, \\ |\Psi^\infty_{\hat\epsilon}(W) - W + 2\pi i a(\epsilon)| < M^\infty N(\beta)\exp(2\pi(Y_0+\beta-\mathrm{Im}(W))), & \mathrm{Im}(W) > Y_0 + 2\beta. \end{cases} \tag{3.3}$$

(ii) *For any* $\beta > 0$*, the maps* Ψ^0 *and* Ψ^∞ *are uniformly continuous in the region* $\{|\mathrm{Im}W| > Y_0 + \beta\} \times V_{\delta,\rho}$.

(iii) *The image of* $\{\mathrm{Im}W < -Y_0\}$ *(resp.* $\{\mathrm{Im}W > Y_0\}$*) under* $\Psi^0_{\hat\epsilon}$ *(resp.* $\Psi^\infty_{\hat\epsilon}$*) contains some half-plane of the form* $\{\mathrm{Im}W < -Y_1\}$ *(resp.* $\{\mathrm{Im}W > Y_1\}$*).*

PROOF.

(i) This follows from the fact that

$$b_n = \int_{X_0 - i(Y_0+\beta)}^{X_0+1-i(Y_0+\beta)} (\Psi^0_{\hat\epsilon} - id)(X - i(Y_0+\beta))\exp(-2\pi i n(X - i(Y_0+\beta)))dX,$$

and similarly for c_n.

(ii) This follows from the fact that the maps commute with T_1 and have a definite limit as $|\mathrm{Im}W| \to \infty$ or $\hat\epsilon \to 0$.

(iii) This follows from the fact that $\Psi^{0,\infty}_{\hat\epsilon}$ commute with T_1. □

In the rest of the paper we will choose our different sectors in z-space (corresponding to strips in W-space), so that any region where we need to consider $\Psi^0_{\hat\epsilon}$ (resp. $\Psi^\infty_{\hat\epsilon}$) is located inside $\mathrm{Im}W < -Y_0 - 2\beta$ (resp. $\mathrm{Im}W > Y_0 + 2\beta$) for some suitable β, so that the estimates of Lemma 3.2 will always be valid.

PROOF OF THEOREM 3.1. We choose any $\delta' \in (0,\delta)$. Working with δ' instead of δ allows to consider $\arg(\hat\epsilon)$ to vary inside a compact set and hence to yield uniform estimates in $\arg(\hat\epsilon)$. We look for a neighborhood $U = B(0,r)$ of the origin in z-space. The final choice of r and ρ' considered before will be done in several steps throughout the proof. We consider the regions R^0 and R^∞ in W-space and the multivalued mapping:

$$W = q^{-1}_{\hat\epsilon}(z) = \begin{cases} \frac{1}{2\sqrt{\hat\epsilon}}\ln\frac{z-\sqrt{\hat\epsilon}}{z+\sqrt{\hat\epsilon}} + \frac{a(\epsilon)}{2}\ln(z^2-\epsilon), & \hat\epsilon \ne 0, \\ -\frac{1}{z} + a(0)\ln(z), & \hat\epsilon = 0. \end{cases} \tag{3.4}$$

While the inverse $q_{\hat\epsilon}$ exists, it cannot be described by a simple formula.

Note that the function $q^{-1}_{\hat\epsilon}(z)$ is simply the time of the vector field (2.7). The map $q^{-1}_{\hat\epsilon}$ has the property that the restriction of $q^{-1}_{\hat\epsilon} \circ p_{\hat\epsilon}$ to a translation domain is a Fatou coordinate of the model family, namely a conjugacy of $p^{-1}_{\hat\epsilon}$ applied to the model with the translation by 1. (Recall that the model is the time one map of the vector field (2.7)).

The function $q^{-1}_{\hat\epsilon}(z)$ is a multi-valued analytic function of two variables outside the set $\{(z,\epsilon)|z^2 - \epsilon = 0\}$. For $\epsilon = 0$, the function $q^{-1}_{\hat\epsilon}$ is not a global diffeomorphism if $a(0) \ne 0$. So we should not consider it over the whole complex plane and it is better to limit ourselves to sectors in a small neighborhood $U = B(0,r)$ of the origin in z-space. The function $q_{\hat\epsilon}^{-1}$ is ramified both at $\pm\sqrt{\hat\epsilon}$. Moreover when $a \neq 0$ a cut cannot simply be taken between $-\sqrt{\hat\epsilon}$ and $\sqrt{\hat\epsilon}$ since there is a global ramification when one makes a turn on $C(0,r)$.

Although it is difficult to visualize the map $q_{\hat\epsilon}^{-1}$ directly, it can be pictured more easily when lifted to the Z-plane via $p_{\hat\epsilon}$. Here it will be a multi-valued function, whose difference in value when continued around any of the holes in the Z-plane is just $2\pi i a(\epsilon)$. The absolute difference between W and Z in a simply connected region is bounded by $2|a(\epsilon)|\ln(r)$. Thus, if we restrict our attention to a simply connected region, W-space can be thought of as a small distortion of Z-space.

The distance vector between the centers of two holes is of the order

$$\alpha = \alpha_{\hat\epsilon} = \frac{\pi i}{\sqrt{\hat\epsilon}}. \tag{3.5}$$

Hence, the distance between two consecutive holes is of the order of $|\alpha|$ and the radius of holes is of the order of $\frac{1}{r}$ for small ϵ.

As suggested above, we will limit ourselves to simply connected regions on which $q_{\hat\epsilon}^{-1}$ and its inverse $q_{\hat\epsilon}$ are well defined. We choose two strips $S_{\hat\epsilon}^{\pm}$ located on each side of the principal hole as in Figure 8. The choice of the strips and of r and ρ' is given in the following Lemma.

Lemma 3.3 *For $\delta \in (0, \frac{\pi}{2})$ there exists $\rho' > 0$ sufficiently small such that for $|\hat\epsilon| < \rho'$ and $\arg(\hat\epsilon) \in (-\delta, 2\pi + \delta)$ there exist adjusted strips constructed as follows.*

- *The total width of the union of the two strips in the direction of the line of holes is $\frac{3|\alpha|}{2}$.*
- *The horizontal width of the intersection is fixed and equal to $2h$ for some positive constant $h < \frac{1}{2r}$ (recall that the radius of the holes is approximately $\frac{1}{r}$).*
- *Let*

  $$\theta = \frac{1}{2}\left(\frac{\pi}{2} + \arg(\sqrt{\hat\epsilon})\right). \tag{3.6}$$

  The strips are bounded on one side by a slanted line of slope

  $$t = -\tan\theta. \tag{3.7}$$

  On the other side they are bounded by a vertical segment $\mathrm{Re}W = \pm h$ of total height $|\alpha|/4$. From the two endpoints of the segment we continue with two half lines with slope $-\tan\theta$ as drawn in Figure 8.
- *The radius r is chosen sufficiently small so that the intersection part of the strips outside the fundamental holes is located in the region $|\mathrm{Im}W| > Y_0 + 2\beta$ where we can apply the estimates of Lemma 3.2.*

PROOF. We only discuss the range $\theta \in (\frac{\pi}{8}, \frac{\pi}{4})$ where both the strip and the line of holes have negative slope. The case $\theta \in (\frac{3\pi}{8}, \frac{7\pi}{8})$ is similar.

If the holes are of negligible width, then it is a simple matter of geometry that the construction above is valid if $\tan(\theta) > 1/3$, and is therefore satisfied in our range. By choosing

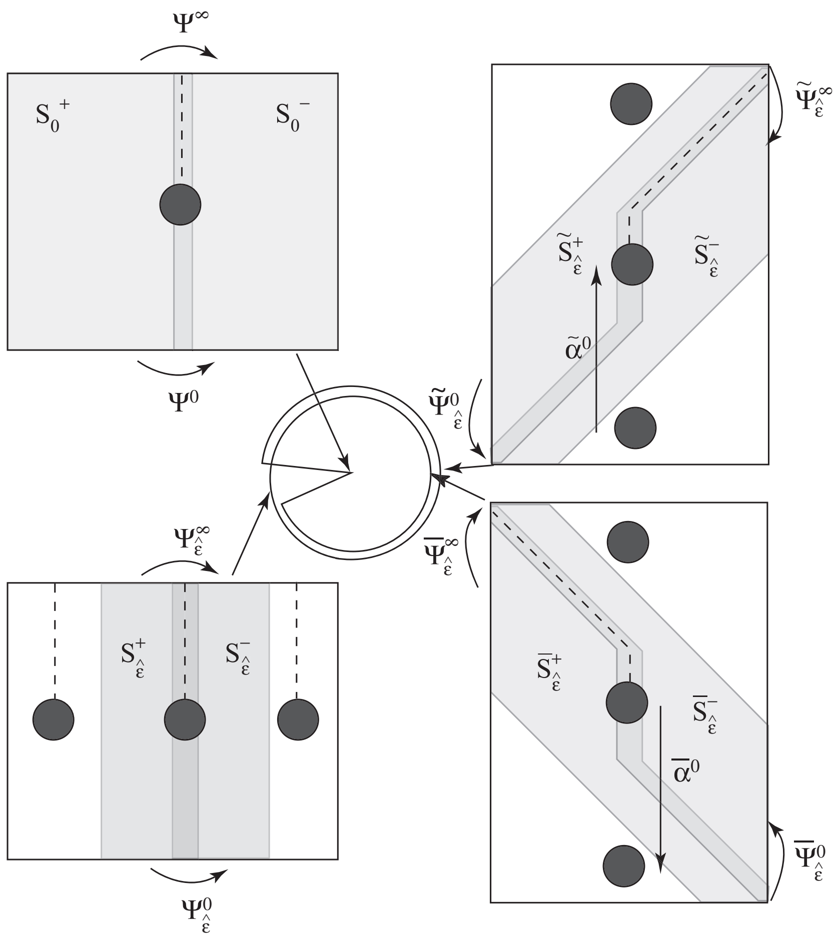


Figure 8: The choice of strips. The dotted lines represent the cuts.

ρ' sufficiently small, we can make the effective size of the holes arbitrarily small and hence the result follows. □

We now consider the images of the two strips, $S^\pm_{\hat\epsilon}$, under $q_{\hat\epsilon}$. These yield two sectors $U^\pm_{\hat\epsilon}$ whose union is $U \setminus \{\pm\sqrt{\hat\epsilon}\}$. For $\epsilon = 0$ the intersection $U_0^+ \cap U_0^-$ is formed of two narrow sectors U_0^0 and U_0^∞ with vertex at 0 and ending on the boundary of U, while for $\hat\epsilon \neq 0$ the intersection is formed of three parts: two sectors $U^0_{\hat\epsilon}$ (resp. $U^\infty_{\hat\epsilon}$) with vertex at $-\sqrt{\hat\epsilon}$ (resp. $\sqrt{\hat\epsilon}$) and ending on the boundary of U and one crescent $U^C_{\hat\epsilon}$ with its two endpoints at $\pm\sqrt{\hat\epsilon}$ (Figure 9). The crescent $U^C_{\hat\epsilon}$ comes from the fact that $q^{-1}_{\hat\epsilon}$ is multivalued and approximately periodic with a period of the order of $\alpha_{\hat\epsilon} = \frac{\pi i}{\sqrt{\hat\epsilon}}$ and the width of the union of the two strips in the direction of $\alpha_{\hat\epsilon}$ is $\frac{3}{2}\alpha_{\hat\epsilon}$.

On $U^\pm_{\hat\epsilon}$ we can define $q^{-1}_{\hat\epsilon}$ in a uniform way, which we call $q^{-1}_{\hat\epsilon,\pm}$. The determinations are chosen so that $q^{-1}_{\hat\epsilon,\pm}$ agree on $U^0_{\hat\epsilon}$. If we take the analytic extension of $q^{-1}_{\hat\epsilon,-}$ after making one turn in the positive direction around $-\sqrt{\hat\epsilon}$, then the extension has the form $T_{\frac{2\pi i}{\mu^0}} \circ q^{-1}_{\hat\epsilon,-}$.

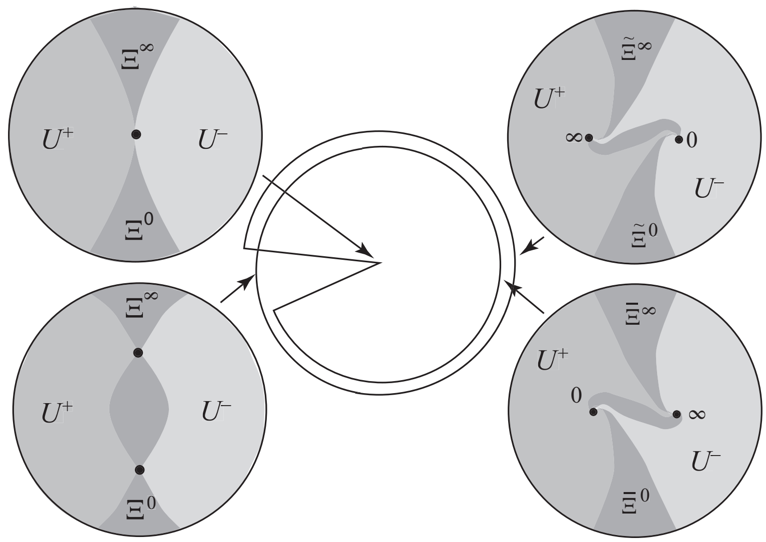


Figure 9: The sectors $U^\pm_{\hat{\epsilon}}$ and their intersection

Let

$$\begin{cases} \Xi^0_{\hat{\epsilon}} = \mathrm{id} + \xi^0_{\hat{\epsilon}} = q_{\hat{\epsilon},+} \circ \Psi^0_{\hat{\epsilon}} \circ q^{-1}_{\hat{\epsilon},+}, \\ \Xi^\infty_{\hat{\epsilon}} = \mathrm{id} + \xi^\infty_{\hat{\epsilon}} = q_{\hat{\epsilon},+} \circ (\Psi^\infty_{\hat{\epsilon}} + 2\pi i a(\epsilon)) \circ q^{-1}_{\hat{\epsilon},+}, \end{cases} \tag{3.8}$$

which are defined respectively in regions containing $U^{0,\infty}_{\hat{\epsilon}}$. For future reference, we also take $\Xi^C_{\hat{\epsilon}} = \mathrm{id}$.

We construct an abstract complex manifold $M_{\hat{\epsilon}}$ by gluing $U^\pm_{\hat{\epsilon}}$ along their intersection. More precisely, let $z^\pm$ be the coordinates on $U^\pm_{\hat{\epsilon}}$. Then we identify

$$z^- = \begin{cases} z^+ + \xi^0_{\hat{\epsilon}}(z^+) = \Xi^0_{\hat{\epsilon}}(z^+), & z^+ \in U^0_{\hat{\epsilon}}, \\ z^+ + \xi^\infty_{\hat{\epsilon}}(z^+) = \Xi^\infty_{\hat{\epsilon}}(z^+), & z^+ \in U^\infty_{\hat{\epsilon}}, \\ z^+ = \Xi^C_{\hat{\epsilon}}(z^+), & z^+ \in U^C_{\hat{\epsilon}}, \end{cases} \tag{3.9}$$

deleting those points in $U^-_{\hat{\epsilon}}$ which are in $U^{0,\infty}_{\hat{\epsilon}}$ but are not in the image of $\Xi^{0,\infty}_{\hat{\epsilon}}$ to ensure that the space we get is Hausdorff.

This gluing is well-defined, since near $\mathrm{Im}(W) = \pm\infty$, $\Psi^{0,\infty}_{\hat{\epsilon}}$ is close to a translation. It is easy to take r and $|\hat{\epsilon}|$ sufficiently small so that this translation is very small compared to the width of the strips: the first condition (r small) ensures that the balls of Figure 8 are sufficiently large, while the second ($|\hat{\epsilon}|$ small) guarantees that the strips and their intersection can be chosen wide.

The map T_1 on the strips lifts to a well-defined holomorphic map $F_{\hat{\epsilon}}$ on $M_{\hat{\epsilon}}$, due to the fact that $\Psi^{0,\infty}_{\hat{\epsilon}}$ commute with T_1. We want to show that $M_{\hat{\epsilon}}$ is conformally equivalent to a disk in $\mathbb{C}$, $D_{\hat{\epsilon}}$, punctured at $\pm\sqrt{\hat{\epsilon}}$. For this we first find a smooth map from $M_{\hat{\epsilon}}$ to $\mathbb{C}$, and then use the Ahlfors-Bers theorem to correct this to a holomorphic map.

Having done this, the image of the map $F_{\hat{\epsilon}}$ is just the diffeomorphism $f_{\hat{\epsilon}}$ we are seeking. Indeed, the W coordinate considered as a multi-valued function in the Z-plane gives Fatou coordinates for $f_{\hat{\epsilon}}$, and our gluings $\Xi_{\hat{\epsilon}}$ can be written as

$$\begin{cases} \Xi^0_{\hat{\epsilon}} = \mathrm{id} + \xi^0_{\hat{\epsilon}} = q_{\hat{\epsilon},+} \circ \Psi^0_{\hat{\epsilon}} \circ q^{-1}_{\hat{\epsilon},+} = q_{\hat{\epsilon},-} \circ \Psi^0_{\hat{\epsilon}} \circ q^{-1}_{\hat{\epsilon},+}, \\ \Xi^\infty_{\hat{\epsilon}} = \mathrm{id} + \xi^\infty_{\hat{\epsilon}} = q_{\hat{\epsilon},+} \circ (\Psi^\infty_{\hat{\epsilon}} + 2\pi i a(\epsilon)) \circ q^{-1}_{\hat{\epsilon},+} = q_{\hat{\epsilon},-} \circ \Psi^\infty_{\hat{\epsilon}} \circ q^{-1}_{\hat{\epsilon},+}, \\ \Xi^C_{\hat{\epsilon}} = \mathrm{id} = q_{\hat{\epsilon},+} \circ q^{-1}_{\hat{\epsilon},+} = q_{\hat{\epsilon},-} \circ T_{\frac{2\pi i}{\mu^0_{\hat{\epsilon}}}} \circ q^{-1}_{\hat{\epsilon},+}. \end{cases} \tag{3.10}$$

That is, the gluings correspond exactly to the fact that the modulus of $f_{\hat{\epsilon}}$ is $(a(\epsilon), [\Psi^0_{\hat{\epsilon}}, \Psi^\infty_{\hat{\epsilon}}])$. The punctures in the disc $D_{\hat{\epsilon}}$, correspond to the critical points of the map $f_{\hat{\epsilon}}$, and their multipliers and thence $a(\epsilon)$ can be similarly derived from $\Xi^{0,\infty,C}_{\hat{\epsilon}}$. The rest of the statements of Theorem 3.1 follow.

We therefore wish to map $M_{\hat{\epsilon}}$ to $\mathbb{C}$ in a smooth way. We express this map via the coordinate patches of $M_{\hat{\epsilon}}$ on the W-plane. We work first with a fixed $\hat{\epsilon}$.

Let $\varphi : \mathbb{R} \to [0,1]$ be a C^∞ monotonic increasing function such that

$$\varphi \equiv \begin{cases} 0, & x \le 0, \\ 1, & x \ge 1. \end{cases}$$

Hence for each n there exists a constant C_n such that

$$|\varphi^{(n)}| \le C_n. \tag{3.11}$$

Writing $W = X + iY$, we take two C^∞ curves $X = \ell_i(Y)$ with $\ell_2(Y) = \ell_1(Y) + h$ which lie within the intersection of the two strips outside the holes, and take

$$N_{\hat{\epsilon}}(X + iY) = \varphi\left(\frac{X - \ell_1(Y)}{h}\right) \tag{3.12}$$

and

$$\begin{cases} \Theta^-_{\hat{\epsilon}}(x,y) = N_{\hat{\epsilon}} \circ q^{-1}_{\hat{\epsilon}}, \\ \Theta^+_{\hat{\epsilon}}(x,y) = 1 - \Theta^-_{\hat{\epsilon}}(x,y), \end{cases}$$

on $q^{-1}_{\hat{\epsilon}}(U^+) \cup q^{-1}_{\hat{\epsilon}}(U^-)$.

For $m \in M_{\hat{\epsilon}}$, we define

$$\chi_{\hat{\epsilon}}(m) = z^+ \Theta^+_{\hat{\epsilon}} + z^- \Theta^-_{\hat{\epsilon}}, \tag{3.13}$$

where m has coordinates $z^+ \in U^+_{\hat{\epsilon}}$ and/or $z^- \in U^-_{\hat{\epsilon}}$.

In this way we realize (via $\chi_{\hat{\epsilon}}$) $M_{\hat{\epsilon}}$ as a neighborhood of the origin, punctured at $\pm\sqrt{\hat{\epsilon}}$. However, the conformal structure of $M_{\hat{\epsilon}}$ is not preserved, but is rather expressed by the Beltrami differential $\mu_{\hat{\epsilon}} = \frac{\partial \chi_{\hat{\epsilon}}/\partial \bar{z}^+}{\partial \chi_{\hat{\epsilon}}/\partial z^+}$. We want to show that there exists $K \in (0,1)$ such that $|\mu_{\hat{\epsilon}}| < K$. We can then correct the map $\chi_{\hat{\epsilon}}$ to a conformal map via the Ahlfors-Bers theorem.

We shall only study what happens on $U^{0,\infty}_{\hat{\epsilon}}$ as $\mu \equiv 0$ outside these sectors. We rewrite:

$$\begin{aligned} \chi_{\hat{\epsilon}}(z_+) &= z^+(\Theta^+_{\hat{\epsilon}} + \Theta^-_{\hat{\epsilon}}) + (z^- - z^+)\Theta^-_{\hat{\epsilon}} \\ &= z^+ + (z^- - z^+)\Theta^-_{\hat{\epsilon}} \\ &= z^+ + \xi^{0,\infty}_{\hat{\epsilon}}(z^+)\Theta^-_{\hat{\epsilon}}. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial \chi_{\hat{\epsilon}}}{\partial z^+} &= 1 + \left[\Theta^-_{\hat{\epsilon}} \frac{\partial \xi^{0,\infty}_{\hat{\epsilon}}}{\partial z^+} + \xi^{0,\infty}_{\hat{\epsilon}} \frac{\partial \Theta^-_{\hat{\epsilon}}}{\partial z^+}\right], \\ \frac{\partial \chi_{\hat{\epsilon}}}{\partial \bar{z}^+} &= \xi^{0,\infty}_{\hat{\epsilon}} \frac{\partial \Theta^-_{\hat{\epsilon}}}{\partial \bar{z}^+}. \end{aligned}$$

The derivatives of $\Theta_{\hat\epsilon}^{\pm}$ satisfy (for $z^{\pm}=x+iy$) near $\pm\sqrt{\hat\epsilon}$:

$$\left|\frac{\partial^{n_1+n_2}\Theta_{\hat\epsilon}^{\pm}}{\partial x^{n_1}\partial y^{n_2}}\right| \leq K_n'\left|z^{\pm}\pm\sqrt{\hat\epsilon}\right|^{-\gamma(n_1+n_2)} \tag{3.14}$$

for some positive constant $\gamma>0$. Indeed, the estimate (3.14) comes directly from the fact that the derivatives of φ are uniformly bounded by (3.11) and that $(q_{\hat\epsilon}^{-1})'(z)=\frac{1+a(\epsilon)z}{z^2-\epsilon}$.

We start by considering $\epsilon=0$. It is known that $\xi_0^{0,\infty}$ is exponentially flat in z^+ (see for instance [7], but the argument is similar to the argument below for the case $\hat\epsilon\neq 0$).

We choose $r>1$ sufficiently small so that we have for $|z^+|<r$

$$\begin{cases}\frac{\partial\chi_0}{\partial\bar z^+}<\frac{1}{8},\\ \frac{\partial\chi_0}{\partial z^+}>\frac{7}{8}.\end{cases}$$

Using the continuity in $\hat\epsilon$ and estimates on $\xi_{\hat\epsilon}^{0,\infty}$, to be proved below, we will choose $\rho'>0$ sufficiently small so that for $|\hat\epsilon|<\rho'$ we have

$$\begin{cases}\frac{\partial\chi_{\hat\epsilon}}{\partial\bar z^+}<\frac{1}{4},\\ \frac{\partial\chi_{\hat\epsilon}}{\partial z^+}>\frac{3}{4}.\end{cases}$$

For that we need to bound the functions $\xi_{\hat\epsilon}^{0,\infty}=\Xi_{\hat\epsilon}^{0,\infty}-id$ and their derivatives. We use the fact that the $\Xi_{\hat\epsilon}^{0,\infty}$ are conjugate to $\Psi_{\hat\epsilon}^{0,\infty}$ through $q_{\hat\epsilon}$. Of course Y_0 can be chosen so that $\left|\Psi_{\hat\epsilon}^{0,\infty}-id\right|$ is uniformly bounded. Moreover we have that

$$\xi_{\hat\epsilon}^{0,\infty}(z)=v_\epsilon^{(\Psi_{\hat\epsilon}^{0,\infty}-id)(q^{-1}(z))}(z)$$

where $v_\epsilon^{(\Psi_{\hat\epsilon}^{0,\infty}-id)(q^{-1}(z))}$ is the flow of v_ϵ (see (2.7)) for the time $(\Psi_{\hat\epsilon}^{0,\infty}-id)(q^{-1}(z))$, which is uniformly bounded. It follows from the theorems on the flow and its dependence on parameters that $\xi_{\hat\epsilon}^{0,\infty}$ and its derivative with respect to z are uniformly bounded for $|\hat\epsilon|$ sufficiently small. To show that $|\mu_{\hat\epsilon}|=\left|\frac{\partial\chi_{\hat\epsilon}/\partial\bar z^+}{\partial\chi_{\hat\epsilon}/\partial z^+}\right|<K<1$ for $|z|<r$ and $|\hat\epsilon|<\rho$ we need to ensure that the derivatives of $\xi_{\hat\epsilon}^{0,\infty}$ are sufficiently flat at $\pm\sqrt{\epsilon}$. So we will show that

$$\left|\xi_{\hat\epsilon}^{0,\infty}(z)\right|<C(\hat\epsilon)\left|z\mp\sqrt{\hat\epsilon}\right|^{\frac{A}{|\sqrt{\hat\epsilon}|}}, \tag{3.15}$$

holds for the values $z\in U^{0,\infty}$ which correspond to values $W=q_{\hat\epsilon}^{-1}(z)$ in the slanted part of the intersection of the strips. Here A is a positive constant which is independent of $\hat\epsilon$.

We will prove (3.15) for $\xi_{\hat\epsilon}^{0}$, the case $\xi_{\hat\epsilon}^{\infty}$ being similar and only sketched.

In the slanted part of the intersection of the strips we have $\mathrm{Im}W<-\frac{B}{|2\sqrt{\hat\epsilon}|}$ for some $B>0$. This yields for the corresponding part of U^0

$$\left|\exp(-2\pi i q^{-1}(z))\right|<e^{\frac{-2\pi B}{2|\sqrt{\hat\epsilon}|}} \tag{3.16}$$

when $\mathrm{Im}(q^{-1}(z))<-\frac{B}{|2\sqrt{\hat\epsilon}|}$. Let

$$g(z)=(z-\sqrt{\hat\epsilon})^{\frac{-2\pi i(1+a\sqrt{\hat\epsilon})}{2\sqrt{\hat\epsilon}}}(z+\sqrt{\hat\epsilon})^{\frac{2\pi i(1-a\sqrt{\hat\epsilon})}{2\sqrt{\hat\epsilon}}}. \tag{3.17}$$

Then $g(z)=\exp(-2\pi i q^{-1}(z))$ and $|g(z)|=\exp(2\pi \mathrm{Im}(q^{-1}(z)))<e^{\frac{-2\pi B}{2|\sqrt{\hat\epsilon}|}}$.

We have

$$\Psi^0_{\hat\epsilon}\circ q_{\hat\epsilon}^{-1}(z)=q_{\hat\epsilon}^{-1}(z)+\sum_{n\le -1} b_n g(z)^{-n},$$

yielding that

$$\left|\Psi^0_{\hat\epsilon}\circ q_{\hat\epsilon}^{-1}(z)-q_{\hat\epsilon}^{-1}(z)\right|=O\left(|g(z)|\right)=O\left(\left|\exp(-2\pi i q_{\hat\epsilon}^{-1}(z))\right|\right).$$

Since

$$\frac{dq}{dz}=\frac{1+az}{z^2-\epsilon},\tag{3.18}$$

if we join two points z_1 and z_2 in the neighborhood of $-\sqrt{\hat\epsilon}$ by a path $\gamma(t)$, $t\in[0,1]$, of length bounded by $c|z_1-z_2|$ for some $c>0$, so that $|\gamma(t)|\ge \min(|z_1+\sqrt{\hat\epsilon}|,|z_2+\sqrt{\hat\epsilon}|)$ for all $t\in[0,1]$, then $|q(z_1)-q(z_2)|\le c|z_1-z_2|\max_{t\in[0,1]}\left|\frac{dq}{dz}(\gamma(t))\right|$. It follows that

$$\left|q_{\hat\epsilon}\circ\Psi^0_{\hat\epsilon}\circ q_{\hat\epsilon}^{-1}(z)-z\right|=O(|g(z)|).$$

This holds uniformly in all the region because of Lemma 3.2 and the constant C is an upper bound for $\left|(z-\sqrt{\hat\epsilon})^{\frac{-2\pi i(1+a\sqrt{\hat\epsilon})}{2\sqrt{\hat\epsilon}}}\right|$ in the region corresponding to the slanted part of the strip near $-\sqrt{\hat\epsilon}$.

In the same way it is possible, using the chain rule, to show that the derivatives of $\xi^0_{\hat\epsilon}$ at $-\sqrt{\hat\epsilon}$ remain bounded when $\hat\epsilon\to 0$. Indeed for the derivatives of q or q^{-1} we use (3.18), while for the derivatives of $\Psi^0_{\hat\epsilon}$ we use (3.16).

For the case of $\xi^\infty_{\hat\epsilon}$ defined in (3.8), the only difference with the previous one is the presence of the translation term in $\Psi^\infty_{\hat\epsilon}$, which comes from the comparison between the two maps, $q_{\hat\epsilon,\pm}^{-1}$, on $U^\infty_{\hat\epsilon}$. Indeed the map $q_{\hat\epsilon}^{-1}$ corresponds to the time for the vector field (2.7). We have two times $q_{\hat\epsilon,\pm}^{-1}$ defined respectively over $U^\pm_{\hat\epsilon}$. While they can be chosen to coincide on $U^0_{\hat\epsilon}$ we have that $q_{\hat\epsilon,-}^{-1}=q_{\hat\epsilon,+}^{-1}-2\pi i a$ over $U^\infty_{\hat\epsilon}$. Then the gluing corresponds to

$$q_{\hat\epsilon,-}^{-1}=\Psi^\infty_{\hat\epsilon}(q_{\hat\epsilon,+}^{-1})+2\pi i a=q_{\hat\epsilon,+}^{-1}+\sum_{n\ge 1}c_n\exp(2\pi i n q_{\hat\epsilon,+}^{-1})$$

(see also (3.10)). The rest of the argument is as in the case of $\xi^0_{\hat\epsilon}$.

Hence $\mu_{\hat\epsilon}$ is a Beltrami field which we extend by $\mu_{\hat\epsilon}(\pm\sqrt{\hat\epsilon})=0$ in a C^1 way. By the Ahlfors-Bers theorem there exists a 1-1 map $\sigma_{\hat\epsilon}:\chi_{\hat\epsilon}(M_{\hat\epsilon})\to\mathbb{C}$ which is holomorphic in the sense of this structure and whose image is the disk $r\mathbb{D}$. Since this construction is continuous in $\hat\epsilon$ up to the limit $\epsilon=0$, we can always suppose that the boundary point r of $M_{\hat\epsilon}$ is sent to the boundary point r of $r\mathbb{D}$ by the composition $\sigma_{\hat\epsilon}\circ\chi_{\hat\epsilon}$. Then

$$\zeta_{\hat\epsilon}=\sigma_{\hat\epsilon}\circ\chi_{\hat\epsilon}\tag{3.19}$$

is holomorphic, yielding that the manifold $M_{\hat\epsilon}$ is conformally equivalent to the disk $r\mathbb{D}$ punctured in two points: $\mathbb{D}\setminus\{x_1,x_2\}$. We conjugate with the unique Möbius transformation $\tau_{\hat\epsilon}$ sending x_1, x_2 and r respectively on $-\sqrt{\hat\epsilon}$, $\sqrt{\hat\epsilon}$ and r. The image of $r\mathbb{D}$ is a disk $D_{\hat\epsilon}$ not necessarily centered at the origin and whose boundary contains $\{r\}$. Let us now consider the case $\epsilon = 0$: there exists a one-parameter family of Möbius transformations τ sending the double point $x_1 = x_2$ and r to 0 and r respectively. Each one is uniquely determined by the derivative at x_1. We choose the one such that $\zeta_0'(0)\tau'(0) = 1$. Indeed we have

$$\lim_{\hat{\epsilon}\to 0} \frac{(\tau_{\hat{\epsilon}} \circ \zeta_{\hat{\epsilon}})(\sqrt{\hat{\epsilon}}) - (\tau_{\hat{\epsilon}} \circ \zeta_{\hat{\epsilon}})(-\sqrt{\hat{\epsilon}})}{2\sqrt{\hat{\epsilon}}} \equiv 1.$$

The construction of $\mu_{\hat{\epsilon}}$ is continuous in $\hat{\epsilon}$ and has a limit when $\hat{\epsilon} \to 0$ on radial rays, yielding the same property for the construction above. We will show below how to modify it slightly so as to ensure that it is also holomorphic in $\hat{\epsilon} \neq 0$ and with a uniform limit on all rays.

Let us start by looking at the different limits we get for $\epsilon = 0$ along the different rays $\arg(\hat{\epsilon}) = \mathrm{Const}$. When constructing an abstract manifold by charts and transition maps between charts, the size of the charts is not intrinsic and it is possible to modify them as long as the new transition maps are analytic extensions of the previous ones. So we get different presentations of a unique manifold as long as the total underlying set is the same. We must be careful at the boundary. Indeed the outer boundary of $U_{\hat{\epsilon}}^+$ is not in general sent into the outer boundary of $U_{\hat{\epsilon}}^-$ under the gluing map. This is why we have taken so much care so that the intersection of the strips be constant near the boundary of the hole in W-space (see Figure 8). With this property the limit is independent of $\arg(\hat{\epsilon})$ since the different $\xi_0^{0,\infty}$ obtained with different slopes are all analytic extensions one of the other.

Let us now show that the map $f_{\hat{\epsilon}}$ depends analytically on $\hat{\epsilon}$. We start by considering a small sector $\arg\hat{\epsilon} \in (\theta_0 - \eta, \theta_0 + \eta)$ for some fixed θ_0 and some small η. It is possible over such a sector to reproduce the same construction as above, but with strips having a fixed slope (for instance that chosen for $\arg\hat{\epsilon} = \theta_0$) and a fixed intersection domain. Since the intersection is fixed, it is possible to choose a fixed $N_{\hat{\epsilon}}$ in (3.12), hence depending analytically on $\hat{\epsilon}$. In this way we locally get maps $\sigma_{\hat{\epsilon}}$ and $\zeta_{\hat{\epsilon}}$ which are analytic in $\hat{\epsilon}$. But these maps have just been instrumental in constructing a unique disk $D_{\hat{\epsilon}}$ endowed with a unique map $f_{\hat{\epsilon}}$. It follows that $f_{\hat{\epsilon}}$ depends analytically on $\hat{\epsilon}$. The analytic dependence on the auxiliary multi-parameter ν, is an immediate application of the analytic dependence on parameters in the Ahlfors-Bers theorem. $\square$

4 The compatibility condition

In Section 3 we have realized the modulus $(a(\epsilon), [\psi_{\hat{\epsilon}}^0, \psi_{\hat{\epsilon}}^\infty])_{\hat{\epsilon}\in V_{\rho,\delta}}$ in a family $f_{\hat{\epsilon}}$ which is ramified in $\hat{\epsilon}$ over some sectorial neighborhood $V_{\rho,\delta}$. We are now interested in the condition that the family $(a(\epsilon), [\psi_{\hat{\epsilon}}^0, \psi_{\hat{\epsilon}}^\infty])_{\hat{\epsilon}\in V_{\rho,\delta}}$ must satisfy in order that there exists a realization in a uniform family f_ϵ defined for $\epsilon \in B(0, \rho)$.

We limit our discussion to the sector

$$V_G = V_G(\rho) = \{\epsilon; 0 < |\epsilon| < \rho, \arg\epsilon \in (-\delta, \delta)\}, \tag{4.1}$$

which is covered in $V_{\rho,\delta}$ by two small sectors

$$\begin{cases} \widetilde{V} = \{\hat{\epsilon}; 0 < |\hat{\epsilon}| < \rho, \arg\hat{\epsilon} \in (2\pi - \delta, 2\pi + \delta)\} \\ \overline{V} = \{\hat{\epsilon}; 0 < |\hat{\epsilon}| < \rho, \arg\hat{\epsilon} \in (-\delta, +\delta)\}. \end{cases} \tag{4.2}$$

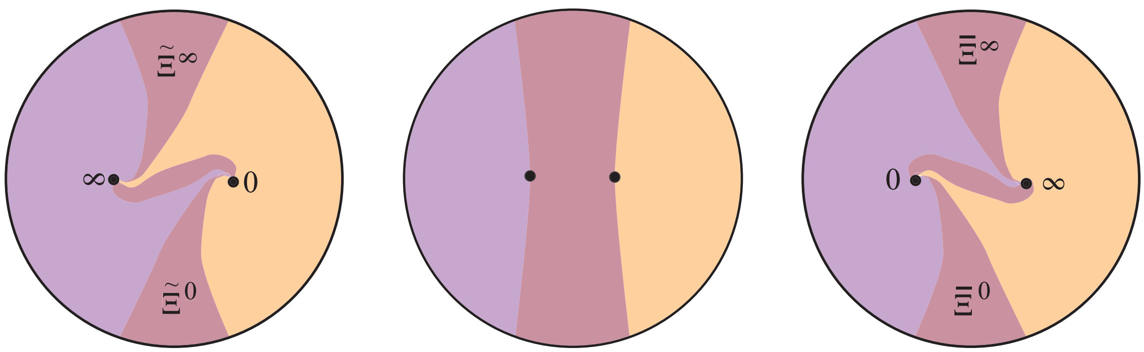
We remark that the Glutsyuk modulus exists for $\epsilon \in V_G$, and ρ sufficiently small. Depending on the context and whether we want to concentrate on ρ or not we will use either the notation V_G or $V_G(\rho)$.

A necessary condition for the existence of a uniform realization is that the functions $f_{\hat{\epsilon}}$ and $f_{\hat{\epsilon}e^{2\pi i}}$ be conjugate. In order to simplify the notation we will write

$$\begin{cases} \overline{\epsilon} = \hat{\epsilon}, & \hat{\epsilon} \in \overline{V}, \\ \tilde{\epsilon} = \hat{\epsilon}e^{2\pi i}, & \hat{\epsilon}e^{2\pi i} \in \widetilde{V}. \end{cases} \tag{4.3}$$

Hence $\overline{\epsilon}$ and $\tilde{\epsilon}$ project on the same $\epsilon \in V_G$. These functions have their moduli presented in different ways. We need to find a compatibility condition (in terms of the modulus) which expresses the fact that the two presentations encode the same dynamics up to conjugacy.

In order to investigate this further we use the notation $\overline{\Psi}^{0,\infty}$ and $\overline{\Xi}^{0,\infty} = id + \overline{\xi}^{0,\infty}$ when $\hat{\epsilon} \in \overline{V}$ and $\widetilde{\Psi}^{0,\infty}$ and $\widetilde{\Xi}^{0,\infty} = id + \tilde{\xi}^{0,\infty}$ when $\hat{\epsilon} \in \widetilde{V}$. We work for a fixed value $\overline{\epsilon} = \hat{\epsilon} \in \overline{V}$ and the corresponding $\tilde{\epsilon} = \hat{\epsilon}e^{2\pi i} \in \widetilde{V}$. Because we work with two fixed values of $\hat{\epsilon}$ we will omit mentioning these values in the indices. In the point of view corresponding to $\overline{V}$, the left



(a) $\hat{\epsilon} \in \widetilde{V}$ (b) The Glutsyuk sectors (c) $\hat{\epsilon} \in \overline{V}$

Figure 10: The different sectors

(resp. right) singular point is $-\sqrt{\hat{\epsilon}}$ (resp. $\sqrt{\hat{\epsilon}}$) and $\overline{\Xi}^0$ (resp. $\overline{\Xi}^\infty$) describes the gluing when turning around it. In the point of view corresponding to $\widetilde{V}$, the left (resp. right) singular point is $\sqrt{\hat{\epsilon}}$ (resp. $-\sqrt{\hat{\epsilon}}$) and $\widetilde{\Xi}^\infty$ (resp. $\widetilde{\Xi}^0$) describes the gluing when turning around it. Remark that in all cases ∞ (resp. 0) will represent $\sqrt{\hat{\epsilon}}$ (resp. $-\sqrt{\hat{\epsilon}}$).

The idea is to derive the Glutsyuk modulus from these two Lavaurs moduli and to equate them. This is done in considering the darkened (striped) regions of the two pictures on the right in Figure 11.

We define the following quantities related to the periods of $q^{\pm}_{\hat{\epsilon}}$ near the inverse images of $\pm\sqrt{\hat{\epsilon}}$.

$$\begin{cases} \alpha^\infty = -\frac{2\pi i(1+a(\epsilon)\sqrt{\hat{\epsilon}})}{2\sqrt{\hat{\epsilon}}} = -\frac{2\pi i}{\mu^\infty}, \\ \alpha^0 = -\frac{2\pi i(1-a(\epsilon)\sqrt{\hat{\epsilon}})}{2\sqrt{\hat{\epsilon}}} = \frac{2\pi i}{\mu^0}. \end{cases} \tag{4.4}$$

Hence

$$\alpha^\infty = \alpha^0 - 2\pi i a. \tag{4.5}$$

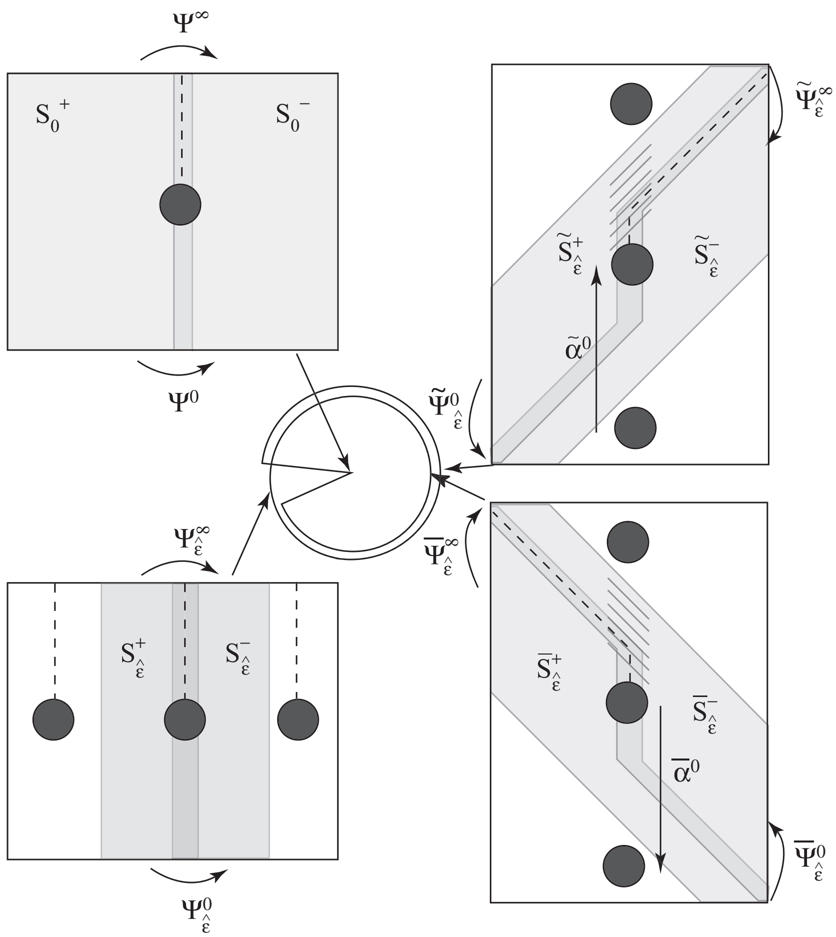


Figure 11: The darkened (striped) region above the principal hole where we compare the two points of view.

We will have $\tilde\alpha^0$ and $\tilde\alpha^\infty$ over $\widetilde{V}$ and $\overline{\alpha}^0$ and $\overline{\alpha}^\infty$ over $\overline{V}$. Moreover

$$\begin{cases} \tilde\alpha^0 = -\overline{\alpha}^\infty, \\ \tilde\alpha^\infty = -\overline{\alpha}^0. \end{cases} \tag{4.6}$$

We define

$$\begin{cases} \widetilde{C}^{0,\infty} = \exp(-2\pi i\tilde\alpha^{0,\infty}), \\ \overline{C}^{0,\infty} = \exp(-2\pi i\overline{\alpha}^{0,\infty}). \end{cases} \tag{4.7}$$

In particular $\widetilde{C}^{0,\infty} = \exp(-2\pi i\tilde\alpha^{0,\infty})$ are exponentially large in $\sqrt{\hat\varepsilon}$ while $\overline{C}^{0,\infty} = \exp(-2\pi i\overline{\alpha}^{0,\infty})$ are exponentially small in $\sqrt{\hat\varepsilon}$.

Theorem 4.1 *(i) There exists $Y_1 > 0$ such that for all $\hat\varepsilon \in \widetilde{V}_\eta$ there exists a map $\widetilde{H}^0$ defined in a region $\mathrm{Im}(W) < -Y_1$, commuting with T_1, and such that*

$$\widetilde{H}^0 \circ T_{\tilde\alpha^0} \circ \widetilde{\Psi}^0 = T_{\tilde\alpha^0} \circ \widetilde{H}^0. \tag{4.8}$$

In the new coordinate $\widetilde{W}^0 = \widetilde{H}^0(W)$ the renormalized return map $T_{\tilde{\alpha}^0} \circ \widetilde{\Psi}^0$ is a translation. Then $\widetilde{W}^0$ is one Fatou Glutsyuk coordinate. Similarly there exists a map $\widetilde{H}^\infty$ defined in the region $\mathrm{Im}(W) > Y_1$, commuting with T_1, and such that

$$\widetilde{H}^\infty \circ T_{\tilde{\alpha}^0} \circ \widetilde{\Psi}^\infty = T_{\tilde{\alpha}^\infty} \circ \widetilde{H}^\infty. \tag{4.9}$$

In the new coordinate $\widetilde{W}^\infty = \widetilde{H}^\infty(W)$ the renormalized return map is a translation and $\widetilde{W}^\infty$ is the second Fatou Glutsyuk coordinate. The Glutsyuk modulus is then given by $\widetilde{H}^\infty \circ (\widetilde{H}^0)^{-1}$.

(ii) Similarly there exists $Y_2 > 0$ such that for all $\overline{\epsilon} \in \overline{V}_\eta$ there exists $\overline{H}^{0,\infty}$ commuting with T_1 and such that

$$\overline{H}^0 \circ \overline{\Psi}^0 \circ T_{\overline{\alpha}^0} = T_{\overline{\alpha}^0} \circ \overline{H}^0 \tag{4.10}$$

on $\mathrm{Im}(W) < -Y_2$ and

$$\overline{H}^\infty \circ \overline{\Psi}^\infty \circ T_{\overline{\alpha}^0} = T_{\overline{\alpha}^\infty} \circ \overline{H}^\infty \tag{4.11}$$

on $\mathrm{Im}(W) > Y_2$. The Glusyuk modulus is then given in this context by $\overline{H}^0 \circ (\overline{H}^\infty)^{-1}$. Considering (i) and (ii) together we can of course suppose that $Y_1 = Y_2$.

(iii) The maps $\overline{H}^{0,\infty}$ and $\widetilde{H}^{0,\infty}$ are unique up to left composition with a translation. In particular they are unique if we ask that their limits for $\mathrm{Im} W \to \pm\infty$ be the identity.

(iv) The functions $\widetilde{H}^0$ and $\widetilde{H}^\infty$ (resp. $\overline{H}^0$ and $\overline{H}^\infty$) have analytic extensions defined on domains which intersect.

(v) A necessary condition for the family $(\Psi^0_{\hat{\epsilon}}, \Psi^\infty_{\hat{\epsilon}})$ to be the modulus of an analytic family f_ϵ of diffeomorphisms is that for corresponding values of $\overline{\epsilon} \in \overline{V}_\eta$ and $\tilde{\epsilon} \in \widetilde{V}_\eta$ there exist constants D_ϵ and D'_ϵ (depending on ϵ, not on $\hat{\epsilon}$!) such that

$$\widetilde{H}^\infty \circ (\widetilde{H}^0)^{-1} = T_{D_\epsilon} \circ \overline{H}^0 \circ (\overline{H}^\infty)^{-1} \circ T_{D'_\epsilon}. \tag{4.12}$$

This condition is called the compatibility condition*.*

(vi) The functions $\widetilde{H}^{0,\infty}$ and $\overline{H}^{0,\infty}$ can be chosen to depend analytically on the auxiliary multiparameter ν, as can the constants D and D' in (4.12).

PROOF.

(i) and (ii) Conjugating $T_{\tilde{\alpha}^0} \circ \widetilde{\Psi}^{0,\infty}$ under $E(W) = \exp(-2\pi i W)$ yields maps $\tilde{\kappa}^{0,\infty} = E \circ T_{\tilde{\alpha}^0} \circ \widetilde{\Psi}^{0,\infty} \circ E^{-1}$ with multiplier of modulus different from one. Hence $\tilde{\kappa}^0$ (resp. $\tilde{\kappa}^\infty$) is linearizable in the neighborhood of 0 (resp. ∞): there exists $\tilde{h}^{0,\infty}$ such that

$$\tilde{h}^{0,\infty} \circ \tilde{\kappa}^{0,\infty} = L_{\exp(-2\pi i \tilde{\alpha}^{0,\infty})} \circ \tilde{h}^{0,\infty}.$$

The maps $\widetilde{H}^{0,\infty}$ are simply $E^{-1} \circ \tilde{h}^{0,\infty} \circ E$. It then follows that they commute with T_1.

The existence of $\overline{H}^{0,\infty}$ is similar.

(iii) This is obvious since $\overline{h}^{0,\infty}$ and $\tilde{h}^{0,\infty}$ are unique up to left composition with linear maps.

(iv) The relation (4.8) allows to extend $\widetilde{H}^0$ by means of $\widetilde{H}^0 \circ T_{\tilde{\alpha}^0} = T_{\tilde{\alpha}^0} \circ \widetilde{H}^0 \circ (\widetilde{\Psi}^0)^{-1} \circ T_{-\tilde{\alpha}^0}$, so its domain becomes the image of $\widetilde{\Psi}^0$ augmented of a strip of width $\tilde{\alpha}^0$. Similarly for $\widetilde{H}^\infty$, $\overline{H}^0$ and $\overline{H}^\infty$.

We claim the existence of a uniform domain. The intuitive idea is that there are no recurrent points for $f_{\hat{\epsilon}}$ for these values of $\hat{\epsilon}$. In practice, the relations (4.8), (4.9), (4.10) and (4.11) allow to extend the maps in the direction of α. The fact that the maps commute with T_1 allows to extend them until the holes. Hence the claim.

(v) The compatibility condition comes from the fact that each Fatou Glutsyuk coordinate is uniquely determined up to a translation.

(vi) This is clear from the nature of the proofs above. □

The compatibility condition was found independently by Reinhard Schäfke [20] in the case $\Psi^\infty_{\hat{\epsilon}} \equiv id$.

Remark 4.2 For each translation T_A there exists a unique $\widetilde{H}^0$, such that $\lim_{Im(W)\to-\infty} \widetilde{H}^0 = W + A$. Similar statements are valid for $\widetilde{H}^\infty$, $\overline{H}^0$, $\overline{H}^\infty$.

Proposition 4.3 *We consider the modulus* $\left(a(\epsilon), [\Psi^0_{\hat{\epsilon}}, \Psi^\infty_{\hat{\epsilon}}]\right)\big|_{\hat{\epsilon}\in V_{\rho,\delta}}$ *attached to a germ of one-parameter prepared analytic family of diffeomorphisms of the form* (2.9) *and hence satisfying the compatibility condition* (4.12)*. Then there exists an analytic function defined by* $\hat{\epsilon} \mapsto \gamma_{\hat{\epsilon}}$ *on* $V_{\rho,\delta}$*, such that on* V_η *we have*

$$\gamma_{\tilde{\epsilon}} - \gamma_{\overline{\epsilon}} = D_\epsilon - 2\pi i a$$

and

$$\lim_{\hat{\epsilon}\to 0} \gamma_{\hat{\epsilon}} = \gamma_0,$$

for some constant γ_0.

Corollary 4.4 *Given a modulus* $\left(a(\epsilon), [\Psi^0_{\hat{\epsilon}}, \Psi^\infty_{\hat{\epsilon}}]\right)\big|_{\hat{\epsilon}\in V_{\rho,\delta}}$*, it is possible to choose a representative* $\left(a(\epsilon), \Psi^0_{\hat{\epsilon}}, \Psi^\infty_{\hat{\epsilon}}\right)\big|_{\hat{\epsilon}\in V_{\rho,\delta}}$ *so that that* $D_\epsilon \equiv 2\pi i a$ *in* (4.12) *and*

$$D'_\epsilon = -2\pi i a + O(\exp(-2\pi i \overline{\alpha}^0)). \tag{4.13}$$

PROOF. It is possible to represent the modulus by the family

$$(\Upsilon^0_{\hat{\epsilon}}, \Upsilon^\infty_{\hat{\epsilon}}) = (T_{-\gamma_{\hat{\epsilon}}} \circ \Psi^0_{\hat{\epsilon}} \circ T_{\gamma_{\hat{\epsilon}}}, T_{-\gamma_{\hat{\epsilon}}} \circ \Psi^\infty_{\hat{\epsilon}} \circ T_{\gamma_{\hat{\epsilon}}}). \tag{4.14}$$

In the equation (4.12) the maps $\widetilde{H}^0$, $\widetilde{H}^\infty$, $\overline{H}^0$, $\overline{H}^\infty$ are then replaced by

$$\begin{cases} \widetilde{H}^{0,\infty}_1 = T_{-\gamma_{\tilde{\epsilon}}} \circ \widetilde{H}^{0,\infty} \circ T_{\gamma_{\tilde{\epsilon}}}, \\ \overline{H}^{0,\infty}_1 = T_{-\gamma_{\overline{\epsilon}}} \circ \overline{H}^{0,\infty} \circ T_{\gamma_{\overline{\epsilon}}}. \end{cases} \tag{4.15}$$

They satisfy the compatibility condition

$$\widetilde{H}^\infty_1 \circ (\widetilde{H}^0_1)^{-1} = T_{2\pi i a} \circ \overline{H}^0_1 \circ (\overline{H}^\infty_1)^{-1} \circ T_{D''}. \tag{4.16}$$

We postpone the proof that

$$D'' = -2\pi i a + O(\exp(-2\pi i\overline{\alpha}^0)) \tag{4.17}$$

after the proof of Lemma 4.5. □

Proof of Proposition 4.3. When we define Fatou coordinates we have one degree of freedom per Fatou coordinate. One degree of freedom has been used when we asked that $\lim_{\mathrm{Im}(W)\to-\infty}\Psi^0_{\hat\epsilon} = \mathrm{id}$, the other degree of freedom can be used to fix a base point for the Fatou coordinate $\Phi^-_{\hat\epsilon}$. Consider Figure 8: we can choose a base point Z_0 located on the right of the principal hole and we can choose the Fatou coordinate $\Phi^-_{\hat\epsilon}$ such that $\Phi^-_{\hat\epsilon}(Z_0) = Z_0$. This is done via the composition $T_{-\gamma_{\hat\epsilon}}\circ\Phi^-_{\hat\epsilon}$. Then Φ^+ is completely determined by $\lim_{\mathrm{Im}(W)\to-\infty}\Psi^0_{\hat\epsilon} = \mathrm{id}$. This yields the new representative of the modulus in (4.14).

Once the Lavaurs Fatou coordinates are chosen, the Fatou Glutsyuk coordinates are completely determined by the limit conditions on the functions $\overline{H}^{0,\infty}$ and $\widetilde{H}^{0,\infty}$. So for the new Fatou coordinate and representative (4.14), the new Fatou Glutsyuk coordinates are simply given in (4.15) (i.e. by $\widetilde{H}_1^{0,\infty}(W)$ and $\widetilde{H}_1^{0,\infty}(W)$). At the limit when $\epsilon = 0$, the Fatou Lavaurs and Fatou Glutsyuk coordinates coincide.

The only thing we need to take care of is that the darkened regions of Figure 8 lie in different sheets due to the sweep of the cut as ϵ made a full turn. Indeed when we adjust the constant D we compare the domains of $\overline{H}^0$ and $\widetilde{H}^\infty$. $\overline{H}^0$ conjugates $\overline{\Psi}^0\circ T_{\overline{\alpha}^0}$ to a translation. We have $T_{\overline{\alpha}^0}:\overline{S}^-\to\overline{S}^+$, while $\overline{\Psi}^0:\overline{S}^+\to\overline{S}^-$. Hence $\overline{\Psi}^0\circ T_{\overline{\alpha}^0}:\overline{S}^-\to\overline{S}^-$ and $\overline{H}^0$ is defined on $\overline{S}^-$. On the other hand $\widetilde{H}^\infty$ conjugates $T_{\tilde\alpha^0}\circ\widetilde{\Psi}^\infty:\widetilde{S}^+\to\widetilde{S}^+$ to a translation. Hence $\widetilde{H}^\infty$ is defined on $\widetilde{S}^+$. Because of the definition of $\overline{S}^\pm$ and $\widetilde{S}^\pm$ the passage map $\overline{S}^-\to\widetilde{S}^+$ is $T_{2\pi i a}$. □

Lemma 4.5 *We consider the maps $\widetilde{H}^0$, $\widetilde{H}^\infty$, $\overline{H}^0$, $\overline{H}^\infty$ of Theorem 4.1. We let*

$$\begin{cases}\widetilde{\Psi}^0 = \mathrm{id}+\widetilde{\Lambda}^0, & \widetilde{\Psi}^\infty = T_{-2\pi i a}+\widetilde{\Lambda}^\infty,\\ \overline{\Psi}^0 = \mathrm{id}+\overline{\Lambda}^0, & \overline{\Psi}^\infty = T_{-2\pi i a}+\overline{\Lambda}^\infty,\\ \widetilde{H}^{0,\infty} = \mathrm{id}+\widetilde{G}^{0,\infty}, & \overline{H}^{0,\infty} = \mathrm{id}+\overline{G}^{0,\infty}.\end{cases}$$

(i) The functions $\widetilde{G}^{0,\infty}$ are given by the following series which are absolutely convergent for $|\mathrm{Im}W| > Y_0+2\beta$ (see Lemma 3.2) and $\hat\epsilon\in\widetilde{V}$

$$\widetilde{G}^0 = -\sum_{n=1}^{\infty}\widetilde{\Lambda}^0\circ(T_{\tilde\alpha^0}\circ\widetilde{\Psi}^0)^{-n}, \tag{4.18}$$

$$\widetilde{G}^\infty = \sum_{n=0}^{\infty}\widetilde{\Lambda}^\infty\circ(T_{\tilde\alpha^0}\circ\widetilde{\Psi}^\infty)^{n}. \tag{4.19}$$

Similarly the functions $\overline{G}^{0,\infty}$ are given by the following series which are absolutely convergent for $|\mathrm{Im}W| > Y_0+2\beta$ and $\hat\epsilon\in\bar{V}$

$$\overline{G}^0 = \sum_{n=0}^{\infty}\overline{\Lambda}^0\circ T_{\overline{\alpha}^0}\circ(\overline{\Psi}^0\circ T_{\overline{\alpha}^0})^{n}, \tag{4.20}$$

$$\overline{G}^{\infty} = -\sum_{n=1}^{\infty} \overline{\Lambda}^{\infty} \circ T_{\overline{\alpha}^0} \circ (\overline{\Psi}^{\infty} \circ T_{\overline{\alpha}^0})^{-n}. \tag{4.21}$$

For $\hat{\varepsilon} \to 0$ we have the following limits

$$\begin{cases} \lim_{\hat{\varepsilon}\to 0} \widetilde{H}^0_{\hat{\varepsilon}} = \lim_{\hat{\varepsilon}\to 0} \overline{H}^0_{\hat{\varepsilon}} = id, \\ \lim_{\hat{\varepsilon}\to 0} \widetilde{H}^{\infty}_{\hat{\varepsilon}} = T_{2\pi i a} \circ \Psi^{\infty}_0, \\ \lim_{\hat{\varepsilon}\to 0} (\overline{H}^{\infty}_{\hat{\varepsilon}})^{-1} = \Psi^{\infty}_0 \circ T_{2\pi i a}. \end{cases}$$

(ii) For $\hat{\varepsilon} \in \widetilde{V}$ we have

$$\begin{cases} \widetilde{H}^0 = id + O(\overline{C}^0), \\ \widetilde{H}^{\infty} = \widetilde{\Psi}^{\infty} + 2\pi i a + O(\overline{C}^0), \end{cases}$$

while for $\hat{\varepsilon} \in \bar{V}$ we have

$$\begin{cases} \overline{H}^0 = id + O(\overline{C}^0), \\ (\overline{H}^{\infty})^{-1} = \overline{\Psi}^{\infty} \circ T_{2\pi i a} + O(\overline{C}^0), \end{cases}$$

where

$$\overline{C}^0 < \exp\left(-\frac{2\pi(2\pi - \gamma^*)}{\sqrt{\hat{\varepsilon}}}\right),$$

for some $\gamma^* \in (0, \frac{1}{2})$.

PROOF. (i) Let us derive (4.18). The function $\widetilde{G}^0$ satisfies $\widetilde{G}^0 \circ T_{\widetilde{\alpha}^0} \circ \widetilde{\Psi}^0 = \widetilde{G}^0 - \widetilde{\Lambda}^0$, which we rewrite

$$\widetilde{G}^0 = \widetilde{G}^0 \circ (T_{\widetilde{\alpha}^0} \circ \widetilde{\Psi}^0)^{-1} - \widetilde{\Lambda}^0 \circ (T_{\widetilde{\alpha}^0} \circ \widetilde{\Psi}^0)^{-1}. \tag{4.22}$$

We obtain an infinite set of equations by composing (4.22) on the right with $(T_{\widetilde{\alpha}^0} \circ \widetilde{\Psi}^0)^{-n}$. Adding these equations yields a telescopic sum. The formula (4.20) is checked in the same manner. For the formulas (4.19), and (4.21) we also use (4.5). To prove the convergence we use Lemma 3.2. Indeed let $\overline{\Psi}^0 = id + \overline{\Lambda}$. Let us look at (4.20). If $W = X + iY$ and $Y < -Y_0 - 2\beta$, then

$$\left|\overline{\Lambda}^0(W)\right| \le \overline{M}^0 N(\beta) \exp(2\pi(Y_0 + \beta + Y)) = \overline{N}^0 \exp(2\pi Y),$$

where $\overline{N}_0 = \overline{M}^0 N(\beta)$, and $N(\beta)$ is a positive function as in Lemma 3.2. For $\arg(\hat{\varepsilon}) \in (-\delta, \delta)$, then $\mathrm{Im}(\overline{\alpha}^0) = -\frac{2\pi - \gamma(\hat{\varepsilon})}{|\sqrt{\hat{\varepsilon}}|}$ for some $\gamma(\hat{\varepsilon}) \in (0, \frac{1}{2})$. We can show by induction that

$$\left|\mathrm{Im}(\overline{\Psi}^0 \circ T_{\overline{\alpha}^0})^n(W) - \mathrm{Im}(T_{\overline{\alpha}^0})^n(W)\right| < n\overline{N}^0 \exp\left(2\pi\left(Y - \frac{n(2\pi - \gamma(\hat{\varepsilon}))}{|\sqrt{\hat{\varepsilon}}|}\right)\right). \tag{4.23}$$

Hence

$$\mathrm{Im}(\overline{\Psi}^0 \circ T_{\overline{\alpha}^0})^n(W) < \mathrm{Im} W - \frac{n(2\pi - \gamma)}{|\sqrt{\hat{\varepsilon}}|} + n\overline{B}^0$$

for some positive constant $\overline{B}^0$. The convergence of $\overline{G}^0$ follows.

(ii) The fact that $\widetilde{H} = id + O(\overline{C}^0)$ comes from (4.23).

To derive that $(\overline{H}^{\infty})^{-1} = \overline{\Psi}^{\infty} \circ T_{2\pi i a} + O(\overline{C}^{0})$ we calculate $(\overline{H}^{\infty})^{-1}$ directly from

$$\overline{\Psi}^{\infty} \circ T_{\overline{\alpha}^{0}} \circ (\overline{H}^{\infty})^{-1} = (\overline{H}^{\infty})^{-1} \circ T_{\overline{\alpha}^{\infty}}. \tag{4.24}$$

□

END OF PROOF OF COROLLARY 4.4. We now need to prove (4.13) (i.e. (4.17)). This follows from calculation of the constant terms on both sides of (4.16), using the fact that $\overline{H}^{0}$ and $\widetilde{H}^{0}$ are almost the identity. □

Remark 4.6 It is remarkable that, although the functions $\widetilde{H}^{0}_{\hat{\epsilon}}$, $\widetilde{H}^{\infty}_{\hat{\epsilon}}$, $\overline{H}^{0}_{\hat{\epsilon}}$, $\overline{H}^{\infty}_{\hat{\epsilon}}$ have no geometric meaning for $\hat{\epsilon} = 0$, the limits however exist.

Theorem 4.7 *We consider a family $(\Psi^{0}_{\hat{\epsilon}}, \Psi^{\infty}_{\hat{\epsilon}})$ for which the compatibility condition*

$$\widetilde{H}^{\infty} \circ (\widetilde{H}^{0})^{-1} = T_{2\pi i a} \circ \overline{H}^{0} \circ (\overline{H}^{\infty})^{-1} \circ T_{D'}. \tag{4.25}$$

is met for $\hat{\epsilon} \in \overline{V}$ and the corresponding $\hat{\epsilon} e^{2\pi i} \in \widetilde{V}$ and such that

$$D' = -2\pi i a + O(\exp(-2\pi i \overline{\alpha}^{0})). \tag{4.26}$$

Then if we use the notation

$$\begin{cases} \overline{\Psi}^{0,\infty} = \Psi^{0,\infty}_{\hat{\epsilon}}, \\ \widetilde{\Psi}^{0,\infty} = \Psi^{0,\infty}_{\hat{\epsilon} e^{2\pi i}}, \end{cases}$$

we have

$$\overline{\Psi}^{0} - \widetilde{\Psi}^{0} = O(\exp(-2\pi i \overline{\alpha}^{0})) \tag{4.27}$$

and

$$\overline{\Psi}^{\infty} - \widetilde{\Psi}^{\infty} = O(\exp(-2\pi i \overline{\alpha}^{0})). \tag{4.28}$$

PROOF. We have seen in the proof of Lemma 3.2 that

$$\widetilde{H}^{\infty} = \widetilde{\Psi}^{\infty} + 2\pi i a + O(\exp(2\pi i \overline{\alpha}^{\infty}))$$

and

$$\overline{H}^{0} = id + O(\exp(-2\pi i \overline{\alpha}^{0})). \tag{4.29}$$

$(\overline{H}^{\infty})^{-1}$ has been calculated in Lemma 4.5. Since $\widetilde{H}^{0} = id + O(\exp(2\pi i \tilde{\alpha}^{0}))$, we also have

$$(\widetilde{H}^{0})^{-1} = id + O(\exp(2\pi i \tilde{\alpha}^{0})). \tag{4.30}$$

Replacing in (4.12) we show that we get (4.28) and (4.26).

From the expression of $\widetilde{\Psi}^{\infty}$ (resp. $\overline{\Psi}^{\infty}$) in term of $\widetilde{H}^{\infty}$ (resp. $\overline{H}^{\infty}$) it suffices to show that $\left|\widetilde{H}^{\infty}(W) - T_{2\pi i a} \circ (\overline{H}^{\infty})^{-1} \circ T_{-2\pi i a}(W)\right| = O(\exp(-2\pi i \overline{\alpha}^{0}))$ follows from (4.25). Indeed let $\widetilde{W} = (\widetilde{H}^{0})^{-1}(W)$, then

$$\begin{aligned}
&\left|\widetilde{H}^{\infty}(W) - T_{2\pi i a} \circ (\overline{H}^{\infty})^{-1} \circ T_{-2\pi i a}(W)\right| \\
&\qquad \leq \left|\widetilde{H}^{\infty}(W) - \widetilde{H}^{\infty}(\widetilde{W})\right| + \left|\widetilde{H}^{\infty}(\widetilde{W}) - T_{2\pi i a} \circ \overline{H}^{0} \circ (\overline{H}^{\infty})^{-1} \circ T_{D'}(W)\right| \\
&\qquad\quad + \left|T_{2\pi i a} \circ \overline{H}^{0} \circ (\overline{H}^{\infty})^{-1} \circ T_{D'}(W) - T_{2\pi i a} \circ (\overline{H}^{\infty})^{-1} \circ T_{-2\pi i a}(W)\right|.
\end{aligned}$$

The second term vanishes from (4.25), and the first and third terms are small from Lemma 3.2(ii).

We have obtained (4.28) by studying the equations (4.8), (4.9), (4.10) and (4.11), which come from comparing the two presentations on the right of Figure 8 on a region located on top of the fundamental hole. To obtain (4.27) we instead compare on a region located at the bottom and we replace the four equations (4.8), (4.9), (4.10) and (4.11) by the four equations

$$\begin{cases} \widetilde{K}^0 \circ \widetilde{\Psi}^0 \circ T_{\tilde{\alpha}^0} = T_{\tilde{\alpha}^0} \circ \widetilde{K}^0, \\ \widetilde{K}^\infty \circ \widetilde{\Psi}^\infty \circ T_{\tilde{\alpha}^0} = T_{\tilde{\alpha}^\infty} \circ \widetilde{K}^\infty, \end{cases} \qquad \begin{cases} \overline{K}^0 \circ T_{\overline{\alpha}^0} \circ \overline{\Psi}^0 = T_{\overline{\alpha}^0} \circ \overline{K}^0, \\ \overline{K}^\infty \circ T_{\overline{\alpha}^0} \circ \overline{\Psi}^\infty = T_{\overline{\alpha}^\infty} \circ \overline{K}^\infty, \end{cases}$$

which have the solutions

$$\begin{cases} \widetilde{K}^0 = T_{-\tilde{\alpha}^0} \circ \widetilde{H}^0 \circ T_{\tilde{\alpha}^0}, \\ \widetilde{K}^\infty = T_{-\tilde{\alpha}^0} \circ \widetilde{H}^\infty \circ T_{\tilde{\alpha}^0}, \end{cases} \qquad \begin{cases} \overline{K}^0 = T_{\overline{\alpha}^0} \circ \overline{H}^0 \circ T_{-\overline{\alpha}^0}, \\ \overline{K}^\infty = T_{\overline{\alpha}^0} \circ \overline{H}^\infty \circ T_{-\overline{\alpha}^0}. \end{cases} \tag{4.31}$$

We verify that:

$$\begin{cases} (\widetilde{K}^0)^{-1} = \widetilde{\Psi}^0 + O(\exp(2\pi i\tilde{\alpha}^0)), \\ \widetilde{K}^\infty = \mathrm{id} + O(\exp(2\pi i\tilde{\alpha}^0)), \end{cases} \qquad \begin{cases} \overline{K}^0 = \overline{\Psi}^0 + O(\exp(-2\pi i\overline{\alpha}^0)), \\ \overline{K}^\infty = \mathrm{id} + O(\exp(-2\pi i\overline{\alpha}^0)). \end{cases} \tag{4.32}$$

Replacing (4.31) in the compatibility condition (4.25) yields

$$\widetilde{K}^\infty \circ (\widetilde{K}^0)^{-1} = T_{-\tilde{\alpha}^0 - \overline{\alpha}^0 + 2\pi i a} \circ \overline{K}^0 \circ (\overline{K}^\infty)^{-1} \circ T_{\tilde{\alpha}^0 + \overline{\alpha}^0 + D'_\epsilon}.$$

Finally using that $\tilde{\alpha}^0 + \overline{\alpha}^0 = 2\pi i a$ we have

$$\widetilde{K}^\infty \circ (\widetilde{K}^0)^{-1} = \overline{K}^0 \circ (\overline{K}^\infty)^{-1} \circ T_{2\pi i a + D'_\epsilon}.$$

Since $D'_\epsilon + 2\pi i a = O(\exp(-2\pi i\overline{\alpha}^0))$, we get (4.27). □

Let us recall the following theorem which is a well-known generalization of a corollary of the Ramis-Sibuya Theorem [14]. This theorem will be used to show the 1/2-summability of $\Psi^0_{\hat{\epsilon}}$ and $\Psi^\infty_{\hat{\epsilon}}$ in $\hat{\epsilon}$.

Theorem 4.8 *Let $\{S_1, \dots, S_k\}$ be a covering of a punctured disk $\mathbb{D}_\epsilon = \{\epsilon; 0 < |\epsilon| < r\}$ by k sectors arranged so that only consecutive sectors overlap (taking $S_{k+1} = S_1$). Let $\Psi_i(\epsilon,\nu)$ be holomorphic and bounded functions defined on $S_i \times U$, where U is a neighborhood of the origin in ν-space and ν is a multi-parameter. Moreover let the functions Ψ_j satisfy*

$$|\Psi_i(\epsilon,\nu) - \Psi_{i+1}(\epsilon,\nu)| \le a \exp\left(-\frac{b}{|\epsilon|^s}\right)$$

on $(S_i \cap S_{i+1}) \times U$, with a and b positive numbers. Then there exists a power series

$$\hat{\Psi}(\epsilon,\nu) = \sum_{n=0}^{\infty} \beta_n(\nu)\epsilon^n,$$

where the $\beta_n(\nu)$ are analytic on U, and positive numbers A and C such that

1. *for all* $n \geq 0$
$$|\beta_n(\nu)| \leq CA^n\,(n!)^{1/s};$$

2. *for each subsector* S *of* S_j, $j = 1, \dots, k$, *there exist constants* $A_S, C_S > 0$ *such that for all* $\nu \in S$
$$\left|\Psi_j(\epsilon,\nu) - \sum_{m=0}^{N-1}\beta_n(\nu)\epsilon^n\right| < C_S A_S^N |\epsilon|^N\,(N!)^{1/s}.$$

Moreover, if one of the $\Psi_i(\epsilon,\nu)$ *can be extended to a sector* S *of opening greater than* π/s, *then* $\hat{\Psi}$ *is* s*-summable in* ϵ *in the sector* S.

Corollary 4.9 *The components* $\Psi^0_{\hat{\epsilon}}$ *and* $\Psi^\infty_{\hat{\epsilon}}$ *of the modulus of a germ of family of diffeomorphisms normalized so that the compatibility condition is satisfied in the form* (4.25) *are* $1/2$*-summable in* ϵ. *The direction of non-summability is the Glutsyuk direction* $\mathbb{R}^+$.

PROOF. This follows directly from Theorem 4.8 above using the estimates (4.27) and (4.28) of Theorem 4.7. □

We can now refine Theorem 3.1.

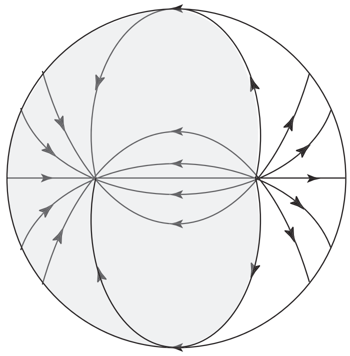
Theorem 4.10 *Under the hypotheses of Theorem* 3.1, *we also have the further conclusion*

- *For* $\epsilon \in V_G$
$$|\overline{f}_{\overline{\epsilon}}(z) - \tilde{f}_{\tilde{\epsilon}}(z)| < B\exp\left(-\frac{A}{|\sqrt{\hat{\epsilon}}|}\right). \tag{4.33}$$

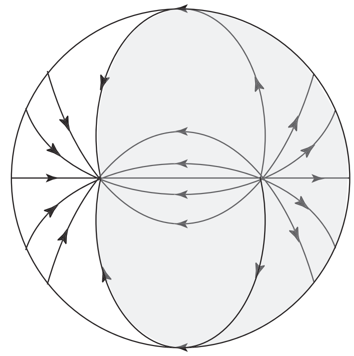
  The estimate is uniform in the ν_i. *Thus* f *is* $\frac{1}{2}$*-summable in* $\hat{\epsilon}$.

PROOF. The only thing to prove is the estimate (4.33). We use the shape of the strips as in Figure 11 so that the functions $\xi^{0,\infty}$ defined in (3.8) satisfy
$$\begin{cases} \left|\overline{\xi}^0 - \tilde{\xi}^0\right| < B_1\exp\left(-\frac{A_1}{|\sqrt{\hat{\epsilon}}|}\right), \\ \left|\overline{\xi}^\infty - \tilde{\xi}^\infty\right| < B_2\exp\left(-\frac{A_2}{|\sqrt{\hat{\epsilon}}|}\right). \end{cases} \tag{4.34}$$

The functions $\overline{\xi}^{0,\infty}$ and $\tilde{\xi}^{0,\infty}$ come from conjugating $\overline{\Psi}^{0,\infty}$ and $\widetilde{\Psi}^{0,\infty}$. The vertical part of the strips are common for $\hat{\epsilon} \in \overline{V}$ and $\hat{\epsilon} \in \widetilde{V}$. Then it is clear that (4.34) follows from the analyticity of $q_{\hat{\epsilon}}$ in the region corresponding to the vertical parts of the strips, so we can use (4.27) and (4.28). The other parts are included in regions corresponding to $|\mathrm{Im} W| > \frac{Y_2}{|\sqrt{\hat{\epsilon}}|}$ for some $Y_2 > 0$ independent of ϵ by Lemma 3.3 where Lemma 3.2 allows to conclude that $\left|\overline{\Psi}^{0,\infty}\right|, \left|\tilde{\Psi}^{0,\infty}\right| < B_2\exp\left(-\frac{A_2}{|\sqrt{\hat{\epsilon}}|}\right)$ from which $\left|\overline{\xi}^{0,\infty}\right|, \left|\tilde{\xi}^{0,\infty}\right| < B_3\exp\left(-\frac{A_3}{|\sqrt{\hat{\epsilon}}|}\right)$ follows for some positive constants A_j, B_j. Indeed, we proved before that the solution of the Beltrami equation depends analytically on $\hat{\epsilon}$. Moroever the solutions of two Beltrami equations where the Beltrami fields satisfy $|\overline{\mu} - \tilde{\mu}| < B_4\exp\left(-\frac{A_4}{|\sqrt{\hat{\epsilon}}|}\right)$ and same values at 3 chosen points also satisfy such type of estimate, from which the result follows. □



(a) Domain of $\overline{\gamma}^0_{\overline{\epsilon}}$ and $\tilde{\gamma}^\infty_{\tilde{\epsilon}}$



(b) Domain of $\overline{\gamma}^\infty_{\overline{\epsilon}}$ and $\tilde{\gamma}^0_{\tilde{\epsilon}}$

Figure 12: The domains of definition of the normalizing maps

Lemma 4.11 *Under the hypotheses of Theorem* 4.10 *and the compatibility condition* (4.12), *there exists a neighborhood* U' *of the origin such that for each* $\hat{\epsilon} \in \overline{V}$ *there exists a conjugacy* $J_{\hat{\epsilon}}$ *between* $\overline{f} = f_{\hat{\epsilon}}$ *and* $\tilde{f} = f_{\hat{\epsilon}e^{2\pi i}}$ *over* U'. *The conjugacy depends analytically on* $\hat{\epsilon}$ *and tends to the identity as* $\hat{\epsilon} \to 0$. *Moreover there exists constants* $A', B' > 0$ *such that* $J_{\hat{\epsilon}}$ *satisfies*

$$|J_{\hat{\epsilon}} - id| < B' \exp\left(-\frac{A'}{|\sqrt{\hat{\epsilon}}|}\right).$$

PROOF. Let us recall that for $\arg\hat{\epsilon} \in (-\delta, \delta)$ we consider $\overline{f}_{\overline{\epsilon}}$. We compare with the point of view for $\arg\hat{\epsilon} \in (2\pi - \delta, 2\pi + \delta)$ in which we consider $\tilde{f}_{\tilde{\epsilon}}$. In both cases the singular point $-\sqrt{\hat{\epsilon}}$ (resp. $\sqrt{\hat{\epsilon}}$) is attached to the upper index 0 (resp. ∞).

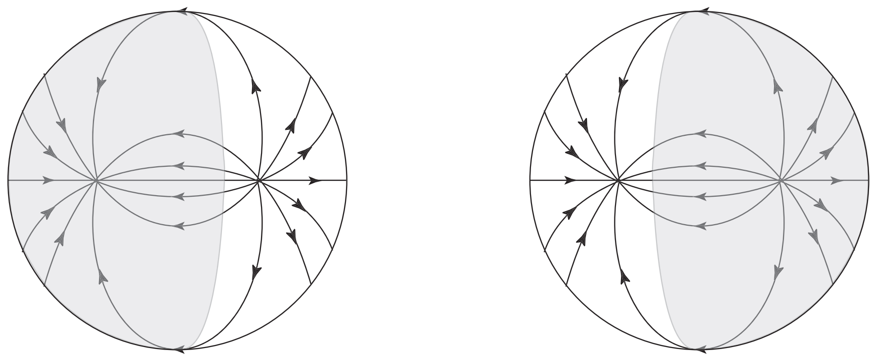
We consider the "normalizing maps" in the neighborhoods of the two singular points given by $\overline{\gamma}^0_{\overline{\epsilon}}$, $\overline{\gamma}^\infty_{\overline{\epsilon}}$, (resp. $\tilde{\gamma}^0_{\tilde{\epsilon}}$, $\tilde{\gamma}^\infty_{\tilde{\epsilon}}$), which are tangent to the identity. These are the maps which transform $\overline{f}$ (resp. $\tilde{f}$) to the model, i.e. the time one map of the vector field (2.7). The advantage of these maps over the linearizing maps is that their limits exist when $\hat{\epsilon} \to 0$ and that they do not explode at the other singular point. It is known [6] that the union of the domains of $\overline{\gamma}^0_{\overline{\epsilon}}$ and $\overline{\gamma}^\infty_{\overline{\epsilon}}$ (resp. $\tilde{\gamma}^0_{\tilde{\epsilon}}$ and $\tilde{\gamma}^\infty_{\tilde{\epsilon}}$) is a whole covering of $U_{\overline{\epsilon}}$ (resp. $U_{\tilde{\epsilon}}$) and that they overlap: indeed the domains have a form as in Figure 12.

We will restrict to smaller domains as in Figure 13 whose union covers U. On these smaller domains we will show that there exist positive constants A_0, B_0 such that

$$\begin{cases} \left|\overline{\gamma}^0_{\overline{\epsilon}}(z) - \tilde{\gamma}^\infty_{\tilde{\epsilon}}(z)\right| < B_0 \exp\left(-\frac{A_0}{|\sqrt{\hat{\epsilon}}|}\right), \\ \left|\overline{\gamma}^\infty_{\overline{\epsilon}}(z) - \tilde{\gamma}^0_{\tilde{\epsilon}}\right| < B_0 \exp\left(-\frac{A_0}{|\sqrt{\hat{\epsilon}}|}\right). \end{cases} \tag{4.35}$$

From the maximum principle it suffices to prove that these estimates hold on an annulus extending to the boundary of these subdomains. To get the result we need to pass to the Fatou Glutsyuk coordinates. Indeed these normalizing maps come from conjugating the Fatou Glustsyuk coordinates with the map $q_{\hat{\epsilon}}^{-1}$. The Fatou Glutsyuk coordinates are constructed as follows. We lift the map $f_{\hat{\epsilon}}$ to

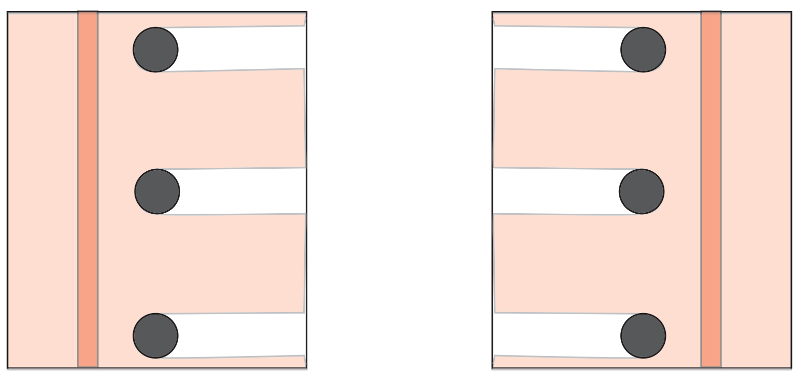
$$F_{\hat{\epsilon}} = q_{\hat{\epsilon}}^{-1} \circ f_{\hat{\epsilon}} \circ q_{\hat{\epsilon}}.$$



(a) Subdomain of $\overline{\gamma}^{0}_{\overline{\epsilon}}$ and $\tilde{\gamma}^{\infty}_{\tilde{\epsilon}}$

(b) Subdomain of $\overline{\gamma}^{\infty}_{\overline{\epsilon}}$ and $\tilde{\gamma}^{0}_{\tilde{\epsilon}}$

Figure 13: Smaller domains of definition of the normalizing maps whose union covers U



(a) Domain of Fatou Glustyuk coordinate on the side of the attracting point

(b) Domain of Fatou Glustyuk coordinate on the side of the repelling point

Figure 14: The domains of definition (translation domains) of the Fatou Glutsyuk coordinates. The darkened strip is where the construction is first performed.

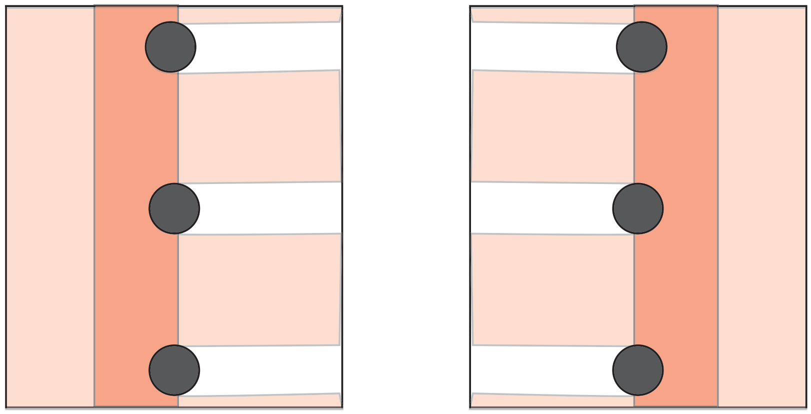
The Fatou Glutsyuk coordinates $\Phi^{0,\infty}_{\hat{\epsilon}}$ satisfy

$$\Phi^{0,\infty}_{\hat{\epsilon}} \circ F_{\hat{\epsilon}} = T_1 \circ \Phi^{0,\infty}_{\hat{\epsilon}}, \tag{4.36}$$

i.e. they conjugate $F_{\hat{\epsilon}}$ with T_1 which is the time-one map of the vector field $\frac{\partial}{\partial W}$. They are first constructed on a strip of horizontal width N and parallel to the line of holes, and then extended to the maximal domain of definition (called translation domain in [11]) by means of (4.36) (see Figure 14). Both $F_{\hat{\epsilon}}$ and $\Phi^{0}_{\hat{\epsilon}}$ (resp. $F_{\hat{\epsilon}}$ and $\Phi^{\infty}_{\hat{\epsilon}}$) commute with T_{α^0} (resp. T_{α^∞}) on the side of $-\sqrt{\hat{\epsilon}}$ (resp. $\sqrt{\hat{\epsilon}}$).

From the relation (4.33) it follows that there holds a similar relation between $\overline{F}_{\overline{\epsilon}} = q_{\overline{\epsilon}}^{-1} \circ \overline{f}_{\overline{\epsilon}} \circ q_{\overline{\epsilon}}$ and $\widetilde{F}_{\tilde{\epsilon}} = q_{\tilde{\epsilon}}^{-1} \circ \tilde{f}_{\tilde{\epsilon}} \circ q_{\tilde{\epsilon}}$:

$$|\overline{F}_{\overline{\epsilon}}(W) - \widetilde{F}_{\tilde{\epsilon}}(W)| < B_1 \exp\left(-\frac{A_1}{|\sqrt{\hat{\epsilon}}|}\right) \tag{4.37}$$



(a) Strip of Fatou Glustyuk coordinate on the side of the attracting point

(b) Strip of Fatou Glustyuk coordinate on the side of the repelling point

Figure 15: Strips whose projections by $q_{\hat{\epsilon}}$ yield annular regions up to the boundary of subdomains as in Figure 13

for some positive constants A_1, B_1. An easy way to check (4.37) is the following: the map $f_{\hat{\epsilon}}$ is the sum of a 1/2-summable series in $\hat{\epsilon}$ with analytic coefficients in z. Hence so is the case of its composition with analytic maps. Moreover it is shown in [11] that $|F_{\hat{\epsilon}}(W) - W - 1| < \frac{1}{4}$ for r, ρ sufficiently small. Hence $|\overline{F}_{\overline{\epsilon}} - \widetilde{F}_{\tilde{\epsilon}}|$ is bounded, from which we can conclude to the existence of a bound independent of W of the special form appearing in (4.37).

The construction of $\Phi^{0,\infty}_{\hat{\epsilon}}$ by means of Alhfors-Bers theorem ([11]) yields to estimates similar to (4.37) for Fatou Glutsyuk coordinates on the strips with positive constants A_2, B_2:

$$\begin{cases} |\overline{\Phi}^0_{\overline{\epsilon}}(W) - \widetilde{\Phi}^\infty_{\tilde{\epsilon}}(W)| < B_2 \exp\left(-\frac{A_2}{|\sqrt{\hat{\epsilon}}|}\right) & \text{on the right strip,} \\ |\overline{\Phi}^\infty_{\overline{\epsilon}}(W) - \widetilde{\Phi}^0_{\tilde{\epsilon}}(W)| < B_2 \exp\left(-\frac{A_2}{|\sqrt{\hat{\epsilon}}|}\right) & \text{on the left strip,} \end{cases} \tag{4.38}$$

as long as we take the same normalization, for instance $\overline{\Phi}^0_{\hat{\epsilon}}(Z_0) = Z_0 = \widetilde{\Phi}^\infty_{\hat{\epsilon}}(Z_0)$ (resp. $\overline{\Phi}^\infty_{\hat{\epsilon}}(Z_1) = Z_1 = \widetilde{\Phi}^0_{\hat{\epsilon}}(Z_1)$) on the right (resp. left) strip. The relation (4.36) implies that for all $n \in \mathbb{Z}$

$$\Phi^{0,\infty}_{\hat{\epsilon}} \circ F^n_{\hat{\epsilon}} = T_n \circ \Phi^{0,\infty}_{\hat{\epsilon}}. \tag{4.39}$$

This in turn ensures that for any N there exist constant a_N, b_N such that estimates of the form (4.38) with A_2 (resp. B_2) replaced by a_N (resp. b_N) are valid in a strip parallel to the holes of horizontal width N. We take N sufficiently large so as to get the estimates on a strip of the form as in Figure 15. Then the projection of these strips by $q_{\hat{\epsilon}}$ yield annular regions up to the boundary of subdomains as in Figure 13. Finally (4.35) follows by conjugating $\Phi^{0,\infty}_{\hat{\epsilon}}$ with $q_{\hat{\epsilon}}$.

There exists a constant $t(\hat{\epsilon})$ such that the map J_ϵ defined by:

$$J_\epsilon = \begin{cases} (\tilde{\gamma}^\infty_{\tilde{\epsilon}})^{-1} \circ \nu_{\hat{\epsilon}}^{t(\hat{\epsilon})} \circ \overline{\gamma}^0_{\overline{\epsilon}} \\ (\tilde{\gamma}^0_{\tilde{\epsilon}})^{-1} \circ \overline{\gamma}^\infty_{\overline{\epsilon}} \end{cases} \tag{4.40}$$

is a conjugacy between $\overline{f}_{\overline{\epsilon}}$ and $\tilde{f}_{\tilde{\epsilon}}$, where $\nu_{\hat{\epsilon}}^{t(\hat{\epsilon})}$ is the flow of the vector field (2.7) for the time $t(\hat{\epsilon})$. The compatibility condition ensures that this map is well defined for an adequate choice of $t(\hat{\epsilon})$.

To determine the constant $t(\hat{\epsilon})$ we take a point $z_0 \in i\mathbb{R}^+$ on the imaginary axis close to the boundary of U. Let us call

$$\begin{cases} J_1 = (\tilde{\gamma}^{\infty}_{\tilde{\epsilon}})^{-1} \circ \overline{\gamma}^{0}_{\overline{\epsilon}} \\ J_2 = (\tilde{\gamma}^{0}_{\tilde{\epsilon}})^{-1} \circ \overline{\gamma}^{\infty}_{\overline{\epsilon}} \\ J_t = (\tilde{\gamma}^{\infty}_{\tilde{\epsilon}})^{-1} \circ \nu_{\hat{\epsilon}}^{t(\hat{\epsilon})} \circ \overline{\gamma}^{0}_{\overline{\epsilon}} \end{cases} \tag{4.41}$$

The constant $t(\hat{\epsilon})$ is uniquely determined by the condition that $J_t(z_0) = J_2(z_0)$. From their boundedness the maps J_1 and J_2 are uniformly continuous and equi-continuous because of the existence of the limit when $\hat{\epsilon} \to 0$. Then (4.35) implies that

$$|J_1(z) - J_2(z)| < B_5 \exp\left(-\frac{A_5}{|\sqrt{\hat{\epsilon}}|}\right) \tag{4.42}$$

in the overlapping region. Moreover there exist positive constants A_6, B_6 such that

$$|t(\hat{\epsilon})| < B_6 \exp\left(-\frac{A_6}{|\sqrt{\hat{\epsilon}}|}\right).$$

The conclusion follows. □

5 The global realization

In Section 3 we have shown how to realize a germ of family $\Psi = (\Psi^0_{\hat{\epsilon}}, \Psi^{\infty}_{\hat{\epsilon}})_{\hat{\epsilon} \in V_{\rho,\delta}}$ as the modulus of a germ of family of diffeomorphisms $f_{\hat{\epsilon}}$ and in Theorem 4.7 of Section 4 we have identified a necessary compatibility condition so that the family Ψ be realizable in a uniform family g_ϵ.

We want to show that this condition is also sufficient. The idea is the same as for the local realization: we realize the family as a 2-dimensional family of diffeomorphisms on an abstract 2-dimensional manifold and we show that this manifold is holomorphically equivalent to a neighborhood of the origin minus $\{\epsilon = 0\}$ via the Newlander-Nirenberg theorem.

When dealing with the global realization we must work with open sets. So we will consider open sectors in $\hat{\epsilon}$-space. We consider the sector $V_{\rho',\delta'}$ constructed in the proof of Theorem 3.1. Let $\delta \in (0, \delta')$ such that the Glutsyuk modulus is defined for $\arg \hat{\epsilon} \in (-\delta, \delta)$ and $\arg \hat{\epsilon} \in (2\pi - \delta, 2\pi + \delta)$. We call

$$V_{\rho'} = \{\hat{\epsilon} \in V_{\rho',\delta'} \setminus \{0\} | \arg \hat{\epsilon} \in (-\delta, 2\pi + \delta)\}.$$

We have the two subsectors $\overline{V}$ and $\widetilde{V}$ defined in (4.2).

Theorem 5.1 *We consider a germ of function $a(\epsilon)$ analytic in ϵ and a germ of family $(\Psi^0_{\hat{\epsilon}}, \Psi^{\infty}_{\hat{\epsilon}})$ for $\hat{\epsilon}$ in some $V_{\rho,\delta}$ satisfying the hypotheses of Theorem 3.1 and the compatibility condition (4.25). We suppose that δ is chosen sufficiently small so that the conclusion of Theorem 4.10 holds. Then there exists a germ of an analytic family of diffeomorphisms*

$$g_\epsilon = z + (z^2 - \epsilon)(1 + O(\epsilon) + O(z)) \tag{5.1}$$

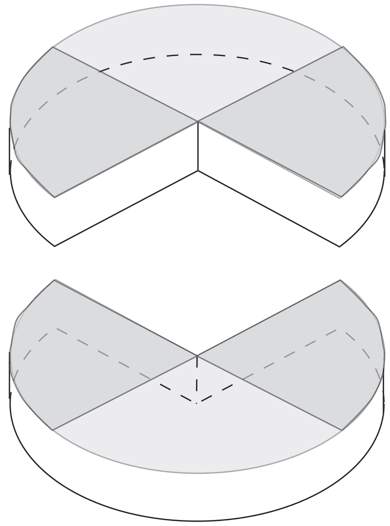


Figure 16: The two sectors $\mathbb{U}_1$ and $\mathbb{U}_2$

whose modulus is given by $(a(\epsilon),[\Psi^0_{\hat\epsilon},\Psi^\infty_{\hat\epsilon}])$ *in some* $V_{\rho',\delta'}$. *Moreover, if the functions* $a(\epsilon,\nu)$ *and* $\Psi^{0,\infty}_{\hat\epsilon,\nu}$ *depend analytically on* $(k-1)$*-parameters* ν*, then the function* $g_{\epsilon,\nu}$ *depends analytically on* ν.

PROOF. We consider the sector $V_{\rho',\delta'}$ (with $\delta'=\delta$) constructed in the proof of Theorem 3.1. We can of course suppose that $\delta\in(0,\frac{\pi}{4})$ and that δ is sufficiently small so that the Glutsyuk modulus is defined for $\arg\hat\epsilon\in(-2\delta,2\delta)$ and $\arg\hat\epsilon\in(2\pi-2\delta,2\pi+2\delta)$. (To realize this requirement it suffices to take $\delta=\frac{\delta'}{2}$ where δ' is constructed in Theorem 3.1.)

For each $\hat\epsilon\in V_{\rho'}$ we have realized the modulus over an open set $U_{\hat\epsilon}$ of $\mathbb{C}$ constructed as in the proofs of Theorem 3.1 and Theorem 4.10. For all $\hat\epsilon\in V_{\rho'}$, $U_{\hat\epsilon}$ contains a fixed disk $B(0,r)$ and the two fixed points lie inside $B(0,r)$. We can suppose r sufficiently small so that $B(0,r)\subset U'$ where U' is the open neighborhood of $\pm\sqrt{\hat\epsilon}$ in Lemma 4.11. So for the rest of the proof we will suppose $U_{\hat\epsilon}=B(0,r)$.

We consider the open set of $\mathbb{C}\times\hat{\mathbb{C}}$ defined by

$$\mathbb{U}=\cup_{\hat\epsilon\in V_{\rho'}}(U_{\hat\epsilon}\setminus\{\pm\sqrt{\hat\epsilon}\},\hat\epsilon).$$

This space is endowed with a projection $\Pi:\mathbb{U}\to V_\rho$.

We cover $V_{\rho'}$ with the two sectors $V^1_{\rho'}$ and $V^2_{\rho'}$ defined by

$$\begin{cases}V^1_{\rho'}=\{\hat\epsilon\in V_{\rho'}|\arg\hat\epsilon\in(-\delta,\pi+\delta)\}\\ V^2_{\rho'}=\{\hat\epsilon\in V_{\rho'}|\arg\hat\epsilon\in(\pi-\delta,2\pi+\delta)\}\end{cases}$$

Their inverse images in $\mathbb{U}$ are called $\mathbb{U}_1=\Pi^{-1}(V^1_{\rho'})$ and $\mathbb{U}_2=\Pi^{-1}(V^2_{\rho'})$ (Figure 16).

We construct a complex manifold $\mathscr{M}$ with atlas given by $\{\mathbb{U}_1,\mathbb{U}_2\}$. The transition function on $\mathbb{U}_1\cap\mathbb{U}_2$ (i.e. when $\arg\hat\epsilon\in(\pi-\delta,\pi+\delta)$) is the identity. The other transition function is obtained as follows: we make the gluing of $\Pi^{-1}(\overline{V})$ with $\Pi^{-1}(\widetilde{V})$ in the following way: we identify $(z,\overline{\epsilon})\in\Pi^{-1}(\overline{\epsilon})$ with $(J_\epsilon(z),\tilde\epsilon)\in\Pi^{-1}(\tilde\epsilon)$ defined in (4.40). With this gluing $\overline{\epsilon}$ and $\tilde\epsilon$ simply become ϵ. On $\mathscr{M}$ a global function f_ϵ is defined. It is given by $f_{\hat\epsilon}$ on each $\mathbb{U}_j$ and the definitions match because J_ϵ conjugates $f_{\overline{\epsilon}}$ and $f_{\tilde\epsilon}$.

On each of $\mathbb{U}_1$ and $\mathbb{U}_2$ we have respective coordinates (z_1, ϵ) and (z_2, ϵ). We want to show that the complex manifold $\mathscr{M}$ is holomorphically equivalent to a neighborhood of the origin in $\mathbb{C}^2$ minus $\{\epsilon = 0\}$.

Let (Θ_1, Θ_2) be a partition of unity associated to the covering $\{\mathbb{U}_1, \mathbb{U}_2\}$. As in Theorem 3.1, we can suppose that the derivatives of Θ_j grow no faster than a negative power of the variables. We can also suppose that the Θ_j depend on ϵ alone. Let us first construct a C^∞-diffeomorphism

$$\Omega : \mathscr{M} \to (\mathbb{C}^2, 0) \setminus \{\epsilon = 0\}$$

defined by

$$\Omega = \Theta_1 \cdot (z_1, \epsilon) + \Theta_2 \cdot (z_2, \epsilon) = (\Theta_1 z_1 + \Theta_2 z_2, \epsilon).$$

This map is C^∞. We will extend it by the identity on $\epsilon = 0$. To show that the extension is C^∞ we use the fact that the map $(z, \hat{\epsilon}) \mapsto J_{\hat{\epsilon}}(z)$ has $J_{\hat{\epsilon}} - \mathrm{id}$ exponentially small in $\sqrt{\hat{\epsilon}}$ near $\hat{\epsilon} = 0$ (see Lemma 4.11). This endows $\Omega(\mathscr{M})$ of two complex coordinates (Z, ϵ) where

$$Z = \Theta_1 z_1 + \Theta_2 z_2. \tag{5.2}$$

We now show that Ω induces an integrable almost complex structure on $\Omega(\mathscr{M})$. Let us recall that an almost complex structure is given by two forms ω, ξ which are $\mathbb{C}$-linear in the sense of this structure.

The almost complex structure is integrable when there exist coordinates (w_1, w_2) such that

$$\langle dw_1, dw_2 \rangle_{\mathbb{C}} = \langle \omega, \xi \rangle_{\mathbb{C}}.$$

In that case there exists a 2×2 invertible matrix A whose entries are C^∞ functions in (Z, ϵ) such that

$$\begin{pmatrix} \omega \\ \xi \end{pmatrix} = A \begin{pmatrix} dw_1 \\ dw_2 \end{pmatrix} = A\, dw.$$

In particular, $d \begin{pmatrix} \omega \\ \xi \end{pmatrix} = dA \wedge dw$ contains no $(0,2)$ component. The Newlander-Nirenberg Theorem asserts that this necessary condition is also sufficient for integrability.

For the second form of the complex structure we take $\xi = d\epsilon$. The other form ω should play the role of dZ. It will be given by

$$\omega = (\Omega^{-1})^*(\widetilde{\omega}) \tag{5.3}$$

for some form $\widetilde{\omega}$ defined on $\mathscr{M}$. The form $\widetilde{\omega}$ is given by $\widetilde{\omega}_j$ on the chart $\mathbb{U}_j$. On $\mathbb{U}_2$ we take $\widetilde{\omega}_2 = dz_2$. On $\mathbb{U}_1 \cap \mathbb{U}_2$ we have $dz_1 = dz_2$. So we want $\widetilde{\omega}_1 = dz_1$ on $\mathbb{U}_1 \cap \mathbb{U}_2$. On the region of the gluing we have

$$\begin{aligned} dz_2 &= \tfrac{\partial J}{\partial \epsilon} d\epsilon + \tfrac{\partial J}{\partial z_1} dz_1 \\ &= \tau_{\epsilon,1} d\epsilon + (1 + \tau_{\epsilon,2}) dz_1, \end{aligned}$$

where the two functions $\tau_{\epsilon,j}$ are exponentially flat in $|\sqrt{\epsilon}|^{-1}$ near $\epsilon = 0$. The gluing is done in the following way: δ has been chosen sufficiently small so that J_ϵ, and then $\tau_{\epsilon,j}$ exist for $\arg(\epsilon) \in (-2\delta, 2\delta)$. We take an increasing C^∞ function $\varphi : \mathbb{R} \to [0,1]$ such that

$$\varphi(x) \equiv \begin{cases} 0 & x < -2\delta \\ 1 & x > -\delta. \end{cases}$$

Then

$$\widetilde{\omega}_1 = dz_1 + \varphi(\arg\epsilon)(\tau_{\epsilon,1}d\epsilon + \tau_{\epsilon,2}dz_1).$$

From its construction the form $\widetilde{\omega} = \widetilde{\omega}_j$ on $\mathbb{U}_j$ is well defined on $\mathscr{M}$, C^∞ and of type $(1,0)$.

Let us now remark that the difference $\omega - dZ$ decreases exponentially fast as $\epsilon \to 0$. This comes from the fact that $\tau_{\epsilon,j}$, $j = 1,2$, are exponentially flat in $|\sqrt{\epsilon}|^{-1}$ near $\epsilon = 0$.

This allows to extend the almost complex structure $\{d\epsilon, \omega\}$ to $\epsilon = 0$, by taking the two forms $d\epsilon$ and dz. The resulting almost complex structure is C^∞ in a neighborhood of the origin in $\mathbb{C}^2$.

To show that this complex structure satisfies the necessary condition for integrability we need to show that $\{d(d\epsilon), d\omega\}$ contains no terms of type $(0,2)$. Obviously $d(d\epsilon) = 0$, so we only need to study $d\omega$. From its construction $d\widetilde{\omega}$ has no terms of type $(0,2)$. But ω is obtained from the pull-back of $\widetilde{\omega}$. Note that no terms containing $d\overline{\epsilon}$ may exist outside the region $\arg\epsilon \in (-2\delta, \delta)$, since φ is constant there and either the $\Theta_j \equiv 1$ or $z_1 = z_2$. In the region $\arg\epsilon \in (-2\delta, \delta)$ the maps $\tau_{\epsilon,j}$ are holomorphic in Z and the maps Θ_j depend on ϵ alone so there is no possibility of a term in $d\overline{Z}$.

Since the almost complex structure satisfies the necessary condition for integrability, we can apply the Newlander-Nirenberg Theorem [13] to the manifold $\overline{\Omega(\mathscr{M})}$, where $\overline{\Omega(\mathscr{M})}$ is the closure of $\Omega(\mathscr{M})$ obtained by adding $\epsilon = 0$, $z \in U_0$. Indeed the complex structure is integrable on $\Omega(\mathscr{M})$ and hence on $\overline{\Omega(\mathscr{M})}$ by continuity. Then the local charts which are holomorphic in the sense of this complex structure are C^∞. Hence there exists a diffeomorphism $\Gamma : \overline{\Omega(\mathscr{M})} \cap \mathscr{U} \to \mathbb{C}^2$, where $\mathscr{U}$ is a neighborhood of the origin in $\mathbb{C}^2$, which is holomorphic with respect to this structure and whose image is a neighborhood of the origin in $\mathbb{C}^2$. From the form of the complex structure it is clear that ϵ can be taken as one of the complex coordinates. So we can suppose that Γ preserves ϵ. The composition $\Gamma \circ \Omega$ is an analytic diffeomorphism of an open set of $\mathscr{M}$ with a neighborhood of the origin in $\mathbb{C}^2$. The map Γ is not unique. We can always choose it in such a way that it sends the curve $z^2 - \epsilon = 0$ to the same curve.

We now conjugate the map (f_ϵ, ϵ) with $\Gamma \circ \Omega$ yielding

$$(g_\epsilon, \epsilon) = (\Gamma \circ \Omega) \circ (f_\epsilon, \epsilon) \circ (\Gamma \circ \Omega)^{-1}.$$

Since g_ϵ is bounded in the neighborhood of $\epsilon = 0$, it is possible to extend it to $\epsilon = 0$ in an analytic way. For each fixed ϵ the map g_ϵ is conjugated to f_ϵ defined on the slice $\mathscr{M}_\epsilon$. By continuity it is clear that g_0 is conjugated to $f_0 = \lim_{\hat{\epsilon}\to 0} f_{\hat{\epsilon}}$ where $f_{\hat{\epsilon}}$ was the family of Theorem 3.1. □

6 Examples

In this section we consider the realization problem for a family $(\Psi^0_{\hat{\epsilon}}, \Psi^\infty_{\hat{\epsilon}})$ which is conjugate under the map $w = E(W) = \exp(-2\pi i W)$ to a family of functions

$$\begin{cases} \psi^0_{\hat{\epsilon}}(w) = m_{A(\hat{\epsilon}),n}(w) = \frac{w}{(1+A(\hat{\epsilon})w^n)^{1/n}} \\ \psi^\infty_{\hat{\epsilon}}(w) = L_{\exp(-4\pi^2 a(\epsilon))} \circ T_{B(\hat{\epsilon}),n'}(w) = \exp(-4\pi^2 a(\epsilon)) \left(w^{n'} + B(\hat{\epsilon})\right)^{1/n'}. \end{cases} \tag{6.1}$$

When $n=1$, we drop the index n. For $n=n'=1$, such a modulus is obtained for instance in the modulus of the holonomy of an unfolding of a Riccati equation with a saddle-node ([18] or [8]), so we will call it the "Riccati case".

6.1 The general case

Let

$$\beta=\exp(-4\pi^2 a(\epsilon)), \tag{6.2}$$

and

$$\begin{cases}\overline{C}=\exp(-2\pi i\overline{\alpha}^0),\\ \widetilde{C}=\exp(-2\pi i\tilde{\alpha}^0).\end{cases} \tag{6.3}$$

Then we have

$$\overline{C}\beta=(\widetilde{C})^{-1}. \tag{6.4}$$

As before, we compare the modulus at values $\overline{\epsilon}=\hat{\epsilon}$ and $\tilde{\epsilon}=\hat{\epsilon}e^{2\pi i}$, which we denote by

$$\begin{cases}\overline{\psi}^0=m_{\overline{A},n},\\ \overline{\psi}^\infty=L_\beta\circ T_{\overline{B},n'},\end{cases}\qquad \begin{cases}\widetilde{\psi}^0=m_{\widetilde{A},n},\\ \widetilde{\psi}^\infty=L_\beta\circ T_{\widetilde{B},n'}.\end{cases}$$

Let

$$\begin{cases}\tilde{h}^0=E\circ\widetilde{H}^0\circ E^{-1},\\ \tilde{h}^\infty=E\circ\widetilde{H}^\infty\circ E^{-1},\end{cases}\qquad \begin{cases}\overline{h}^0=E\circ\overline{H}^0\circ E^{-1},\\ \overline{h}^\infty=E\circ\overline{H}^\infty\circ E^{-1}.\end{cases}$$

They satisfy respectively

$$\begin{cases}\tilde{h}^0\circ L_{\widetilde{C}}\circ\widetilde{\psi}^0=L_{\widetilde{C}}\circ\tilde{h}^0,\\ \tilde{h}^\infty\circ L_{\widetilde{C}}\circ\widetilde{\psi}^\infty=L_{\widetilde{C}\beta}\circ\tilde{h}^\infty,\end{cases}\qquad \begin{cases}\overline{h}^0\circ\overline{\psi}^0\circ L_{\overline{C}}=L_{\overline{C}}\circ\overline{h}^0,\\ \overline{h}^\infty\circ\overline{\psi}^\infty\circ L_{\overline{C}}=L_{\overline{C}\beta}\circ\overline{h}^\infty.\end{cases} \tag{6.5}$$

To calculate $\tilde{h}^0$, $\tilde{h}^\infty$, $\overline{h}^0$, and $\overline{h}^\infty$ we use the following proposition

Proposition 6.1 *The functions $m_{A,n}$ and $T_{B,n}$ satisfy:*

(i) $m_{A,n}\circ L_C=L_C\circ m_{AC^n,n}$;

(ii) $T_{B,n}\circ L_C=L_C\circ T_{B/C^n,n}$;

(iii) $m_{A,n}\circ m_{A',n}=m_{A+A',n}$;

(iv) $T_{B,n}\circ T_{B',n}=T_{B+B',n}$.

Theorem 6.2 *(i) The maps $\tilde{h}^0$, $\tilde{h}^\infty$, $\overline{h}^0$, $\overline{h}^\infty$ are given by*

$$\begin{cases}\tilde{h}^0=m_{\tilde{d},n}, & \textit{with}\quad \tilde{d}=\frac{\widetilde{A}}{1-\widetilde{C}^n},\\ \tilde{h}^\infty=T_{\tilde{e},n'}, & \textit{with}\quad \tilde{e}=\frac{(\widetilde{C}\beta)^{n'}\widetilde{B}}{(\widetilde{C}\beta)^{n'}-1}=\frac{\widetilde{B}}{1-\overline{C}^{n'}},\\ \overline{h}^0=m_{\overline{d},n}, & \textit{with}\quad \overline{d}=\frac{\overline{A}\,\overline{C}^n}{1-\overline{C}^n},\\ \overline{h}^\infty=T_{\overline{e},n'}, & \textit{with}\quad \overline{e}=\frac{\beta^{n'}\overline{B}}{(\overline{C}\beta)^{n'}-1}=\frac{(\widetilde{C}\beta)^{n'}\overline{B}}{1-\widetilde{C}^{n'}}.\end{cases}$$

(ii) The compatibility condition can only be satisfied when either A *or* B *vanish or we have* $n = n'$. *The compatibility condition in the latter case is given by the condition* $\widetilde{A}\widetilde{B} = \overline{A}\,\overline{B}$, *so that the analytic invariant* AB *depends analytically on* ϵ. *The linear changes of Glutsyuk coordinates* L_F *and* L_G *allowing to realize the compatibility condition*

$$\tilde{h}^\infty \circ (\tilde{h}^0)^{-1} = L_F \circ \overline{h}^0 \circ (\overline{h}^\infty)^{-1} \circ L_G \tag{6.6}$$

are given by

$$\begin{cases} F^n = -\frac{\widetilde{B}}{\overline{B}}\, \frac{(1-\overline{C}^n)((\beta\overline{C})^n-1)-AB(\overline{C}\beta)^n}{\beta^n(1-\overline{C}^n)^2}, \\ G^n = -\frac{\widetilde{A}}{\overline{A}}\, \frac{\beta^n\left[(1-\overline{C}^n)((\beta\overline{C})^n-1)-AB(\overline{C}\beta)^n\right]}{((\beta\overline{C})^n-1)^2}. \end{cases}$$

Then

$$F^nG^n = 1 + 2AB\widetilde{C}^{-n}(1+O(\overline{C}^n)) = 1 + 2AB\widetilde{C}^{-n} + o(\widetilde{C}^{-n}).$$

This yields a geometric interpretation of the analytic invariant AB *as a shift between the two constants* F *and* G.

(iii) If $A(\hat{\epsilon}) \equiv 0$ *(resp.* $B(\hat{\epsilon}) \equiv 0$*), then the compatibility condition is given by* $\overline{B}/\widetilde{B}$ *(resp.* $\overline{A}/\widetilde{A}$*) bounded and bounded away from 0. In particular* $\overline{B}$ *and* $\widetilde{B}$ *(resp.* $\overline{A}$ *and* $\widetilde{A}$*) vanish at the same values of* ϵ*, with same multiplicity. In that case* $FG = 1$.

PROOF.

(i) The result follows by applying Proposition 6.1 in (6.5) and using (6.4).

(ii) The compatibility condition is that there exist nonzero constants F and G such that $\tilde{h}^\infty \circ (\tilde{h}^0)^{-1} = L_F \circ \overline{h}^0 \circ (\overline{h}^\infty)^{-1} \circ L_G$, i.e

$$T_{\tilde{e},n'} \circ m_{-\tilde{d},n} = L_{FG} \circ m_{G^n\overline{d},n} \circ T_{-\overline{e}/G^n,n'}.$$

Such an equation can obviously only be satisfied for $n = n'$, unless $A \equiv 0$ or $B \equiv 0$.

Let us calculate both sides when $n = n'$:

$$T_{\tilde{e},n} \circ m_{-\tilde{d},n}(w) = \left(\frac{w^n(1-\tilde{d}\tilde{e})+\tilde{e}}{1-\tilde{d}w^n}\right)^{1/n}$$

and

$$L_{FG} \circ m_{G^n\overline{d},n} \circ T_{-\overline{e}/G^n,n} = \left(\frac{\frac{F^nG^n}{1-\overline{d}\overline{e}}w^n - \frac{F^n\overline{e}}{1-\overline{d}\overline{e}}}{1+\frac{G^n\overline{d}}{1-\overline{d}\overline{e}}w^n}\right)^{1/n}.$$

Then the compatibility conditions become

$$\begin{cases} 1-\tilde{d}\tilde{e} = \frac{F^nG^n}{1-\overline{d}\overline{e}}, \\ \tilde{e} = -\frac{F^n\overline{e}}{1-\overline{d}\overline{e}}, \\ \tilde{d} = -\frac{G^n\overline{d}}{1-\overline{d}\overline{e}}. \end{cases} \tag{6.7}$$

From this we get

$$\begin{cases} F^n = -\frac{\widetilde{B}}{\overline{B}} \frac{\overline{C}^n[(1-\overline{C}^n)(1-\widetilde{C}^n)-\overline{A}\,\overline{B}]}{(1-\overline{C}^n)^2}, \\ G^n = -\frac{\widetilde{A}}{\overline{A}} \frac{(1-\overline{C}^n)(1-\widetilde{C}^n)-\overline{A}\,\overline{B}}{\overline{C}^n(1-\widetilde{C}^n)^2}, \end{cases}$$

and the compatibility condition linking $\overline{A}\,\overline{B}$ and $\widetilde{A}\widetilde{B}$ becomes $\tilde{d}\tilde{e} = \overline{d}\overline{e}$ which is equivalent to

$$\overline{A}\,\overline{B} = \widetilde{A}\widetilde{B}.$$

Since this product is an invariant, we can simply note it by AB. Note that

$$F^nG^n = (1-\tilde{d}\tilde{e})^2 = 1 + 2AB\widetilde{C}^{-n}(1 + O(\overline{C}^n)) = 1 + 2AB\widetilde{C}^{-n} + o(\widetilde{C}^{-n}).$$

In the particular case $F = 1/\beta$, i.e. the modulus family has been normalized so as to satisfy (4.25), then we get that $G = \beta + O(\overline{C})$, which ensures $\tilde{A} - \overline{A} = O(\overline{C})$ and similarly $\tilde{B} - \overline{B} = O(\overline{C})$ as proved in Theorem 4.7.

(iii) If $A \equiv 0$, then $\overline{d} = \tilde{d} = 0$ in (6.7), from which the conclusion follows. □

Corollary 6.3 *No family* $\left(a(\epsilon), [m_{A(\hat{\epsilon}),n}, L_\beta \circ T_{B(\hat{\epsilon}),n'}]\right)\big|_{\hat{\epsilon}\in V_{\rho,\delta}}$ *is realizable as the modulus of a prepared family unfolding a diffeomorphism with a parabolic fixed point when* $n \neq n'$ *and neither* $A(\epsilon)$ *or* $B(\epsilon)$ *are identically zero.*

Remark 6.4 The Corollary 6.3 shows the strength of the compatibility condition. Indeed, while $(a(0), [m_{A(0),n}, L_\beta \circ T_{B(0),n'}])$ is realizable as the modulus of a single diffeomorphism, its unfolding can never keep this simple form.

Theorem 6.5 *We consider a realizable family of triples* $\left(a(\epsilon), [m_{A(\hat{\epsilon}),n}, L_\beta \circ T_{B(\hat{\epsilon}),n}]\right)\big|_{\hat{\epsilon}\in V_{\rho,\delta}}$. *It is possible to choose analytic representatives of the modulus. The different equivalence classes have a unique representative composed of a triple of germs of analytic functions* $(a(\epsilon), A(\epsilon), B(\epsilon))$, *with* $a(\epsilon)$ *arbitrary and* $A(\epsilon)$, $B(\epsilon)$ *of one of the following type for some choice of* N_A, $N_B \in \mathbb{N} = \{0, 1, \ldots\}$.

(i) $A(\epsilon) = \epsilon^{N_A}$, $B(\epsilon) = \epsilon^{N_B} B_1(\epsilon)$, *with* B_1 *analytic satisfying* $B_1(0) \neq 0$;

(ii $A(\epsilon) \equiv 0$, $B(\epsilon) = \epsilon^{N_B}$;

(iii) $A(\epsilon) = \epsilon^{N_A}$, $B(\epsilon) \equiv 0$;

(iv) $A(\epsilon) = B(\epsilon) \equiv 0$.

PROOF. The compatibility condition shows that AB is analytic in ϵ. Moreover we have shown in Theorem 4.10 that $A(\epsilon)$ and $B(\epsilon)$ can be chosen to have 1/2-summable power series in ϵ. These power series have sums that are analytic in the sector $V_{\rho,\delta}$ with continuous limit at $\epsilon = 0$. When they are not identically zero, they have the form $\epsilon^N c(\epsilon)$ with $c(\epsilon)$ nonzero, analytic in the sector with continuous nonzero limit at $\epsilon = 0$. Dividing A by such a function (and multiplying B by the same amount) is allowed in the equivalence class for the modulus. Thus, in the case when $A \neq 0$ we can take a scaling so that $A \equiv \epsilon^{N_A}$, for some $N_A \in \mathbb{N}$. This gives cases (i) or (iii). In the case where $A \equiv 0$ we can perform a similar division on B to give (ii) or (iv). It is clear that no more scalings are allowed within the equivalence classes, and so the representations (i) to (iv) are unique. □

6.2 The Riccati case

Here we use the following notation

$$\begin{cases} m_A = m_{A,1}, \\ T_B = T_{B,1}. \end{cases}$$

Theorem 6.6 *For any germs of analytic functions $a(\epsilon), A(\epsilon), B(\epsilon)$, the modulus $(a(\epsilon), [m_{A(\epsilon)}, L_\beta \circ T_{B(\epsilon)}])$ can be realized as the modulus of the unfolding of the holonomy of the strong separatrix of a Riccati equation*

$$\begin{aligned} \dot{x} &= x^2 - \epsilon, \\ \dot{y} &= f_{0,\epsilon}(x) + y f_{1,\epsilon}(x) + y^2 f_{2,\epsilon}(x), \end{aligned} \tag{6.8}$$

with $f_{j,\epsilon}$ a germ of analytic family of functions in x.

PROOF. It is proved in [18] that the modulus of the unfolding of the holonomy of such a family is formed by Möbius functions, hence by analytic functions $m_{A(\epsilon)}$, $T_{B(\epsilon)}$ as in Theorem 6.5. It is also shown there that the spherical coordinates (called w) on the fundamental domains of Figure 2 can be obtained by first integrals of the saddle-node model family

$$\begin{aligned} \dot{x} &= x^2 - \epsilon \\ \dot{y} &= y(1 + ax), \end{aligned} \tag{6.9}$$

which is the point of view in [7] and [19]. For this reason, we will be brief with the details. We intend to treat in full detail the general case of a saddle-node in a forthcoming paper.

We first discuss the local realization of a family with modulus $(a(\epsilon), [m_{A(\epsilon)}, L_\beta \circ T_{B(\epsilon)}])$, i.e. of a ramified (in $\hat{\epsilon}$) family realizing this modulus. For the local construction (local in $\hat{\epsilon}$), we consider the two same sectors $U_{\hat{\epsilon}}^{\pm}$ of Figure 9 and their intersection which is formed of the three (resp. two) sectors $U_{\hat{\epsilon}}^{0,\infty,C}$ (resp. $U_0^{0,\infty}$) for $\hat{\epsilon} \neq 0$ (resp. $\epsilon = 0$). Note that r can be chosen arbitrarily large since $\psi_{\hat{\epsilon}}^{0,\infty}$ are global diffeomorphisms. Let

$$\mathscr{U}_{\hat{\epsilon}}^{\#} = U_{\hat{\epsilon}}^{\#} \times \mathbb{CP}^1$$

for $\# \in \{+, -, 0, \infty, C\}$. On each $\mathscr{U}_{\hat{\epsilon}}^{\pm}$ we take the model family (6.9) in coordinates $(x, y^{\pm})$. We glue together the two models over $\mathscr{U}_{\hat{\epsilon}}^{\#}$, $\# \in \{0, \infty, C\}$. Over $\mathscr{U}_{\hat{\epsilon}}^{\pm}$ we have first integrals $H_{\hat{\epsilon}}^{\pm}(x, y^{\pm}) = y^{\pm} g_{\hat{\epsilon}}(x)$ with $g_{\hat{\epsilon}}$ given in (3.17). We need to write the change of coordinates over $\mathscr{U}_{\hat{\epsilon}}^{0,\infty,C}$. It comes from the change in first integral

$$H_{\hat{\epsilon}}^{-} = \begin{cases} \psi_{\hat{\epsilon}}^{0}(H_{\hat{\epsilon}}^{+}) = m_{A(\epsilon)}(H_{\hat{\epsilon}}^{+}), & \text{on} \quad \mathscr{U}_{\hat{\epsilon}}^{0}, \\ \psi_{\hat{\epsilon}}^{\infty}(H_{\hat{\epsilon}}^{+}) = L_{\beta(\epsilon)} T_{B(\epsilon)}(H_{\hat{\epsilon}}^{+}), & \text{on} \quad \mathscr{U}_{\hat{\epsilon}}^{\infty}, \\ L_{\overline{C}(\hat{\epsilon})}(H_{\hat{\epsilon}}^{+}), & \text{on} \quad \mathscr{U}_{\hat{\epsilon}}^{C}, \end{cases} \tag{6.10}$$

and yields

$$(x, y^{-}) = \begin{cases} \left(x, \frac{y^{+}}{1 + \frac{A(\epsilon)}{g_{\hat{\epsilon}}(x)} y^{+}}\right), & \text{on} \quad \mathscr{U}_{\hat{\epsilon}}^{0}, \\ (x, y^{+} + B(\epsilon) g_{\hat{\epsilon}}(x)), & \text{on} \quad \mathscr{U}_{\hat{\epsilon}}^{\infty}, \\ (x, y^{+}), & \text{on} \quad \mathscr{U}_{\hat{\epsilon}}^{C}. \end{cases} \tag{6.11}$$

Note that $g_{\hat{\epsilon}}(\sqrt{\hat{\epsilon}}) = 0$ and $1/g_{\hat{\epsilon}}(-\sqrt{\hat{\epsilon}}) = 0$, so we can glue in the two lines $\{\pm\sqrt{\hat{\epsilon}}\} \times \mathbb{CP}^1$ to obtain a C^∞ manifold. We show that this manifold is analytic. For this it suffices to see that a cylindrical neighborhood of each line $\{\pm\sqrt{\hat{\epsilon}}\} \times \mathbb{CP}^1$ minus the corresponding line is analytically isomorphic to the product of a pointed disk with $\mathbb{CP}^1$. Let us now write the details for a neighborhood of the line $\{\sqrt{\hat{\epsilon}}\} \times \mathbb{CP}^1$. We consider $\check{U}$ a small disk centered at $\sqrt{\hat{\epsilon}}$ that does not contain $-\sqrt{\hat{\epsilon}}$ and $\check{U}^*$ the pointed disk. We look for global coordinates (x, Y) on $\check{U}^* \times \mathbb{CP}^1$. For this, we look for functions $k^\pm(x)$ such that

$$Y^\pm = y^\pm + k^\pm_{\hat{\epsilon}}(x) \tag{6.12}$$

and $Y^+ \equiv Y^-$ over $\mathscr{U}^+_{\hat{\epsilon}} \cap \mathscr{U}^-_{\hat{\epsilon}}$. Then $k^\pm_{\hat{\epsilon}}$ must satisfy

$$k^+_{\hat{\epsilon}}(x) - k^-_{\hat{\epsilon}}(x) = \begin{cases} 0, & x \in U^0_{\hat{\epsilon}} \cup U^C_{\hat{\epsilon}}, \\ B(\epsilon) g_{\hat{\epsilon}}(x), & x \in U^\infty_{\hat{\epsilon}}. \end{cases} \tag{6.13}$$

There are just found as solutions of the Cousin problem. The explicit formula for the solution allows to show that they have a limit at $\sqrt{\hat{\epsilon}}$. Since $g_{\hat{\epsilon}}(\sqrt{\hat{\epsilon}}) = 0$, they can be taken such that $k^\pm(\sqrt{\hat{\epsilon}}) = 0$. The global coordinate we are looking for is given by $Y = Y^\pm(x, y)$ on $\mathscr{U}^\pm_{\hat{\epsilon}} \cap (\check{U}^* \times \mathbb{CP}^1$ with analytic extension to $\check{U} \times \mathbb{CP}^1$.

A similar proof can be done in a neighborhood of the line $\{-\sqrt{\hat{\epsilon}}\} \times \mathbb{CP}^1$. It can be reduced to the previous proof if we use the change $Y\pm \mapsto 1/Y^\pm$.

So the manifold we have constructed is a 2-dimensional complex analytic manifold which is fibred over a disk with a fiber given by $\mathbb{CP}^1$. Since any vector bundle over a non-compact Riemann surface is holomorphically trivial (see for instance [5]), this bundle must also be holomorphically trivial since it is clear that it can be constructed as the projectivization of a vector bundle, using a suitable lift of the maps (6.11).

Of course, we would have obtained the same result if we had used the Newlander-Nirenberg theorem. There, we could have included $\hat{\epsilon}$ as a parameter and obtained that the construction depends analytically on $\hat{\epsilon}$. And it is of course possible to manage that the limit exists for $\hat{\epsilon} = 0$

Correction to a uniform family. The family we have realized is defined over $B(0, r) \times \mathbb{CP}^1$ for values of $\hat{\epsilon}$ in a sector of radius ρ and of opening greater than 2π. For this correction, we use the Newlander-Nirenberg theorem as in Section 5. Indeed the vector field for $\overline{\epsilon} \in \overline{V}$ is conjugate to that for $\tilde{\epsilon} \in \widetilde{V}$. Let $(x, \overline{Y}, \overline{\epsilon}) \mapsto (x, \Xi(x, \overline{Y}, \overline{\epsilon}), \tilde{\epsilon})$ be this conjugating map. This map can be used to glue the family of vector fields over $\overline{V}$ with the family of vector fields over $\widetilde{V}$. So we realize a family of vector fields over a 3-dimensional analytic manifold $\mathscr{M}$. We glue in $B(0, r) \times \mathbb{CP}^1 \times \{\epsilon = 0\}$, thus obtaining a C^∞-manifold. We must recognize that this manifold is of the form $V \times B(0, r) \times \mathbb{CP}^1$. For this we endow it of an integrable almost complex structure. Two of the forms are given by dx and $d\epsilon$. A form playing the role of dY is constructed as in the proof of Theorem 5.1. The variables x and ϵ remain holomorphic in the new coordinates, and give a projection from the image of the corrected manifold onto a neighborhood of $(x, \epsilon) = (0, 0)$. The inverse image of each point (x, ϵ) close to $(0, 0)$ is clearly isomorphic to the Riemann sphere. We conclude by applying the Fisher-Grauert theorem to conclude that the bundle has a local trivialization [4]. $\square$

For generic $a(\epsilon), A(\epsilon), B(\epsilon)$, the triple $(a(\epsilon), [m_{A(\epsilon)}, L_\beta \circ T_{B(\epsilon)}])$ can be realized as the modulus of the unfolding of the holonomy of the strong separatrix of a Riccati equation given by a quadratic vector field. Since this proof is completely elementary, we add it for completeness.

Theorem 6.7 *Given germs of analytic functions $A(\epsilon)$ and $B(\epsilon)$, then for most a_0 and for a corresponding germ of analytic function $a(\epsilon)$ yielding a realizable family of triple $(a(\epsilon), [m_{A(\epsilon)}, L_\beta \circ T_{B(\epsilon)}])$ as in Theorem 6.5, there exists analytic functions $c(\epsilon)$ and $d(\epsilon)$ such that the triple can be realized as the moduli of the unfolding of the holonomy of the strong separatrix of a Riccati equation of the form*

$$\begin{array}{rcl} \dot{x} & = & x^2-\epsilon \\ \dot{y} & = & c(\epsilon)(x^2-\epsilon)+y(1+a(\epsilon)x)+d(\epsilon)y^2. \end{array} \tag{6.14}$$

There is no restriction on $a(\epsilon)$ when $A(0)B(0)\neq 0$. Also, when $A(0)=B(0)=0$ and $a(0)$ is not an integer, then the triple $(a(\epsilon), [m_{A(\epsilon)}, L_\beta \circ T_{B(\epsilon)}])$ can be realized.

PROOF. For the system

$$\begin{array}{rcl} \dot{x} & = & x^2-\epsilon \\ \dot{y} & = & \alpha(\epsilon)\beta(\epsilon)(x^2-\epsilon)+y(1+(1-\alpha(\epsilon)-\beta(\epsilon))x)+y^2, \end{array} \tag{6.15}$$

it is shown in [8], that the moduli are given (up to a scaling of the form $(A,B)\to(Ak,B/k)$ with k bounded and bounded away from 0) by

$$A(\epsilon)=\frac{2\pi i}{\Gamma(1-\alpha)\Gamma(1-\beta)}, \qquad B(\epsilon)=\frac{-2\pi i e^{\pi i(1-\alpha-\beta)}}{\Gamma(\alpha)\Gamma(\beta)}.$$

We first take $d(\epsilon)=1$ and $c(\epsilon)=\alpha(\epsilon)\beta(\epsilon)$ in (6.14) to obtain (6.15) where $a(\epsilon)=1-\alpha(\epsilon)-\beta(\epsilon)$. Thus,

$$A(\epsilon)=\frac{2\pi i}{\Gamma(a+\beta)\Gamma(1-\beta)}, \qquad B(\epsilon)=\frac{-2\pi i e^{\pi i a}}{\Gamma(1-a-\beta)\Gamma(\beta)}.$$

If $a(0)$ is not an integer, it is clear that we can choose α and β to obtain any values of the parameters we wish (making sure that we have $\beta(0)\neq 0$), except for the cases where A and B both have a zero at $\epsilon=0$. (Recall, that A and B are only defined up to an inessential scaling.) If $a(0)$ is an integer, we can only realize $A(\epsilon)$ and $B(\epsilon)$ when $A(0)B(0)\neq 0$.

To discuss now the cases $A(0)=B(0)=0$, we consider (6.14) with $d(0)=0$ but $d(\epsilon)\not\equiv 0$. For $0\neq\epsilon<<1$ we can substitute $d(\epsilon)y\mapsto y$ to obtain

$$\begin{array}{rcl} \dot{x} & = & x^2-\epsilon \\ \dot{y} & = & \gamma(\epsilon)d(\epsilon)(x^2-\epsilon)+y(1+a(\epsilon)x)+y^2, \end{array} \tag{6.16}$$

and take $\gamma(\epsilon)=\alpha(\epsilon)\overline{\beta}(\epsilon)$, and denote $\beta(\epsilon)=\overline{\beta}(\epsilon)d(\epsilon)$, to obtain (6.15) where $a=1-\alpha-\beta$ as before.

However, this calculation is only for $\epsilon\neq 0$ and we need to make sure that the scaling factor is correct in the limit as ϵ tends to 0.

The values of A and B in [8] are obtained from the first integral, H say, of (6.15) which is of the form

$$\kappa\frac{w_2y+(x^2-\epsilon)w_2'}{w_3y+(x^2-\epsilon)w_3'},$$

where $\kappa = (2\sqrt{\epsilon})^{1-\alpha-\beta} e^{\pi i(\frac{a+b-1}{2}+\frac{1}{2\sqrt{\epsilon}})}$, and w_2 and w_3 are given by hypergeometric functions, and in particular,

$$w_3 = {}_2F_1\left(\alpha, \beta, \frac{1+\alpha+\beta}{2} - \frac{1}{2\sqrt{\epsilon}}, 1 - \frac{x}{\sqrt{\epsilon}}\right).$$

In our case, we have $\beta = \overline{\beta} d$, and hence d divides each term in w_3'. Thus, in original coordinates, we need to replace H by

$$\overline{H} = Hd = \kappa \frac{w_2 d Y + (x^2-\epsilon)w_2'}{w_3 Y + (x^2-\epsilon)w_3'/d},$$

to achieve a uniform limit as ϵ tends to zero. This means a scaling of $d(\epsilon)$ in the modulus given in [8], which gives

$$A(\epsilon) = \frac{d}{\Gamma(a+\overline{\beta}d)\Gamma(1-\overline{\beta}d)}, \qquad B(\epsilon) = \frac{-2\pi i e^{\pi i(1-a-b)}}{\Gamma(1-a-\overline{\beta}d)\Gamma(\overline{\beta}d)d}.$$

We note that $(\Gamma(\overline{\beta}d)d)^{-1} = \overline{\beta} + o(\overline{\beta}, d)$, and hence, if $a(0)$ is not an integer, we can clearly choose $\overline{\beta}$ and d to obtain any germs of functions A and B with $A(0) = B(0) = 0$. □

Remark 6.8 The triple $(a(\epsilon), [m_{A(\epsilon)}, L_\beta \circ T_{B(\epsilon)}])$ cannot be realized in a family of type (6.15), when $a(0) = 2$, $A(0) = 0$ and $B(\epsilon) \neq 0$.

6.3 The only families with continuous representative $\psi_\epsilon^{0,\infty}$ of the modulus

We propose the following conjecture which we prove in a special case.

Conjecture 6.9 The only families with representative $\psi_\epsilon^{0,\infty}$ of the modulus which are analytic in ϵ are the ones presented in Theorem 6.14.

Theorem 6.10 *The conjecture* 6.9 *is valid in the subcase where either* ψ_ϵ^∞ *(or* ψ^0*) is linear.*

This has been proved in the case $a = 0$ by Reinhard Schäfke [20].

PROOF OF THEOREM 6.10. We make the proof in the case where ψ_ϵ^∞ is linear, and thus $\overline{h}^\infty = id$ and $\tilde{h}^\infty = id$. Using the notation of Section 6, the compatibility condition is given by

$$(\tilde{h}^0)^{-1} = L_F \circ \overline{h}^0 \circ L_G, \tag{6.17}$$

where

$$\begin{cases} \tilde{h}^0 \circ L_{(\overline{C}\beta)^{-1}} \circ \widetilde{\psi}^0 = L_{(\overline{C}\beta)^{-1}} \circ \tilde{h}^0, \\ \overline{h}^0 \circ \overline{\psi}^0 \circ L_{\overline{C}} = L_{\overline{C}} \circ \overline{h}^0. \end{cases} \tag{6.18}$$

We note that $G = F^{-1}$ because $(\overline{h}^0)'(0) = (\tilde{h}^0)'(0) = 1$ in (6.17).

If ψ_ϵ^0 depends analytically on ϵ, then $\overline{\psi}^0 = \widetilde{\psi}^0$. From (6.18), we have

$$\begin{cases} \widetilde{\psi}^0 = L_{\overline{C}\beta} \circ (\tilde{h}^0)^{-1} \circ L_{(\overline{C}\beta)^{-1}} \circ \tilde{h}^0, \\ \overline{\psi}^0 = (\overline{h}^0)^{-1} \circ L_{\overline{C}} \circ \overline{h}^0 \circ L_{(\overline{C})^{-1}}. \end{cases} \tag{6.19}$$

Since $\overline{\psi}^0 = \widetilde{\psi}^0$, this yields, after some rearrangement,

$$\overline{h}^0 \circ L_{(\overline{C})^{-1}} \circ (\tilde{h}^0)^{-1} \circ L_{\overline{C}\beta} = L_{(\overline{C})^{-1}} \circ \overline{h}^0 \circ L_{\overline{C}\beta} \circ (\tilde{h}^0)^{-1}. \tag{6.20}$$

Substituting (6.17) yields

$$\overline{h}^0 \circ L_{F(\overline{C})^{-1}} \circ \overline{h}^0 \circ L_{\overline{C}\beta} = L_{(\overline{C})^{-1}} \circ \overline{h}^0 \circ L_{F\overline{C}\beta} \circ \overline{h}^0. \tag{6.21}$$

We now substitute $\overline{h}^0(w) = w + \sum_{j\geq 2} b_j w^j$ and equate coefficients of w^j in (6.21). Let b_s be the first nonzero coefficient. Then we need to choose $F^{s-1} = \frac{(\overline{C}\beta)^{1-s}-1}{(\overline{C})^{1-s}-1}$. We note that $F\beta = 1$ if and only if $a \in \frac{1}{2\pi i}\mathbb{Z}$ (i.e. $\beta = 1$). For $j > s$, the coefficient of w^j is a polynomial in $b_2, \dots, b_j$, where the only monomial in b_j is of the form $c_j b_j$ with

$$c_j = F\beta\left[1 - (F\beta)^{j-1} + (\overline{C}\beta)^{j-1}(F^{j-1} - 1)\right]$$

which does not vanish for $|\epsilon| << 1$ as soon as $F\beta \neq 1$. This means that the solution is unique. Since $m_{(1-s)b_s, s-1}$ is one solution, it is the only one.

We now need to treat the case $a \in \frac{1}{2\pi i}\mathbb{Z}$. In this case $\beta = 1$ and (6.21) gives

$$\overline{h}^0 \circ L_{(\overline{C})^{-1}} \circ \overline{h}^0 \circ L_{\overline{C}} - L_{(\overline{C})^{-1}} \circ \overline{h}^0 \circ L_{\overline{C}} \circ \overline{h}^0 = 0. \tag{6.22}$$

If $b_2 \neq 0$, the terms of degree 2 and 3 yield no constraints, but the terms of degree 4 give $b_3 - b_2^2 = 0$, and the terms of degree j give b_{j-1} uniquely in terms of $b_2, \dots, b_{j-2}$. Since $\overline{h}^0 = m_{-b_2,1}$ is a solution, it must be the solution.

If b_2 vanishes, then we take b_s to be the first non-vanishing coefficient of $\overline{h}^0 - id$ as above. Suppose $b_j \neq 0$ is the first term non-zero coefficient for which $j-1$ is not divisible by $s-1$. We consider the term in w^{j+s-1} in (6.22) to yield a contradiction. Thus, all terms in $\overline{h}^0$ only contains terms in $b_{1+k(s-1)}x^{1+k(s-1)}$. We conjugate (6.22) by $z \to w^{s-1}$ to obtain an equation of the form (6.22) with $\overline{C}$ replaced by $\overline{C}^{s-1}$ and $\overline{h}$ replaced by a power series $\overline{h}'$ in z with the coefficient of z^2 given by $(s-1)b_s$. Therefore, we can proceed as above to obtain $\overline{h}' = m_{(1-s)b_s,1}(z)$, and hence $\overline{h_0} = M_{(1-s)b_s,s-1}$. □

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