Study of the cyclicity of some degenerate graphics inside quadratic systems [*][†]

F. Dumortier, Hasselt University
C. Rousseau, DMS and CRM, Université de Montréal

November 30, 2007

Abstract

In this paper we make essential steps in proving the finite cyclicity of degenerate graphics in quadratic systems, having a line of singular points in the finite plane. In particular we consider the graphics (DF_{1a}), (DF_{2a}) of the program of [DRR] to prove the finiteness part of Hilbert's 16th problem for quadratic vector fields. We make a complete treatment except for one very specific problem that we clearly identify.

1 Introduction

Together with Robert Roussarie, we initiated the DRR program in 1994 (see [DRR]) to prove the finiteness part of Hilbert's 16th problem for quadratic vector fields. The program reduces the global problem of showing the existence of a uniform upper bound for the number of limit cycles of a quadratic vector field to 121 local finiteness problems. Each of these local problems consists in showing the finite cyclicity of a graphic surrounding the origin inside the family

$$\begin{array}{rcl} \dot{x} & = & \lambda x - \mu y + a_1x^2 + a_2xy + a_3y^2 \\ \dot{y} & = & \mu x + \lambda y + b_1x^2 + b_2xy + b_3y^2, \end{array} \tag{1.1}$$

with $\Lambda = (\lambda, \mu) \in \mathbb{S}^1$ and $A = (a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{S}^5$. A graphic Γ surrounding the origin for a given system (1.1) corresponding to parameter values (Λ, A) has finite cyclicity inside the family (1.1) if there exists an integer n, a tubular neighborhood U of Γ and a neighborhood V of (Λ, A) in parameter space, such that for any $(\Lambda', A') \in V$ the corresponding vector field of (1.1) has at most n limit cycles in U. The paper [R] lists the graphics whose finite cyclicity is proved.

The original list of 121 graphics of the DRR program contains 13 degenerate graphics with a line of singular points. These graphics are considered the most difficult and the most interesting ones. Indeed it was recently shown in [DR2] and [DPR] that bifurcations of degenerate graphics can create more limit cycles than usually expected. Also Artes, Llibre and Schlomiuk identify in [ALS] an interesting quadratic system with a line of singular point, which they conjecture to be an organizing center of the bifurcation diagram of quadratic

[*]This work is supported by NSERC in Canada.
[†]AMS classification numbers: 34C07, 34C26, 37G99

systems. Partial results on the cyclicity of the graphic (DI_{2a}) are ready to be presented as a preprint ([ADL]).

In this paper we wish to start a systematic study of the cyclicity of the degenerate graphics appearing in the DRR program, attempting at least to prove finite cyclicity or giving a rather rough upper bound. In this attempt we can rely on recent results and methods in treating the cyclicity of degenerate graphics. The graphics of the DRR program all have the following features: they have one line of singular points, all of which are normally hyperbolic, except for one, called a contact point. In treating such contact points, a technique has been particularly powerful. It consists in desingularizing the family near the contact point by the method of the blow-up of the family. This allows to identify the regions in parameter space where limit cycles are likely to appear. In these regions the displacement map is studied. Under the condition that the slow dynamics is non- zero, its derivative is shown to be C^∞ contact equivalent to a development having the slow divergence integral as leading term. The properties of the development are sufficiently regular to permit a nice control by the leading term. As a consequence, among other results, finite cyclicity is obtained as soon as the slow divergence integral does not vanish identically and in that case the cyclicity is determined by the slow divergence integral. Refinements have been studied in [DD3] for the case when singularities like saddle points or saddle-nodes appear in the slow dynamics. These are used in the generic case. In the center case, we need to make an ad hoc adaptation. We also have to deal with a slow divergence integral that is identically zero when the center conditions are met, requiring extensions of the existing theoretical results. In the presence of a desingularized contact point we are able to deal with this problem by blowing up the parameters in the neighborhood of the center conditions.

It is known for a long time that the space of quadratic systems modulo affine conjugacy and scaling of time is essentially of dimension 5 (see for instance [RS]). It is hence natural to reduce the family (1.1) to a 5-parameter family which is well suited to study the finite cyclicity of a given degenerate graphic. Such a 5-parameter “normal form” is not unique. The surprise is that there is no adequate normal form allowing to perform the program described above for all parameter values. Indeed the blow-up of the family requires at the contact point a weighted blow-up. Depending on the monomials we choose to keep in the normal form they may play an important role in the slow divergence integral or in the family rescaling of the blow-up family. There does not seem to exist a normal form in which one can desingularize the contact point when the slow dynamics has a zero at the contact point. This seems to be a hard problem that we can not yet treat for the moment. Due to this difficulty our study is incomplete. The lack of knowlegde is however limited to a small and precise region in parameter space.

In this paper we study the graphics (DF_{1a}), (DF_{2a}) of the DRR program (see Figure 1 below). The graphic (DF_{2a}) is the center version of the generic graphic (DF_{1a}). We expect that our method can be used for the other degenerate graphics of the DRR program although we do not expect that this will be a trivial exercise, especially for (DH_5). In all these graphics the main difficulty will not be to treat the center graphics, but to overcome the fact that the family cannot be desingularized. For this reason we have given normal forms adequate to study the finite cyclicity of all the degenerate graphics of the DRR program. We had a second surprise when looking for a normal form for the unfolding of a quadratic system with a graphic of type (DH_5): there exists no analytic 5-parameter normal form and a natural analytic normal form needs 7 parameters! The explanation is the following: usually one reduces the quadratic family by means of the action of the affine group and time scaling. This works when the orbits are transversal to the stratum being studied. Here the stratum is invariant under a 2-dimensional subgroup of the affine group.

2 Normal forms for families unfolding degenerate graphics

In this section and in view of treating all degenerate quadratic graphics, we derive good normal forms for quadratic families unfolding the degenerate graphics of [DRR]. There are 13 such graphics but only three normal forms are necessary to treat them all; we give all three. Different normal forms for the families are possible: we choose some where each parameter has its definite role in the perturbed system.

2.1 Quadratic families unfolding the graphics with a line of singular points in the finite plane

These are the graphics (DF_{1a}), (DF_{1b}), (DF_{2a}), (DF_{2b}), (DH_1), (DH_2).

Proposition 2.1 *A quadratic system with a line of singular points in the finite plane, all of which except one are normally hyperbolic, and a focus (strong or weak) or center can always be brought to the form*

$$\begin{aligned} \dot{x} &= y + b_0 xy - y^2 \\ \dot{y} &= xy, \end{aligned} \tag{2.1}$$

with $b_0 \in (-2, 2)$. The general quadratic perturbation of (2.1) can be brought by an affine change of coordinates and time scaling, depending analytically on the parameters, to the form

$$\begin{aligned} \dot{x} &= y + bxy - y^2 + \mu_1 + \mu_2 x + \mu_3 x^2 \\ \dot{y} &= xy + \mu_4, \end{aligned} \tag{2.2}$$

where $b = b_0 + \mu_0$ is a variable parameter inside $(-2, 2)$. Hence the five-parameter quadratic unfolding of (2.1) is parameterized by $(\mu_0, \ldots, \mu_4)$. When $b_0 \neq 0$,using a scaling $(x,t) \mapsto (-x, -t)$ we can consider $b > 0$.

Proof If a quadratic system has a line of singular points we can always suppose that it is the line $y = 0$. All points of the line are normally hyperbolic except one which we can suppose to be the origin. We can suppose that the focus or center is located at $(0, 1)$. So we can always suppose that the system has the form (2.1) with $b_0 \in (-2, 2)$, the later condition guaranteeing that $(0, 1)$ is a focus or center. The general quadratic perturbation is of the form

$$\begin{aligned} \dot{x} &= y + b_0 xy - y^2 + \textstyle\sum_{0 \le i+j \le 2} a_{ij} x^i y^j \\ \dot{y} &= xy + \textstyle\sum_{0 \le i+j \le 2} b_{ij} x^i y^j. \end{aligned} \tag{2.3}$$

We consider a change of coordinates

$$(x, y) = (X + \delta_1 Y + \delta_3, \delta_2 X + Y + \delta_4) \tag{2.4}$$

for the family, which reduces to the identity for (2.1). Such a change of coordinates brings (2.3) to the form

$$\begin{aligned} \dot{X} &= Y + b_0 XY - Y^2 + \textstyle\sum_{0 \le i+j \le 2} A_{ij} X^i Y^j \\ \dot{Y} &= XY + \textstyle\sum_{0 \le i+j \le 2} B_{ij} X^i Y^j. \end{aligned} \tag{2.5}$$

We consider the map $(\delta_1,\delta_2,\delta_3,\delta_4,a_{ij},b_{ij}) \mapsto (B_{10},B_{01},B_{20},B_{02})$. The differential w.r.t. $(\delta_1,\delta_2,\delta_3,\delta_4)$ at $a_{ij}=b_{ij}=0$, namely

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \tag{2.6}$$

is invertible, yielding that we can solve the equations $B_{10}=B_{01}=B_{20}=B_{02}=0$ by the implicit function theorem.

Using scalings in (X,Y,t) allows to take $A_{01}=A_{02}=B_{11}=0$, while we can write $b_0+A_{11}=b$. □

2.2 Quadratic families unfolding the graphics with a line of singular points at infinity

These are the graphics (DI_{1a}), (DI_{1b}), (DI_{2a}), (DI_{2b}), (DH_3), (DH_4).

Proposition 2.2 *A quadratic system with a line of singular points at infinity, all of which except one are normally hyperbolic, a finite invariant line and a focus (strong or weak) or center can always be brought to the form*

$$\begin{array}{rcl} \dot{x} & = & c_0x-y+1+x^2 \\ \dot{y} & = & xy, \end{array} \tag{2.7}$$

with $c_0\in(-2,2)$. The general quadratic perturbation of (2.7) can be brought by an affine change of coordinates and time scaling, depending analytically on the parameters, to the form

$$\begin{array}{rcl} \dot{x} & = & cx-y+1+(1+\mu_2)x^2+\mu_1xy+\mu_0y^2 \\ \dot{y} & = & xy-\mu_3x^2, \end{array} \tag{2.8}$$

where $c=c_0+\mu_4$ is a variable parameter inside $(-2,2)$, yielding the five-parameter unfolding of (2.7) being parameterized by $(\mu_0,\dots,\mu_4)$. When $c_0\neq 0$, using a scaling $(x,t)\mapsto(-x,-t)$ we can consider $c>0$.

Proof We can always suppose that the invariant line is the line $y=0$. The line at infinity is a line of singular points, so is the equator of the Poincaré-sphere, compactifying the finite plane. All points of the equator are normally hyperbolic except for two. By a change of coordinate $X=x+ay$ we can always suppose that the non normally hyperbolic points are located on the y-axis. We can suppose that the focus or center is located at $(0,1)$. So we can always suppose that the system has the form (2.7) with $c_0\in(-2,2)$, the last condition guaranteeing that $(0,1)$ is a focus or center. The general quadratic perturbation is of the form

$$\begin{array}{rcl} \dot{x} & = & c_0x-y+1+x^2+\sum_{0\le i+j\le 2}a_{ij}x^iy^j \\ \dot{y} & = & xy+\sum_{0\le i+j\le 2}b_{ij}x^iy^j. \end{array} \tag{2.9}$$

We consider a change of coordinates

$$(x,y)=(X+\delta_1Y+\delta_3,\delta_2X+Y+\delta_4) \tag{2.10}$$

for the family, which reduces to the identity for (2.7). Such a change of coordinates brings (2.9) to the form

$$\begin{array}{rcl} \dot{X} & = & c_0X - Y + 1 + X^2 + \sum_{0\leq i+j\leq 2} A_{ij}X^iY^j \\ \dot{Y} & = & XY + \sum_{0\leq i+j\leq 2} B_{ij}X^iY^j. \end{array} \tag{2.11}$$

We consider the map $(\delta_1, \delta_2, \delta_3, \delta_4, a_{ij}, b_{ij}) \mapsto (B_{00}, B_{10}, B_{01}, B_{02})$. The differential w.r.t. $(\delta_1, \delta_2, \delta_3, \delta_4)$ at $a_{ij} = b_{ij} = 0$, given by

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & -c & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{2.12}$$

is invertible, yielding that we can solve the equations $B_{00} = B_{10} = B_{01} = B_{02} = 0$ by the implicit function theorem.

Using scalings in (X, Y, t) allows to take $A_{01} = A_{00} = B_{11} = 0$, while we can write $c_0 + A_{10} = c$. □

2.3 Quadratic family unfolding the graphic with two lines of singular points

This is the graphic (DH_5) which occurs in the system

$$\begin{array}{rcl} \dot{x} & = & xy \\ \dot{y} & = & -y + y^2. \end{array} \tag{2.13}$$

In the two previous cases the quadratic family could be reduced to a 5-parameter family of vector fields through the action of the affine group and scaling of time. In this case it is not possible through a change of coordinates, depending analytically on the parameter, to reduce the full quadratic unfolding of (2.13) to a 5-parameter family. Indeed (2.13) is invariant under any change of coordinate $(X, Y) = (\alpha x + \beta(y - 1), y)$. Using a change of coordinate $(x, y) = (X + \delta_3, \delta_2 X + Y + \delta_4)$ and scalings in Y and t we can reduce the full quadratic unfolding to the 7-parameter family

$$\begin{array}{rcl} \dot{x} & = & xy(1 + \mu_4) + \mu_0 + \mu_1 x + \mu_2 x^2 + \mu_3 y^2 \\ \dot{y} & = & -y + y^2 - \mu_6 x^2 + (\mu_2 - \mu_5)xy. \end{array} \tag{2.14}$$

Although we have more parameters than really necessary, the unfolding (2.14) has the advantage that the slow motion on the line $y = 0$ is given by $\dot{x} = \mu_0 + \mu_1 x + \mu_2 x^2$, while the slow motion on the equator, using coordinates $(v, w) = (x/y, 1/y)$ is given by $\dot{v} = \mu_3 + \mu_4 v + \mu_5 v^2 + \mu_6 v^3$, both being simple polynomial expressions.

3 Statement of results on the cyclicity of DF_{1a} and DF_{2a}

We will study the degenerate graphic DF_{1a} and DF_{2a} as they occur in system (2.1) for respectively $b_0 \in (0, 2)$ and $b_0 = 0$ (see Figure 1). Such a graphic consists of a fast orbit, that we denote by $\gamma_{x_0}^{b_0}$, and a critical curve, included in the x-axis, that we denote by $C_{x_0}^{b_0} =$

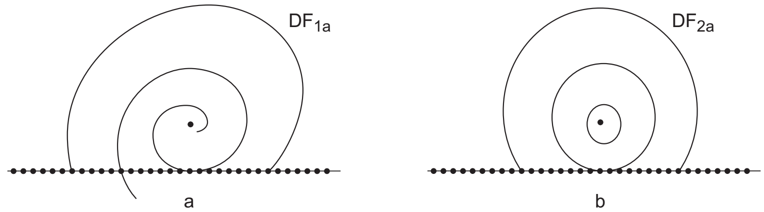


Figure 1: Degenerate graphics under consideration.

$[f_{b_0}(x_0), x_0]$; $x_0 > 0$ is the x-coordinate of the point $p_0 = (x_0, 0)$ representing the α-limit of the fast orbit and $p_1 = (f_{b_0}(x_0), 0)$ represents the ω-limit of the fast orbit. Let $\Gamma_{x_0}^{b_0} = \gamma_{x_0}^{b_0} \cup C_{x_0}^{b_0}$ denote the degenerate graphic.

We will study the cyclicity of $\Gamma_{x_0}^{b_0}$, with $x_0 \in (0, \infty)$, We therefore consider system (2.2), with $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \sim (0, 0, 0, 0)$, for some $b \in [0, 2)$.

From the subsequent calculations it will reveal to be interesting to write (2.2) as

$$\begin{cases} \dot{x} = y + bxy - y^2 + \varepsilon^2(E_0 + E_1 x + E_2 x^2) \\ \dot{y} = xy + \varepsilon^3 D \end{cases} \tag{3.1}$$

with $\varepsilon \geq 0$ small, $b - b_0$ small and keeping (E_0, E_1, E_2, D) in the boundary C_1 of a cylinder, more precisely inside

$$C_1 = B_0 \cup B_1 \cup B_{-1}$$

with

$$\begin{aligned} B_0 &= \{E_0^2 + E_1^2 + E_2^2 = 1, D \in [-1, 1]\} \text{ and} \\ B_i &= \{E_0^2 + E_1^2 + E_2^2 \leq 1, D = i\} \end{aligned} \tag{3.2}$$

for $i = \pm 1$.

Studying the systems (3.1), for $\varepsilon > 0$, subject to $(D, E_0, E_1, E_2) \in C_1$, requires using a number of recent results on singular perturbations, as well as extensions of these results. Unfortunately, for the moment, we are not able to deal with systems (3.1) that are close to $P_* \in B_0 \subset C_1$ with

$$P_* = (D, E_0, E_1, E_2) = (0, 0, 0, 1).$$

In making precise statements of the results that we have obtained till now, we introduce the balls

$$B_\delta(P_*) = \{(D, E_0, E_1, E_2) \in C_1 \mid d((D, E_0, E_1, E_2), P_*) < \delta\},$$

with $\delta > 0$.

We can write the unknown systems with $(D, E_0, E_1, E_2) \in B_\delta(P_*)$, for $\delta > 0$ sufficiently small, as:

$$\begin{cases} \dot{x} = y + bxy - y^2 + \varepsilon^2(e_0 + e_1 x + x^2) \\ \dot{y} = xy + \varepsilon^3 D, \end{cases} \tag{3.3}$$

with $(e_0, e_1, D) \sim (0, 0, 0)$.

As we will see in the further elaboration, it does not seem possible to "desingularize" at

$(e_0, e_1, D) = (0, 0, 0)$, creating theoretical problems that we can not overcome for the moment. A similar problem has also been encountered for the equations (5) in [ADL].
Except for that problem we are able to make a fairly complete study for the degenerate graphics (DF_{1a}) and (DF_{2a}). The results that we obtain are presented in Theorem 3.1.

Theorem 3.1 *Consider a system (3.1) and a set of degenerate graphics $\Gamma_{x_0}^{b_0}$, with $b_0 \in [0, 2)$ and $x_0 \in K$, with $K \subset (0, \infty)$ compact. Then the following statements hold, for arbitrary $\delta > 0$:*

(i) If $b_0 \in (0, 2)$, there exists $\varepsilon_0 > 0$, $\eta_0 > 0$ and $\rho > 0$ such that system (3.1) with $\varepsilon \in [0, \varepsilon_0], b \in (b_0 - \eta_0, b_0 + \eta_0)$ and $(D, E_0, E_1, E_2) \in C_1 \backslash B_\delta(P_*)$ has at most three limit cycles (multiplicity taken into account), lying each within Hausdorff distance ρ of a corresponding graphic $\Gamma_{x_0}^{b_0}$, with $x_0 \in K$. If moreover we keep $E_1 \geq 0$, then, under the same conditions on ε, b, (D, E_0, E_1, E_2), system (3.1) has at most one limit cycle, which is hyperbolic and attracting when it exists.

(ii) If $b_0 = 0$ there exists $\varepsilon > 0$, $\eta_0 > 0$ and $\rho > 0$ such that system (3.1) with $\varepsilon \in [0, \varepsilon_0], b \in [-\eta_0, \eta_0]$ and $(D, E_0, E_1, E_2) \in C_1 \backslash B_\delta(P_*)$ has at most 5 limit cycles (multiplicity taken into account) each within Hausdorff distance ρ of a corresponding graphic $\Gamma_{x_0}^0$ with $x_0 \in K$. If moreover we keep $bE_1 \geq 0$, then, under the same conditions on ε, b, (D, E_0, E_1, E_2), system (3.1) has at most one limit cycle. When it exists it is hyperbolic. It is attracting (resp. repelling) for $E_1 > 0$ or $E_1 = 0$, $b > 0$ (resp. $E_1 < 0$ or $E_1 = 0$, $b < 0$)

(iii) Let $B_{\delta_1}((0, E_0, 0, E_2))$ be a δ_1-neighbourhood of the circle $\{D = E_1 = 0\}$ inside C_1. If $b_0 = 0$ and $\delta_1 > 0$ is arbitrary, then there exists $\varepsilon_0 > 0$ and $\eta_0 > 0$ such that system (3.1) with $\varepsilon \in [0, \varepsilon_0], b \in [-\eta_0, \eta_0]$ and $(D, E_0, E_1, E_2) \in C_1 \backslash (B_\delta(P_*) \cup B_{\delta_1}((0, E_0, 0, E_2)))$ has at most 1 limit cycle and this limit cycle is hyperbolic; it is repelling for $E_1 < 0$ and attracting for $E_1 > 0$.

Remark 3.2 We can state rather precise cyclicity results in (i) and (iii); (i) treats the case $b_0 \in (0, 2)$ and $(D, E_0, E_1, E_2) \neq (0, 0, 0, 1)$ while (iii) treats the case $b_0 = 0$ and $E_1 \neq 0$. In (ii) we only obtain a rather rough cyclicity result when $b_0 = 0$ and $E_1 = 0$, with $(D, E_0, E_2) \neq (0, 0, 1)$. Using (iii), this statement has hence only to be proven for $b \sim 0$ and (D, E_0, E_1, E_2) with $(D, E_1) \sim (0, 0)$. As already mentioned we do not yet see how to treat the cyclicity of systems (3.1) with $(D, E_0, E_1, E_2) = (0, 0, 0, 1)$.

4 Proof of Theorem 3.1

4.1 Blow up at the origin

To prove Theorem 3.1 we will blow up the origin by means of the family blow up

$$(x, y, \varepsilon) = (r\bar{x}, r^2\bar{y}, r\bar{\varepsilon}), \tag{4.1}$$

with $r \geq 0$ small and keeping $(\bar{x}, \bar{y}, \bar{\varepsilon})$ inside S_2^+ with $S_2^+ = \{\bar{x}^2 + \bar{y}^2 + \bar{\varepsilon}^2 = 1, \bar{\varepsilon} \geq 0\}$.

Of course the calculations near the blow-up locus $S_2^+ \times \{0\}$ will be performed, as usual, in charts. We start by the traditional rescaling chart $\{\bar{\varepsilon} = 1\}$ in which the expression of (3.1), after division by r, becomes:

$$\begin{cases} \dot{\bar{x}} = \bar{y} + br\bar{x}\bar{y} - r^2\bar{y}^2 + E_0 + rE_1\bar{x} + r^2E_2\bar{x}^2 \\ \dot{\bar{y}} = \bar{x}\bar{y} + D; \end{cases} \tag{4.2}$$

on $\{r = 0\}$ this gives:

$$\begin{cases} \dot{\bar{x}} = \bar{y} + E_0 \\ \dot{\bar{y}} = \bar{x}\bar{y} + D. \end{cases} \tag{4.3}$$

Hence we see that we have desingularized if either E_0 or D is nonzero. When $E_0 = D = 0$ the case $E_1 \neq 0$ causes no problem as it prevents the existence of limit cycles. But near $E_0 = E_1 = D = 0$ we expect several limit cycles and we cannot desingularize: this new problem is a real challenge.

For the phase-directional rescaling in the $\{\bar{x} = 1\}$-direction we get, after division by r:

$$\begin{cases} \dot{r} = r(\bar{y} + \bar{\varepsilon}^2E_0 + r(b\bar{y} + \bar{\varepsilon}^2E_1) + r^2(-\bar{y}^2 + \bar{\varepsilon}^2E_2)) \\ \dot{\bar{\varepsilon}} = -\bar{\varepsilon}(\bar{y} + \bar{\varepsilon}^2E_0 + r(b\bar{y} + \bar{\varepsilon}^2E_1) + r^2(-\bar{y}^2 + \bar{\varepsilon}^2E_2)) \\ \dot{\bar{y}} = \bar{\varepsilon}^3D + (1 - 2\bar{\varepsilon}^2E_0)\bar{y} - 2\bar{y}^2 - 2r\bar{y}(b\bar{y} + \bar{\varepsilon}^2E_1) + 2r^2\bar{y}(\bar{y}^2 - \bar{\varepsilon}^2E_2). \end{cases} \tag{4.4}$$

On $\{r = \bar{\varepsilon} = 0\}$ we find two singularities, at respectively $\bar{y} = 0$ and $\bar{y} = 1/2$. At the first one the vector field (4.4) is normally repelling, having two-dimensional center bevaviour transverse to the $\bar{y}$-axis; each center manifold contains a line of singularities, defined by $\{\bar{y} = \bar{\varepsilon} = 0\}$. The center manifold on the blow up locus $\{r = 0\}$ is situated at

$$\bar{y} = -D\bar{\varepsilon}^3(1 + O(\bar{\varepsilon})) \tag{4.5}$$

and the dynamics on this center manifold is given by

$$\dot{\bar{\varepsilon}} = -\bar{\varepsilon}^3E_0 + D\bar{\varepsilon}^4(1 + O(\bar{\varepsilon})). \tag{4.6}$$

At the other singularity, the vector field (4.4) has a resonant saddle; but this singularity is unimportant for the study of limit cycles near the graphics under consideration.

The phase-directional rescaling in the $\{\bar{x} = -1\}$-direction leads to a similar situation, given by the expression:

$$\begin{cases} \dot{\bar{r}} = -r(\bar{y} + \bar{\varepsilon}^2E_0 - r(b\bar{y} + \bar{\varepsilon}^2E_1) + r^2(-\bar{y}^2 + \bar{\varepsilon}^2E_2)) \\ \dot{\bar{\varepsilon}} = \bar{\varepsilon}(\bar{y} + \bar{\varepsilon}^2E_0 - r(b\bar{y} + \bar{\varepsilon}^2E_1) + r^2(-\bar{y}^2 + \bar{\varepsilon}^2E_2)) \\ \dot{\bar{y}} = \bar{\varepsilon}^3D - (1 - 2\bar{\varepsilon}^2E_0)\bar{y} + 2\bar{y}^2 - 2r\bar{y}(b\bar{y} + \bar{\varepsilon}^2E_1) - 2r^2\bar{y}(\bar{y}^2 - \bar{\varepsilon}^2E_2). \end{cases} \tag{4.7}$$

At the origin, system (4.7) is now normally attracting, also having two-dimensional center manifolds, transverse to the $\bar{y}$-axis and containing the line of zeroes given by $\{\bar{y} = \bar{\varepsilon} = 0\}$. The center manifolds on the blow up locus $\{r = 0\}$ is situated at

$$\bar{y} = D\bar{\varepsilon}^3(1 + O(\bar{\varepsilon})) \tag{4.8}$$

and the dynamics on this center manifold is given by

$$\dot{\bar{\bar{\varepsilon}}} = E_0\bar{\varepsilon}^3 + D\bar{\varepsilon}^4(1 + O(\bar{\varepsilon})). \tag{4.9}$$

There is no need to use a phase-directional rescaling in the $\bar{y}$-direction.

4.2 Proof of statement (i) of Theorem 3.1, i.e. for $b_0 \in (0,2)$

From now on we will simply write $b_0 = b$, unless it is explicitly needed to underline the difference. Let us start by considering a graphic DF_{1a}, as represented in Figure 1(a). We recall that it consists of a fast orbit, that we denote by $\gamma^b_{x_0}$, and a critical curve, included in the x-axis, that we denote by $C^b_{x_0} = [f_b(x_0), x_0]$; $x_0 > 0$ is the x-coordinate of the point $p_0 = (x_0, 0)$ representing the α-limit of the fast orbit; $p_1 = (f_b(x_0), 0)$ represents the ω-limit of the fast orbit. The relation, expressed by f, i.e. $f(x_0, b) = f_b(x_0)$, is defined by the layer equation (2.1). It can hence also be defined by following an orbit of the linear vector field

$$\begin{cases} \dot{x} = 1 + bx - y \\ \dot{y} = x, \end{cases} \tag{4.10}$$

starting at the initial value p_0, until it hits again, for the first time, the x-axis (at $p_1 = f_b(x_0, 0)$).

Important features in order to study the cyclicity of the graphic $\Gamma^b_{x_0} = \gamma^b_{x_0} \cup C^b_{x_0}$ are the slow dynamics along $C^b_{x_0}$, and the slow divergence integral, along $C^b_{x_0}$. The slow dynamics, defined on $\{y = 0\}$, is given by

$$\dot{x} = P(x, E); \tag{4.11}$$

the slow divergence integral is defined as

$$I_b(x_0, E) = \int_{f_b(x_0)}^{x_0} \frac{x\ dx}{P(x, E)}, \tag{4.12}$$

with $E = (E_0, E_1, E_2)$ and

$$P(x, E) = E_0 + E_1 x + E_2 x^2. \tag{4.13}$$

Indeed the divergence on $y = 0$ is given by x when $\varepsilon = 0$ and $dt = \frac{dx}{P(x,E)}$.

The dynamics on the blow up locus S^+_2 can be obtained by combining (4.3), together with the information on $\{r = 0\}$ given by the phase-directional rescaling. The whole combines to a compactification of (4.3) on a $(1, 2)$-Poincaré-Lyapunov disk (PL-disk). To permit the creation of limit cycles near Γ_{x_0}, we at least need the existence of an invariant curve, connecting the points P_1 and P_0 (see Figure 2), that are situated at $(\bar{\varepsilon}, r, \bar{y}) = (0, 0, 0)$ in respectively the $(\bar{x} = -1)$ and $(\bar{x} = 1)$-charts. It is clear from (4.3), (4.5) and (4.8) that such an invariant curve is only possible for $D = 0$; it can only be part of a limit periodic set (after blow up) if $E_0 \geq 0$.

i) $D = 0$, $E_0 > 0$

In this case (4.3) contains a regular orbit having, in the $(1, 2)$-PL disk, P_1 as α-limit and P_0 as ω-limit. Both near P_1 and P_0, the respective systems (4.7) and (4.4) have a center behaviour which is a hyperbolic saddle, after division by $\bar{\varepsilon}^2$. The parameter D is a "breaking parameter" (see [DD1] and [DD2]) in the sense that the derivative w.r.t. D of the distance

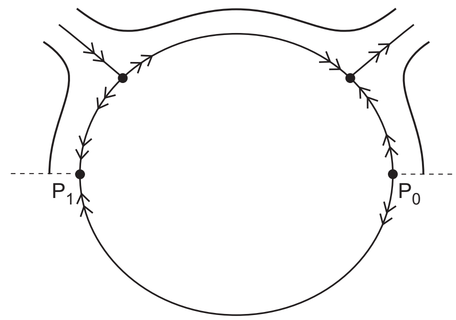


Figure 2: Blow up of (3.1) at $(x, y) = (0, 0)$.

between the respective center manifolds of P_1 and P_0 is nonzero. We can measure this distance using any regular parameter along a section, e.g. $\{\bar{x} = 0\}$, transversally cutting both center manifolds.
Depending on the quadratic function $P(x_0, E)$ we are in a perfect position to use the results given in [DD1] or [DD3].

In case $P(x, E)$ has no zeroes on $[f_b(x_0), x_0]$ and hence is strictly positive there, then we know from [DD1] that the cyclicity of $\Gamma^b_{x_0}$ can be at most one unit higher than the order of the zero of $I_b(x, E)$ at x_0.
If we do not consider a single x_0, but let x_0 vary in a compact interval $[x_1, x_2] \subset]0, \infty[$, then also in this case the total number of limit cycles, close to degenerate graphics $\Gamma^b_{x_0}$, with $x_0 \in [x_1, x_2]$, is at most one unit higher than the total number of zeroes of $I_b(x, E)$ on $[x_1, x_2]$, multiplicities taken into account.

It is hence clear that the function $f(x_0, b)$ will play an important role in the calculations. In the Appendix we will prove the following proposition:

Proposition 4.1 *The function $f(x, b)$ has the following properties:*

(i)

$$f(x, b) < -x \leq 0 \text{ for all } 0 < b < 2 \text{ and } x \geq 0, \tag{4.14}$$

while $f(x, 0) = -x$.

(ii) For $0 < b < 2$ and $x \geq 0$

$$f(x, b) < -x - b \tag{4.15}$$

and

$$f''(x, b) = \frac{\partial^2 f}{\partial x^2}(x, b) < 0. \tag{4.16}$$

(iii) *For* $0 \le b < 2$ *and* $x \ge 0$

$$f'(x,b) = \frac{\partial f}{\partial x}(x,b) = \frac{S(x,b)}{S(f(x,b),b)}, \tag{4.17}$$

with

$$S(x,b) = \frac{x}{x^2 + bx + 1}.$$

(iv) *For* $x \ge 0$

$$\varphi(x) = \frac{\partial f}{\partial b}(x,0) = -1 + \frac{x^2+1}{x}\arctan x - \pi\frac{x^2+1}{x}. \tag{4.18}$$

From now on we will rely on these properties.

Essential information is given by the function $I(x_0, b, E)$ as defined in (4.12); we can write:

$$I(x_0, b, E) = \int_{f(x_0,b)}^{x_0} R(x,E)dx \tag{4.19}$$

with

$$R(x,E) = \frac{x}{P(x,E)} = \frac{x}{E_2x^2 + E_1x + E_0}$$

Proposition 4.2 *Under the condition* $0 < b < 2$, $E_0 > 0$, *and restricting to values* (x_0, E), *with* $x_0 \ge 0$ *and such that* $P(x,E)$ *is strictly positive on* $[f(x_0,b), x_0]$, *we get the following results:*

1) *If* $E_1 \ge 0$, *then* $I(x_0, b, E) < 0$.

2) *If* $E_1 < 0$, *then we have the following cases:*

  (i) $E_1 = bE_2$, *implying that* $I(x_0, b, E)$ *is strictly monotone decreasing w.r.t.* x_0, *with* $I'(x_0, b, E) < 0$ *for* $x_0 > 0$.

  (ii) $E_1 \ne bE_2$
  Either $I(x_0, b, E)$ *is strictly monotone w.r.t.* x_0, *with* I' *non-zero for* $x_0 > 0$, *or it has a unique critical point for* $x_0 > 0$, *which is non-degenerate. Moreover* $I'(x_0, b, E) < 0$ *for* $x_0 \to \infty$, *when* $bE_2 < E_1$.

3) $I(0, b, E) < 0$ *and* $I'(0, b, E) = 0$.

Proof
We have

$$R(x,E) + R(-x,E) = \frac{-2E_1x^2}{P(x,E)P(-x,E)} < 0,$$

when $E_1 > 0$, implying that

$$|R(x,E)| < |R(-x,E)|,$$

when $E_1 > 0$ and $x > 0$.

From (4.14) we know that $|f(x,b)| > x$, for $b > 0$, while $R(x,E)$ has the same sign as x. Hence $I(x_0,b,E) < 0$ when $E_1 > 0$. We clearly also have $I(x_0,b,E) < 0$ when $E_1 = 0$. Moreover, for all E, we have $I(0,b,E) < 0$.

Let us now restrict to the case $E_0 > 0$ and $E_1 < 0$:

$$\begin{aligned} I'(x,b,E) &= \tfrac{\partial I}{\partial x}(x,b,E) = R(x,E) - R(f(x,b),E)f'(x,b) \\ &= R(x,E) - R(f(x,b),E)\tfrac{S(x,b)}{S(f(x,b),b)} \\ &= \tfrac{S(f(x,b),b)R(x,E) - R(f(x,b),E)S(x,b)}{S(f(x,b),b)} \\ &= \tfrac{xf(x,b)(f(x,b)-x)}{S(f(x,b),b)} \cdot \tfrac{K(x,f(x,b))}{L(x,f(x,b))}, \end{aligned} \tag{4.20}$$

where

$$K(x,\alpha) = (bE_2 - E_1)x\alpha + (\alpha + x)(E_2 - E_0) + E_1 - bE_0 \tag{4.21}$$

and

$$L(x,\alpha) = (1 + b\alpha + \alpha^2)(1 + bx + x^2)(E_0 + E_1 x + E_2 x^2)(E_0 + E_1\alpha + E_2\alpha^2).$$

We clearly see, under the conditions of Proposition 4.2, that $L(x,f(x,b)) > 0$, so that $I'(0,b,E) = 0$ and, for $x > 0$, I' has the same zeroes as K; $I'(x,b,E)$ has exactly the opposite sign of $K(x,f(x,b))$, when K is nonzero. Remark that $K(0,0) < 0$.
A thorough study of the set $K(x,a) = 0$ is hence needed and especially its position with respect to the graph of $\alpha = f(x,b)$.

We essentially distinguish two cases:

i) $E_1 = bE_2$
In that case

$$K(x,\alpha) = (E_2 - E_0)(\alpha + x + b).$$

If we restrict to $\alpha = f(x,b)$, then we see that $K(x,f(x,b))$ is strictly positive by (4.15).

ii) $E_1 \neq bE_2$
In that case $K(x,\alpha) = 0$ represents a hyperbola whose axes are parallel to the coordinate-axes. The asymptotes of the hyperbola intersect at

$$\left(\frac{E_0 - E_2}{bE_2 - E_1}, \frac{E_0 - E_2}{bE_2 - E_1}\right).$$

The hyperbola cuts the coordinate-axes at

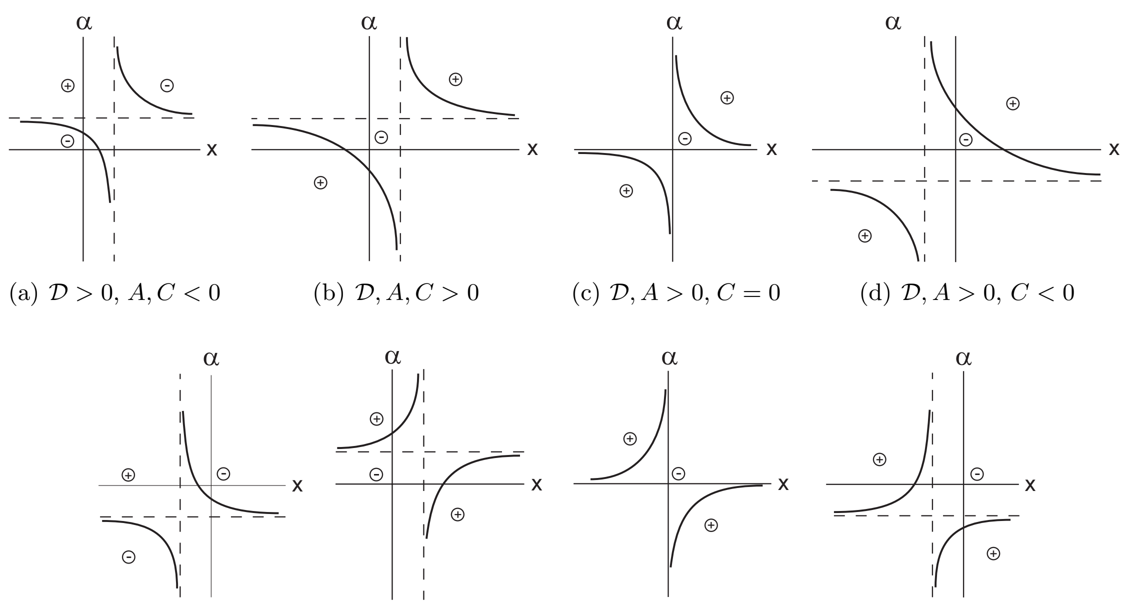
$$\left(\frac{bE_0 - E_1}{E_2 - E_0}, 0\right) \text{ and } \left(0, \frac{bE_0 - E_1}{E_2 - E_0}\right),$$

and we remark that $bE_0 - E_1 > 0$, since $E_0 > 0$ and $E_1 < 0$. The hyperbola is symmetric with respect to the diagonal.

In Lemma 4.3 below we will show that a branch of the hyperbola $K(x, \alpha) = 0$ can have at most one point of intersection with the graph of $\alpha = f(x, b)$, and in forthcoming case the two curves cut transversally. This will follow from the non existence of contact points between $K(x, \alpha) = 0$ and a vector field having $\alpha = f(x, b)$ as a trajectory.

We can now give an overview of the different cases concerning the relative position of the hyperbola and the curve $\alpha = f(x, b)$, and check the consequences for $I'(x, b, E)$, and hence also for $I(x, b, E)$. In each case we will see that there is at most one intersection point of the hyperbola with $\alpha = f(x, b)$, leading to at most one zero of $I'(x, b, E)$, which moreover is simple.

The different cases are represented in Figure 3. We also represent the sign of K. Seen



(a) $\mathcal{D} > 0$, $A, C < 0$ (b) $\mathcal{D}, A, C > 0$ (c) $\mathcal{D}, A > 0$, $C = 0$ (d) $\mathcal{D}, A > 0$, $C < 0$

(e) $\mathcal{D}, C > 0$, $A < 0$ (f) $\mathcal{D}, A, C < 0$ (g) $\mathcal{D}, A < 0$, C=0 (h) $\mathcal{D}, A < 0$, $C > 0$

Figure 3: Different positions of the hyperbola $K(x, \alpha) = 0$ (see (4.22) for the notation).

in the (x, α)-plane, the graph of $\alpha = f(x, b)$ is approximately as in Figure 4. Taking into account the information, pictorially represented in the Figures 3 and 4, together with the result proved in Lemma 4.3, the claim is clear. In Figure 3 the following notation is used

$$\begin{cases} A = bE_2 - E_1 \\ C = E_0 - E_2 \\ \mathcal{D} = (E_2 - E_0)^2 - (E_1 - bE_0)(bE_2 - E_1) \end{cases} \tag{4.22}$$

An intersection point can appear in all cases except for the cases (c) and (d). It will necessarily appear in cases (a), (f) and g. In the remaining cases this depends on the value of the parameters. The sign of K in the different regions is determined from the fact that $K(0, 0) <$

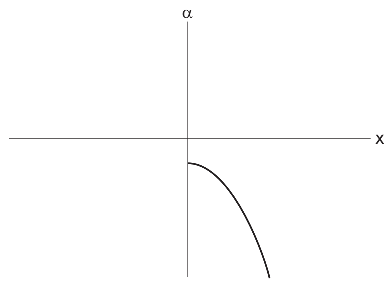


Figure 4: Graph of $\alpha = f(x,b)$.

Figure 5: Graph of $I(x,b,E)$ when it is not monotone.

0. Hence $I'(x,b,E) > 0$ for $x \sim 0$, except possibly in cases (b), (e) and (h). In cases (b) (resp. (e) and (h)) it is possible to have $I(x,b,E)$ monotone increasing (resp. decreasing) with $I'(x,b,E) > 0$ (resp. $I'(x,b,E) < 0$) for $x > 0$, while it is also possible to encounter a function $I(x,b,E)$ having a single critical point which is a non-degenerate minimum (resp. maximum). In cases (a), (f) and (g), $I(x,b,E)$ necessarily has a non-degenerate maximum. In cases (c) and (d), $I(x,b,E)$ is monotonically increasing.
This ends the proof of Proposition 4.2. □
In Figure 5 we represent the two possibilities for $I(x,b,E)$ to have a critical point, not paying attention to the relative position w.r.t. the x-axis, nor to the exact behaviour for $x \to +\infty$. The case of a non-degenerate minimum (case (b) of Figures 4 and 5) can only occur for $E_1 < bE_2$ $(A > 0)$.

Lemma 4.3 *In case $E_2 \neq bE_1$, a branch of the hyperbola $K(x,\alpha) = 0$ has at most one point of intersection with the graph of $\alpha = f(x,b)$. At such point both curves intersect transversally.*

Proof From (4.17) we see that the graph of $\alpha = f(x,b)$ is an orbit of the system

$$\begin{cases} \dot{x} = \alpha(x^2 + bx + 1) \\ \dot{\alpha} = x(\alpha^2 + b\alpha + 1). \end{cases} \tag{4.23}$$

If we would have two intersection points of a branch of the hyperbola with the graph of $\alpha = f(x,b)$, then in between these two intersection points, there would be a point of contact of (4.21) with $K(x,\alpha) = 0$.
The same would happen at a point where the intersection of the two curves is not transverse. We will now show that the hyperbola is never tangent to an orbit of (4.23). At such a contact

point (x_0, α_0) the vector $(\alpha_0(x_0^2 + bx_0 + 1), x_0(\alpha_0^2 + b\alpha_0 + 1))$ has to be perpendicular to

$$\nabla K(x_0, \alpha_0) = ((bE_2 - E_1)\alpha_0 + (E_2 - E_0), (bE_2 - E_1)x_0 + (E_2 - E_0))$$

and moreover $K(x_0, \alpha_0) = 0$.

The former condition implies that

$$\begin{aligned} &((bE_2 - E_1)x_0^2 + (E_2 - E_0)x_0)(\alpha_0^2 + b\alpha_0 + 1) \\ &= (x_0^2 + bx_0 + 1)((bE_2 - E_1)\alpha_0^2 + (E_2 - E_0)\alpha_0). \end{aligned} \tag{4.24}$$

Remark that we may suppose that $(bE_2 - E_1)z + (E_2 - E_0)$ is different from zero for both $z = x_0$ and α_0, since otherwise they both would have to be zero, which is impossible since $x_0 \cdot \alpha_0 < 0$.
It is a straightforward calculation to see that condition (4.24) can be written as:

$$(\alpha_0 - x_0)[(b^2 - 1)E_2 - bE_1 + E_0)\alpha_0 x_0 + (bE_2 - E_1)(\alpha_0 + x_0) + (E_2 - E_0)] = 0.$$

It hence suffices to show that there does not exist a common solution of:

$$\begin{cases} (bE_2 - E_1)x\alpha + (E_2 - E_0)(\alpha + x) + E_1 - bE_0 = 0 \\ (b(bE_2 - E_1) - (E_2 - E_0))x\alpha + (bE_2 - E_1)(\alpha + x) + E_2 - E_0 = 0. \end{cases} \tag{4.25}$$

From the former equation we get an expression for $x\alpha$, and plugging it into the latter one, it implies that a solution of (4.25) necessarily satisfies:

$$\begin{aligned} &(x + \alpha)[-b(E_2 - E_0)(bE_2 - E_1) + (E_2 - E_0)^2 + (bE_2 - E_1)^2] \\ &+[(E_2 - E_0)(bE_2 - E_1) - b(bE_2 - E_1)(E_1 - bE_0) + (E_2 - E_0)(E_1 - bE_0)] = 0. \end{aligned} \tag{4.26}$$

A straightforward calculation, using $E_0^2 + E_1^2 + E_2^2 = 1$ reduces equation (4.26) to:

$$(x + \alpha + b)(1 - bE_0E_1 - bE_1E_2 + (b^2 - 2)E_0E_2) = 0,$$

implying the existence of a solution to

$$\begin{cases} bE_0E_1 + bE_1E_2 + (2 - b^2)E_0E_2 = 1 \\ E_0^2 + E_1^2 + E_2^2 = 1. \end{cases}$$

This is equivalent to proving a non-zero solution to

$$E_0^2 + E_1^2 + E_2^2 - bE_0E_1 - bE_1E_2 + (b^2 - 2)E_0E_2 = 0. \tag{4.27}$$

We only need to restrict to the range of E-values that we consider, namely $E_0 > 0$, $E_1 < 0$ and $E_2 \neq bE_1$.

Since the left hand side of (4.26) is a quadratic form we merely have to calculate the eigenvalues of its matrix; they are respectively $\frac{1}{2}(2 + b^2)$, $\frac{1}{2}(4 - b^2)$ and 0. A straightforward calculation shows that the eigenspace for the eigenvalue 0 is given by

$$(E_0, E_1, E_2) = E_0(1, b, 1),$$

which is not in the range of E-values that we consider. □

This finishes the study for $E_0 > 0$, under the condition that $P(x,E) > 0$ on $C^b_{x_0} = [f_b(x_0), x_0]$, with $x_0 \in K \subset (0,\infty)$.
Of course, the slow dynamics, given by

$$P(x,E) = E_2x^2 + E_1x + E_0,$$

with $E_2^2 + E_1^2 + E_0^2 = 1$, can have zeroes on the segments $[f_b(x_0), x_0]$ under consideration. P has at most 2 zeroes, multiplicity taken into account. It can have a unique double zero in case $E_2 \geq 0$. This double zero will be situated at the origin in case $E_1 = 0 = E_0$, a case that we discarded from the statements in Theorem 3.1. For $(E_0, E_1) \neq (0,0)$ a possible double zero will be situated at a non-zero value $\overline{x}_0$, inducing (see [DD3]) that nearby systems (3.1) can have at most one limit cycle lying near the graphics $\Gamma^b_{x_0}$, with $x_0 \in K$. Such a limit cycle is hyperbolic; it is attracting if $\overline{x}_0 < 0$ and repelling if $\overline{x}_0 > 0$.

Another possibility under which a slow dynamics with zeroes on $[f_b(x_0), x_0]$ can nevertheless permit limit cycles near $\Gamma^b_{x_0}$ is when a simple zero occur at x_0, at $f_b(x_0)$ or at both x_0 and $f_b(x_0)$. All these cases have been studied in [DD3]. The first two have cyclicity one; the third one has cyclicity two if the product of the hyperbolicity ratios at the two saddle points is different from one. Recall that the hyperbolicity ratio of a saddle point is the quotient of minus the negative eigenvalue to the positive eigenvalue. The hyperbolicity ratio at $(x_0, 0)$ is

$$\left|\frac{P'(x_0,E)}{x_0}\right| \tag{4.28}$$

and that at $(f_b(x_0), 0)$ is

$$\left|\frac{f_b(x_0)}{P'(f_b(x_0),E)}\right|. \tag{4.29}$$

Since P' is the derivative of the slow movement, then $P'(x_0,E)$ and $P'(f_b(x_0),E)$ are the "slow" eigenvalues at $(x_0,0)$ and $(f_b(x_0),0)$ respectively.

Since we suppose that $P(x,E^0)$, for the chosen value $E = E^0$, has zeroes at respectively x_0 and $f_b(x_0)$, we have

$$P(x,E^0) = E_2^0(x - x_0)(x - f_b(x_0)),$$

with $E_2^0 > 0$. As such

$$\begin{aligned} P'(x_0,E^0) &= E_2^0(x_0 - f_b(x_0)) \\ P'(f_b(x_0),E^0) &= E_2^0(f_b(x_0) - x_0) \end{aligned}$$

Since $f_b(x_0) < -x_0$ it is clear that the product of the hyperbolicity ratios (4.28) and (4.29) is different from one, thus finishing the proof of statement (i) for $E_0 > 0$.

(ii) $D = 0$, $E_0 = 0$.

The slow dynamics is given by

$$P(x,(0,E_1,E_2)) = E_1x + E_2x^2,$$

showing that no passage at $x = 0$ is possible when $E_1 \neq 0$. Since the condition $E_2 \geq 0$ is necessary to admit limit cycles near the $\Gamma^b_{x_0}$, for $x_0 > 0$, we end up with the case

$$(E_0, E_1, E_2) = (0, 0, 1)$$

that we discarded from the statements in Theorem 3.1.

The unicity of the limit cycle for $E_1 \geq 0$ follows from Proposition 4.2(1), stating that $I < 0$ in that case.

This ends the proof of statement (i) in Theorem 3.1.

4.3 Proof of statement (iii) of Theorem 3.1: the case $b_0 = 0$ and $(D, E_1) \neq (0, 0)$

Like in the proof of statement (i) it is clear that limit cycles are only possible near systems (3.1) for which $D = 0$. In the subsequent calculations we will hence suppose that $D = 0$. In the same way as before we can also suppose that $E_0 > 0$.

In case $b = 0$ we can use that

$$f(x_0, 0) = -x_0$$

and

$$I(x_0, 0, E) = \int_{-x_0}^{x_0} \frac{x}{E_2x^2 + E_1x + E_0} dx. \tag{4.30}$$

This integral can easily be calculated, depending on the specific value of E; recall that we take $E_1 \neq 0$, so that no passage near 0 is possible unless $E_2 \neq 0$. There is however no need to calculate this integral. For $E_1 = 0$ it is clearly identically zero. Moreover we see that

$$\frac{\partial I}{\partial E_1}(x_0, 0, E) = -\int_{-x_0}^{x_0} \frac{x^2}{(E_2x^2 + E_1x + E_0)^2} dx < 0, \tag{4.31}$$

implying that $I(x_0, 0, E)$, for $x_0 > 0$, is everywhere strictly negative for $E_1 > 0$ and everywhere strictly positive for $E_1 < 0$. The same property also holds if we take $x_0 \in K$, K compact, and $b \sim 0$.

As in the proof of statement (i) this induces the occurrence of at most one limit cycle near graphics $\Gamma^0_{x_0}$ with $x_0 \in K \subset (0, \infty)$, for systems (3.1) with $\varepsilon > 0$ sufficiently small, $D \sim 0$ and (E_0, E_1, E_2), with $E_0 > 0$, near values for which $P(x, E) = E_2x^2 + E_1x + E_0$ has no zeroes on $[-x_0, x_0]$.

Since $E_1 \neq 0$ it is not possible for P to have neither a zero at both $-x_0$ and x_0 nor a double zero at the origin, so that statement (iii) follows from [DD3], in case $P(x, E)$ has zeroes on $[-x_0, x_0]$.

4.4 Proof of statement (ii) of Theorem 3.1: the case $b_0 = 0$ and $(D, E_1) = (0, 0)$

Recall that we can keep $E_0 > 0$, so that from [DD2] follows that there exists a smooth function

$$D_0(\varepsilon, b, E, x_0)$$

with the property that periodic orbits of system (3.1), lying near graphics $\Gamma^b_{x_0}$, with $x_0 \in K \subset (0,0)$, can only occur for $D = D_0(\varepsilon, b, E, x_0)$.

For precise understanding of this statement, we consider $\{x = 0, y > 1\}$. Each point on this half-line belongs to a unique graphic $\Gamma^b_{x_0}$, and we can hence parametrize the points on $\{x = 0, y > 1\}$ by means of this unique x_0. System (3.1), for $\varepsilon \sim 0$, will exhibit a closed orbit passing through $(0, y)$, characterized in this way by x_0, if and only if $D = D_0(\varepsilon, b, E, x_0)$.

From this point on we can no longer rely on the existing literature to get results on limit cycles of system (3.1), for $\varepsilon \sim 0$, by merely studying the zeroes of the slow divergence integral. We will have to recall and extend some steps in the proofs given in [DD1], [DD2] and [DD3].

To that end we consider a first section

$$T_1 = \{x = 0, y > 1\}$$

that we parameterize by $(\varepsilon, b, D, E, x_0)$ in the way explained above.

We consider a second section T_2 that we define along the blown-up locus of the origin. More precisely, in the blow-up coodinates $(\overline{x}, \overline{y}, 1)$ as defined in (4.1), we consider

$$T_2 = \{\overline{x} = 0\}.$$

T_2 is hence defined in the traditional rescaling chart $\{\overline{\varepsilon} = 1\}$. We know from [DD2] that, by following the orbits of systems (3.1) both in forward time and in backward time, we can define smooth transition maps from T_1 to T_2, denoted by respectively Δ_1 and Δ_2. Closed orbits of system (3.1) for $\varepsilon > 0$, $\varepsilon \sim 0$, are given by zeroes of the displacement map

$$\Delta = \Delta_1 - \Delta_2.$$

Δ is smooth in the variables $(\varepsilon, b, D, E, x_0)$, also at $\varepsilon = 0$. We can take $b \in (-2, 2), (D, E) \in C_1$, and $x_0 \in K \subset (0, \infty)$.
In this part of the proof we will of course take $b \sim 0$, $\varepsilon \sim 0$, and $(D, E_1) \sim (0, 0)$.
Since $\Delta(0, b, 0, E, x_0) = 0$ and $\frac{\partial\Delta}{\partial D}(0, b, 0, E, x_0) \neq 0$ (as discussed in Section 4.2), it is a consequence of the Implicit Function Theorem that solution of $\Delta = 0$, for $\varepsilon \sim 0$ and $D \sim 0$, can only occur for

$$D = D_0(\varepsilon, b, E, x_0),$$

with D_0 some smooth function.

Moreover we know that the systems (3.1) with $b = D = E_1 = 0$ represent centers since they are invariant under $(x, t) \mapsto (-x, -t)$. As such

$$\Delta(\varepsilon, 0, 0, (E_0, 0, E_1), x_0) = 0, \tag{4.32}$$

as well as

$$D_0(\varepsilon, 0, (E_0, 0, E_1), x_0) = 0. \tag{4.33}$$

Instead of continuing working with Δ, we prefer to work with

$$\widetilde{\Delta}(\varepsilon, b, E, x_0) = \Delta(\varepsilon, b, D_0(\varepsilon, b, E, x_0), E, x_0), \tag{4.34}$$

which is a smooth function. The closed orbits correspond to the solutions of

$$\widetilde{\Delta}(\varepsilon, 0, (E_0, 0, E_2), x_0) = 0. \tag{4.35}$$

From [DD1] improving the results of [DR1], we know that $\frac{\partial\widetilde{\Delta}}{\partial x_0}$ is, on $\{\varepsilon > 0\}$, C^∞-contact equivalent to

$$I(x_0, b, E) + \varphi_1(x_0, \varepsilon, b, E) + \varphi_2(\varepsilon, b, E)\varepsilon \ln \varepsilon, \tag{4.36}$$

for some C^∞ functions φ_1 and φ_2, with $I(x_0, b, E) = I_b(x_0, E)$ the slow divergence integral as defined in (4.12) and φ_1 is $O(\varepsilon)$.
We recall that "C^∞-contact equivalent" means that $\frac{\partial\widetilde{\Delta}}{\partial x_0}$ is the product of a strictly positive C^∞ function with the expression in (4.36). The C^∞-contact equivalence only holds for $\varepsilon > 0$ but the functions φ_1 and φ_2 are C^∞ also at $\varepsilon = 0$.
The function I is analytic and it is identically zero for $b = E_1 = 0$. From (4.35) we even know that $\frac{\partial\widetilde{\Delta}}{\partial x_0}$ is identically zero for $b = E_1 = 0$, so that we can write (4.36) as

$$b(I^b(x_0, b, E) + \Phi^b(x_0, \varepsilon, b, E)) + E_1(I^1(x_0, b, E) + \Phi^1(x_0, \varepsilon, b, E)), \tag{4.37}$$

where both Φ^b and Φ^1 are $O(\varepsilon)$ and ε-regularly smooth in (x_0, b, E), as defined in [DR2]. It means that Φ^b and Φ^1 and all their partial derivatives w.r.t. (x_0, b, E) are continuous in ε, including at $\varepsilon = 0$.

We will now start by keeping $E_0 = 1$, thus writing $\overline{E}_2$ instead of E_2, with $\overline{E}_2$ belonging to an arbitrarily large compact and

$$(b, E_1) = u(\overline{b}, \overline{E}_1) \tag{4.38}$$

with $u \geq 0$ and $\overline{b}^2 + \overline{E}_1^2 = 1$. (This means that we made an abuse of notation and kept the same E_1 when we took the chart $E_0 = 1$.) As already mentioned we can limit our study to $\overline{b} \geq 0$. Expression (4.37) can be written as

$$u\left[\overline{I}(x_0, \overline{b}, \overline{E}_1, \overline{E}_2) + O(u) + O(\varepsilon)\right], \tag{4.39}$$

where $O(u)$ represents an analytic function and $O(\varepsilon)$ an ε-regularly smooth function and

$$\begin{array}{rcl} \overline{I}(x_0, \overline{b}, \overline{E}_1, \overline{E}_2) & = & \overline{b}\frac{\partial I}{\partial b}(x_0, 0, (1, 0, \overline{E}_2)) + \overline{E}_1\frac{\partial I}{\partial E_1}(x_0, 0, (1, 0, \overline{E}_2)) \\ & = & \lim_{u\to 0} \frac{1}{u} I(x_0, u\overline{b}, (1, u\overline{E}_1, \overline{E}_2)). \end{array} \tag{4.40}$$

We see that (see Appendix)

$$\begin{array}{l} \frac{\partial I}{\partial b}(x_0, 0, (1, 0, \overline{E}_2)) = \frac{x_0}{\overline{E}_2 x_0^2 + 1}\left(-1 + \frac{1+x_0^2}{x_0}\arctan x_0 - \pi\frac{1+x_0^2}{x_0}\right) \\ \frac{\partial I}{\partial E_1}(x_0, 0, (1, 0, \overline{E}_2)) = -\int_{-x_0}^{x_0} \frac{x^2\,dx}{(\overline{E}_2 x^2 + 1)^2}. \end{array} \tag{4.41}$$

Both functions in (4.41) are strictly negative.

Therefore the expression inside brackets in (4.39) is strictly negative when we take $\overline{E}_1 \geq 0$ and $\overline{b} \geq 0$ and strictly positive when $\overline{E}_1 \leq 0$ and $\overline{b} \leq 0$. The same is hence true for expression (4.39) itself if we take $u > 0$. This yields the conclusion on the existence of at most one limit cycle, which is hyperbolic if it exists and attracting.

It remains to look what happens with (4.39) when we take $\overline{E}_1 \cdot \overline{b} < 0$. We already know that it suffices to take $\overline{b} \geq 0$ and, to have uniformity in the results, we will take $\overline{E}_1 \leq 0$. Instead of working directly with (4.39) we will consider its derivative with respect to x_0 that can be written as

$$u\left[\frac{\partial\overline{I}}{\partial x_0}(x_0,\overline{b},\overline{E}_1,\overline{E}_2)+O(u)+O(\varepsilon)\right], \tag{4.42}$$

where $O(u)$ represents an analytic function, $O(\varepsilon)$ is an ε-regularly smooth function and

$$\frac{\partial\overline{I}}{\partial x_0}=\lim_{u\to 0}\frac{1}{u}\frac{\partial I}{\partial x_0}(x_0,u\overline{b},(1,u\overline{E}_1,\overline{E}_2)), \tag{4.43}$$

because of (4.40). We will now prove that $\frac{\partial\overline{I}}{\partial x_0}$ can have at most three zeroes, multiplicity taken into account, for the conditions on x_0, $\overline{E}_2$ and $(\overline{b},\overline{E}_1)$ under consideration. This will of course imply a similar statement for the expression in (4.42) for $u>0$, $u\sim 0$.

By (4.43) it suffices to study $\frac{\partial I}{\partial x_0}$. Hence, because of (4.20) we only need to consider $K(x_0,f_b(x_0))$, where K is given in (4.21). Noting that $f_0(x_0)=-x_0$ we write

$$f_b(x_0)=-x_0+b\psi(x_0,b), \tag{4.44}$$

where $\psi(x_0,b)$ is analytic for nonzero x_0 and $\psi(x_0,0)=\varphi(x)$ is the transcendental function defined in (4.18).

From (4.20) and (4.21) we see that $\frac{\partial\overline{I}}{\partial x_0}$ is C^∞-contact equivalent (even C^ω-contact equivalent) to

$$K_1(x_0,\overline{b},\overline{E}_1,\overline{E}_2)=(\overline{b}\,\overline{E}_2-\overline{E}_1)x_0^2-\overline{b}(\overline{E}_2-1)\varphi(x_0)+\overline{b}-\overline{E}_1. \tag{4.45}$$

The function K_1 in (4.45) is a linear combination of the three linearly independent functions $\{1,x_0^2,\varphi(x_0)\}$. Hence it can only be identically zero for $\overline{b}=\overline{E}_1$ and $\overline{E}_2=1$, a case in which we already proved the cyclicity to be less than or equal to 1. For the values of $(\overline{b},\overline{E}_1,\overline{E}_2)$ we are considering now, (4.45) never vanishes identically and hence $\frac{\partial\overline{I}}{\partial x_0}$ will have a uniformly bounded number of zeroes. In the next proposition we show that this number is three.

Proposition 4.4 *Any linear combination of* $\{1,x^2,\varphi(x)\}$ *has at most three zeroes in* $]0,\infty[$, *multiplicity taken into account.*

Proof Let us consider $F(x)=A+Bx^2+C\varphi(x)$, with $(A,B,C)\neq(0,0,0)$. If $B=C=0$ then F does not vanish. Otherwise we consider

$$F'(x)=2Bx+C\left(\frac{x^2-1}{x}\arctan x+\frac{1}{x}-\pi\frac{x^2-1}{x}\right). \tag{4.46}$$

When $C=0$ then $F'(x)$ does not vanish, yielding at most one zero for $F(x)$. If $C\neq 0$, let $G(x)=\frac{F'(x)}{x}$. Then $G'(x)=0$ if and only if

$$\pi+\arctan x=\frac{x(3+x^2)}{(x^2+1)(3-x^2)}. \tag{4.47}$$

Since the left hand side is positive, the only solutions of (4.47) lie in $]0,\sqrt{3}[$. Moeover the derivative of $\frac{x(3+x^2)}{(x^2+1)(3-x^2)}-\pi-\arctan x$ is $\frac{16x^4}{(x^2+1)^2(3-x^2)^2}>0$, yielding that (4.47) has at most one positive solution and hence that F has at most three positive zeroes. □

As such we see that for $\varepsilon > 0$, $\varepsilon \sim 0$, expression (4.36) has at most four zeroes, multiplicity taken into account. This result holds in a uniform way on ε (see [DD1]) as long as the slow dynamics is different from zero on $[-x_0, x_0]$, with $x_0 \in K \subset (0, \infty)$. The slow dynamics is given by

$$\dot{x} = E_0 + E_2 x^2,$$

and since we avoid working at $E_0 = 0$, it can also be expressed as

$$\dot{x} = 1 + \overline{E}_2 x^2.$$

So the only situation where the results from [DD1] do not apply is when we consider $\overline{E}_2 < 0$ and $x_0 = \overline{x}_0 = 1/e_2$ with $e_2 = (-\overline{E}_2)^{1/2}$.

This situation has not been studied in [DD3], since it is too degenerate. We can however develop a similar elaboration as in [DD3] to treat the specific case under consideration. More precisely we will first show that $\overline{I}(x_0, \overline{b}, \overline{E}_1, \overline{E}_2)$ as defined in (4.40), with $\overline{E}_2 = -e_2^2$, has at most a simple zero for $x_0 \sim \overline{x}_0 = 1/e_2$ and $\overline{b}^2 + \overline{E}_1^2 = 1$ and then we will show that the result extends to (4.39) itself, when we add the perturbation $O(u) + O(\varepsilon)$, implying that the cyclicity of $\Gamma^0_{\overline{x}_0}$ is at most two.

Let us start with the first claim. For $x_0 \sim 1/e_2$, but $x_0 < 1/e_2$ we can calculate $\frac{\partial I}{\partial E_1}$ (see (4.41)) and write expression (4.40) as:

$$\begin{aligned} &\overline{b}\tfrac{\partial I}{\partial b}(x_0, 0, (1, 0, -e_2^2)) + \overline{E}_1 \tfrac{\partial I}{\partial E_1}(x_0, 0, (1, 0, -e_2^2)) \\ &= \tfrac{1}{1-e_2^2 x_0^2}\left[\overline{b}(-x_0 + (1+x_0^2)(\arctan x_0 - \pi)) - \tfrac{\overline{E}_1}{2e_2^3}\left(2e_2 x_0 - (1-e_2^2 x_0^2)\ln\left(\tfrac{1+e_2 x_0}{1-e_2 x_0}\right)\right)\right]. \end{aligned} \tag{4.48}$$

Expression (4.48) has only to be considered for $x_0 \in (0, \frac{1}{e_2})$. We will now show that the expression in between brackets in (4.48) has at most one simple zero for $x_0 \sim \overline{x}_0$, and this for any choice of $(\overline{b}, \overline{E}_1) \in \mathbb{S}^1$. We therefore write this expression as

$$\overline{b} f(x_0) - \frac{\overline{E}_1}{2e_2^3} g(x_0), \tag{4.49}$$

with

$$\begin{aligned} f(x_0) &= -x_0 + (1+x_0^2)(\arctan x_0 - \pi), &(4.50)\\ g(x_0) &= 2e_2 x_0 - (1-e_2^2 x_0^2)\ln\left(\frac{1+e_2 x_0}{1-e_2 x_0}\right). &(4.51) \end{aligned}$$

We also see that

$$\begin{aligned} f'(x_0) &= 2x_0(\arctan x_0 - \pi), &(4.52)\\ g'(x_0) &= 2e_2^2 x_0 \ln\left(\frac{1+e_2 x_0}{1-e_2 x_0}\right). &(4.53) \end{aligned}$$

For $x_0 \to 1/e_2$ the function $f(x_0)\cdot g'(x_0) - g(x_0) f'(x_0)$ tends to $+\infty$, inducing that (4.49), and hence also (4.48) can never be identically zero, unless $(\overline{b}, \overline{E}_1) = (0, 0)$. This also implies that (4.48) has at most one simple zero for $x_0 \sim \overline{x}_0$, as claimed.

We will now show that this implies that the cyclicity of $\Gamma^0_{\overline{x}_0}$ is at most two. We recall that we observed that $\frac{\partial\widetilde{\Delta}}{\partial x_0}$ is, on $\{\varepsilon = 0\}$, C^∞ contact equivalent to (4.36), an expression that we can also write as (4.37). This however has only been proved for a regular slow dynamics. We now want to get, in the case under consideration, a similar result for $x_0 \sim \overline{x}_0 = 1/e_2$, to be able to conclude on the number of zeroes of $\widetilde{\Delta}$ for $x_0 \sim \overline{x}_0$. To study $I(x_0, b, E)$ for $x_0 \sim \overline{x}_0$ we multiply it by the positive quantity

$$1 - e_2^2 x_0^2 = 1 - \left(\frac{x_0}{\overline{x}_0}\right)^2, \tag{4.54}$$

where $\pm\overline{x}_0 = \pm 1/e_2$ represent the (simple) zeroes of the slow dynamics along the critical curve $\{y = 0\}$ for $\varepsilon = 0$. The expression in between brackets in (4.48) is in fact I multiplied by $(1 - e_2^2 x_0^2)$, expressed in $(b, E_1) = u(\overline{b}, \overline{E}_1)$, divided by u and evaluated at $u = 0$. We will now define in (4.59) below a smooth extension ψ of (4.54) and will then continue studying $\psi \cdot \frac{\partial\widetilde{\Delta}}{\partial x_0}$ instead of $\frac{\partial\widetilde{\Delta}}{\partial x_0}$ itself. Let us first show how to define ψ.

We are looking at system (3.1) or more precisely at

$$\begin{cases} \dot{x} = y + bxy - y^2 + \varepsilon^2(1 + E_1 x - e_2^2 x^2) \\ \dot{y} = xy + \varepsilon^3 D, \end{cases} \tag{4.55}$$

with $(b, D, E_1) \sim (0, 0, 0)$. (As before we make an abuse of notation by not changing the name of the parameter E_1 when passing to the chart $E_0 = 1$.)
These systems have a hyperbolic saddle near $(x, y) = (\overline{x}_0, 0)$ and $(x, y) = (-\overline{x}_0, 0)$ for $\varepsilon \sim 0, \varepsilon > 0$.
To prove this we use the center manifold theorem near these points providing C^∞ center manifolds given as graphs of functions $y^c(x)$ with

$$y^c(x) = \frac{D}{x}\varepsilon^3\left(-1 + \frac{1}{x^2}(1 + E_1 x - e_2^2 x^2)\varepsilon^2 + O(\varepsilon^3)\right). \tag{4.56}$$

(These are calculated as series in ε with coefficients given by functions of x.) On such a center manifold, always near $(\pm\overline{x}_0, 0)$ system (4.55) is described by

$$\dot{x}_0 = \varepsilon^2\left[(1 + E_1 x_0 - e_2^2 x_0^2) - \frac{(1 + bx_0)}{x_0} D\varepsilon + O(\varepsilon^2)\right], \tag{4.57}$$

and zeroes are given by

$$z^\pm = \pm\frac{1}{e_2} + O(E_1, \varepsilon). \tag{4.58}$$

The expression of z^+ and z^- depends on D, but as before we will restrict D to $D_0(\varepsilon, b, E, x_0)$, as used in (4.34).
Seen in the x_0-coordinate, as used from the start in Section 4.4, it is easy to see that closed orbits are only possible for $x_0 \in]z^-, z^+[$. We now consider

$$\psi(\varepsilon, b, E, x_0) = \left(1 - \frac{x_0}{z^-}\right)\left(1 - \frac{x_0}{z^+}\right) \tag{4.59}$$

as a natural extension of (4.54). As announced, instead of studying directly $\frac{\partial\widetilde{\Delta}}{\partial x_0}$, with $\widetilde{\Delta}$ as introduced in (4.34), we multiply $\frac{\partial\widetilde{\Delta}}{\partial x_0}$ by the function ψ given (4.59).

We will now show that $\psi \cdot \frac{\partial \tilde{\Delta}}{\partial x_0}$ is, for any $r \in \mathbb{N}$, C^r contact equivalent to an ε-regular C^r extension of the slow divergence integral, mutiplied by ψ. This will imply that $\psi \cdot \frac{\partial \tilde{\Delta}}{\partial x_0}$ is for any $r \in \mathbb{N}$, C^r contact equivalent to an ε-regular C^r extension of $\psi \cdot I$, which near $(b, E_1) = (0,0)$ and because of the fact that $\frac{\partial \tilde{\Delta}}{\partial x_0} \equiv 0$ at $(b, E_1) = (0,0)$ can be written as

$$u[(1 - e_2^2 x_0^2)\overline{I}(x_0, \overline{b}, \overline{E}_1, -e_2^2) + O(u) + O(\varepsilon)], \tag{4.60}$$

thus inducing that $\psi \cdot \frac{\partial \tilde{\Delta}}{\partial x_0}$, and hence also $\frac{\partial \tilde{\Delta}}{\partial x_0}$, can have at most one simple zero for $x_0 \sim \overline{x}_0$. The use of ψ only serves to be able to rely on the known results about $(1 - e_2^2 x_0^2)\overline{I}(x_0, \overline{b}, \overline{E}_1, -e_2^2)$. In fact it is possible to prove that, for any $r \geq 1$, $\frac{\partial \tilde{\Delta}}{\partial x_0}$ is C^r-contact equivalent to an ε-regular C^r extension of I. Seen that $\frac{\partial \tilde{\Delta}}{\partial x_0}$ is, for $\varepsilon > 0$, C^∞ contact equivalent to the (full) divergence integral multiplied by ε^2 and taking into account the known results in case of a regular dynamics, the full divergence integral only needs to be studied near the endpoints $x_0 = \pm\overline{x}_0$. Near each of these points we will prove that the full divergence integral multiplied by ε^2 is, for any $r \geq 1$, a C^r extension of the slow divergence integral.

The proof of this fact is based on the use of the normal linearization theorem of Takens [T], permitting to write (4.55) near the points $(z^+, y^c(z^+))$ and $(z^-, y^c(z^-))$ as

$$\begin{cases} \dot{v} = \varepsilon^2 \mu^{\pm}(\varepsilon, b, E) v \\ \dot{w} = \lambda^{\pm}(\varepsilon, b, E, u) w, \end{cases} \tag{4.61}$$

for which $\{w = 0\}$ represent a center manifold, $\{v = 0, w = 0\}$ is the singularity and $\lambda^{\pm}$, $\mu^{\pm}$ are, for given $r \in \mathbb{N}$, C^r functions with the following properties:

$$\lambda^+(0, 0, (1, 0, -e_2^2), 0) = \overline{x}_0$$

represents the unstable eigenvalue of (3.1) for $x_0 \sim \overline{x}_0$, and similarly

$$\lambda^-(0, 0, (1, 0, -e_2^2), 0) = -\overline{x}_0$$

represents the stable eigenvalue of (3.1) for $x_0 \sim -\overline{x}_0$:

$$\mu^+(0, 0, (1, 0, -e_2^2), 0) = -2/\overline{x}_0$$

represents the stable eigenvalue at $\overline{x}_0$ for the slow dynamics and

$$\mu^-(0, 0, 0(1, 0, -e_2^2), 0) = 2/\overline{x}_0$$

represents the unstable eigenvalue $-\overline{x}_0$ for the slow dynamics.
The product of the hyperbolicity ratios at both saddles is equal to 1, so we cannot use the results from [DD3].

The calculations that we intend to make near $(\overline{x}_0, 0)$ are similar to those near $(-\overline{x}_0, 0)$, so we will consider expression (4.61) with λ^- and μ^- and suppose that the side under consideration is $v > 0$. We write λ and μ instead of respectively λ^- and μ^-. In these coordinates (v, w), used in (4.61) we can now compare ε^2 times the divergence integral along orbits from $(v, 1)$ to $(1, w)$ to the slow divergence integral for $b = 0$ and $\overline{E}_1 = 0$. The divergence integral multiplied by ε^2 is given by

$$\frac{1}{\mu(\varepsilon, b, E)} \int_v^1 \frac{\lambda(\varepsilon, b, E, s) + \varepsilon^2 \mu(\varepsilon, b, E)}{s} ds; \tag{4.62}$$

the related slow divergence integral for $b = 0$ and $\overline{E}_1 = 0$ integral is given by

$$\frac{1}{\mu(0,0,(1,0,-e_2^2))} \int_v^1 \frac{\lambda(0,0,(1,0,-e_2^2),s)}{s} ds. \tag{4.63}$$

For simplicity in notation, let us introduce $\mu_0 = \mu(0,0,(1,0,-e_2^2))$,

$$\lambda_0(s) = \lambda(0,0,(1,0,-e_2^2),s) \text{ and } \delta = (\varepsilon, b, \overline{E}_1).$$

The slow divergence integral can then be written as:

$$\frac{1}{\mu_0} \int_v^1 \frac{\lambda_0(s)}{s} ds \tag{4.64}$$

and the full divergence integral, multiplied by ε^2, as:

$$\frac{1}{\mu_0 + 0(\|\delta\|)} \int_v^1 \frac{\lambda_0(s)(1+\phi(\varepsilon,b,E,s))}{s} ds, \tag{4.65}$$

with $\phi(\varepsilon,b,E,s) = O(\|\delta\|)$. We now write ϕ as:

$$\phi(\varepsilon,b,E,s) = \phi_0(\varepsilon,b,E) + s\phi_1(\varepsilon,b,E,s),$$

so that (4.65) can be written as:

$$\begin{aligned} &\tfrac{1}{\mu_0+O(\|\delta\|)} \int_v^1 \tfrac{\lambda_0(s)}{s} \left(1+\phi_0(\varepsilon,b,E)+s\phi_1(\varepsilon,b,E,s)\right)ds \\ &= \left(\tfrac{1+O(\|\delta\|)}{\mu_0+O(\|\delta\|)}\right) \int_v^1 \tfrac{\lambda_0(s)}{s} ds + O(\|\delta\|). \end{aligned} \tag{4.66}$$

This shows that the (full) divergence integral multiplied by ε^2 is a C^r extension of the slow divergence integral, thus proving the claim that we made on $\psi \cdot \frac{\partial\tilde{\Delta}}{\partial x_0}$. □

A Appendix: Proof of Proposition 4.1

In paragraph 3 the function $f(x_0, b)$ has been defined and we know that it plays an important role in the study of the cyclicity of the graphic $\Gamma_{x_0}^b$. It can be calculated, or at least its essential properties can be studied by merely considering the linear system, given by (2.12), that we repeat here:

$$\begin{cases} \dot{x} = 1 + bx - y \\ \dot{y} = x. \end{cases} \tag{A.1}$$

It is defined by considering an orbit of (A.1) starting at some point $(x_0, 0)$, with $x_0 \geq 0$, and following it until it hits again, for the first time, the x-axis at a point whose x-coordinate we define to be $f(x_0, b)$. Recall that $-2 < b < 2$, and that we can limit to $b \geq 0$.

We first introduce a first integral for (A.1), obtained by considering invariant straight lines $F(x,y)=0$, with

$$F(x,y) = 1 - Ax - y. \tag{A.2}$$

Clearly $\dot{F} = -AF$, if A satisfies $A^2 + Ab + 1 = 0$, hence

$$A, \bar{A} = \frac{1}{2}\left(-b \pm i(4-b^2)^{1/2}\right). \tag{A.3}$$

This gives the first integral:

$$H_b(x,y) = (1 - Ax - y)^{i\bar{A}}(1 - \bar{A}x - y)^{-iA}, \tag{A.4}$$

which on $\{y = 0\}$ gives:

$$H_b(x,0) = H_b^0(x) = (1 - Ax)^{i\bar{A}}(1 - \bar{A}x)^{-iA}. \tag{A.5}$$

We also see that

$$\begin{aligned} (H_b^0)'(x) &= H_b^0(x)\tfrac{iA\bar{A}(\bar{A}-A)x}{|A|^2x^2-2ReAx+1} \\ &= H_b^0(x)\tfrac{(4-b^2)^{1/2}x}{x^2+bx+1}, \end{aligned} \tag{A.6}$$

since $ReA = -\frac{b}{2}$, $A\bar{A} = 1$ and $ImA = \frac{1}{2}(4-b^2)^{1/2}$.

Writing x instead of x_0, we know that $f(x,b)$ is given by

$$\begin{aligned} &H_b^0(f(x,b)) = H_b^0(x), \\ &f(x,b) < 0, \end{aligned} \tag{A.7}$$

and hence:

$$f'(x,b) = \frac{\partial f}{\partial x}(x,b) = \frac{S(x,b)}{S(f(x,b),b)}, \tag{A.8}$$

with

$$S(x,b) = \frac{x}{x^2 + bx + 1}. \tag{A.9}$$

Indeed:

$$\begin{aligned} (H_b^0(f(x,b)))' &= (H_b^0)'(f(x,b)) \cdot f'(x,b) \\ &= (4-b^2)^{1/2}S(f(x,b))H_0(f(x,b)) \cdot f'(x,b) \\ &= (4-b^2)^{1/2}H_b^0(x)S(x). \end{aligned}$$

We can also calculate $\frac{\partial f}{\partial b}(x,b)$ by deriving

$$H^0(f(x,b),b) = H^0(x,b), \tag{A.10}$$

introducing $H^0(x,b) = H^0_b(x)$. We get:

$$\frac{\partial f}{\partial b}(x,b) = \frac{\frac{\partial H^0}{\partial b}(x,b) - \frac{\partial H^0}{\partial b}(f(x,b),b)}{\frac{\partial H^0}{\partial x}(f(x,b),b)}. \tag{A.11}$$

We would like to evaluate $\frac{\partial f}{\partial b}$ at $b=0$, where we already know that $f(x,b)=-x$. From (A.5) we get

$$\frac{\partial H^0}{\partial b}(x,b) = H^0(x,b)\left[\frac{-i\bar{A}x}{1-Ax}\frac{dA}{db} + \frac{iAx}{1-\bar{A}x}\frac{d\bar{A}}{db} + i\frac{d\bar{A}}{db}\ln(1-Ax) - i\frac{dA}{db}\ln(1-\bar{A}x)\right] \tag{A.12}$$

From $A^2+Ab+1=0$ we get

$$\frac{dA}{db} = -\frac{A}{2A+b},$$

and since $A=i$ and $\bar{A}=-i$ for $b=0$, this gives

$$\frac{dA}{db}\,|_{b=0} = \frac{d\bar{A}}{db}\,|_{b=0} = -\frac{1}{2}. \tag{A.13}$$

We need to use (A.12) to evaluate $\frac{\partial H^0}{\partial b}(f(x,0),0)$. Of course $H^0(x,0) = H^0(f(x),0)$. The bracket part of (A.12) is odd in x but we must pay attention to the logarithmic terms whose imaginary part depends on the argument of $1-if(x,0)$ and $1+if(x,0)$. The quantity $1-if(x,0)$ (resp. $1+if(x,0)$ is the analytic continuation of $1-ix-y$ (resp $1+ix-y$) along the trajectory joining $(x,0)$ to $(f(x),0)$. This trajectory surrounds the point (0,1) in the positive direction. Let $Y=y-1$. We work in the (x,Y) coordinates. Let $\pm\theta = \arg(\pm ix - Y)|_{y=0}$. When we move along the trajectory of the vector field, $ix-Y$ turns in the positive direction and $\arg(ix-Y)$ increases from θ to $2\pi-\theta$. Similarly $-ix-Y$ turns in the negative direction and $\arg(-ix-Y)$ decreases from $-\theta$ to $-2\pi+\theta$. Hence the contribution to the real part in the bracket expression of (A.12) is $-\theta$. When we evaluate the same expression at $f(x,0)$ the contribution to the real part is $\theta-2\pi$. It follows that

$$\begin{aligned}\tfrac{\partial H^0}{\partial b}(x,0) &= H^0(x,0)\left[\tfrac{x}{1+x^2} + \tfrac{i}{2}\ln\left|\tfrac{1+ix}{1-ix}\right| - \theta\right]\\ &= H^0(x,0)\left[\tfrac{x}{1+x^2} - \arctan x\right]\end{aligned} \tag{A.14}$$

and

$$\begin{aligned}\tfrac{\partial H^0}{\partial b}(f(x,0),0) &= H^0(f(x,0),0)\left[-\tfrac{x}{1+x^2} - \tfrac{i}{2}\ln\left|\tfrac{1+ix}{1-ix}\right| + \theta - 2\pi\right]\\ &= H^0(f(x,0),0)\left[-\tfrac{x}{1+x^2} + \arctan x - \pi\right].\end{aligned} \tag{A.15}$$

Using (A.6) with $b=0$ this implies

$$\begin{aligned}\varphi(x) = \tfrac{\partial f}{\partial b}(x,0) &= \frac{\frac{x}{x^2+1} - \arctan x + \pi}{-\frac{x}{x^2+1}}\\ &= -1 + \tfrac{x^2+1}{x}\arctan x - \pi\tfrac{x^2+1}{x}.\end{aligned} \tag{A.16}$$

Lemma A.1 *When $b>0$, then $f(x,b) < -x-b$ for all $x \geq 0$.*

Proof Consider

$$F(x,y) = x^2 + y^2 - bxy + bx - 2y.$$

F is a Lyapunov function for (A.1), having a minimum at $(0,1)$ with $F(0,1) = -1$. Indeed:

$$\begin{aligned}\dot F(x,y) &= bx^2 + by^2 - b^2xy - 2by + b^2x + b \\ &= G(x,y).\end{aligned}$$

To obtain that $G(x,y) \geq 0$, it suffices to check that G attains its minimum at $(0,1)$.

Now $F(x,0) = x^2 + bx$ is zero iff $x = 0$ or $x = -b$, implying already that

$$f(0,b) < -b.$$

We also see that $f(x,b)$ has to be inferior to $x_1 < 0$, where x_1 satisfies

$$F(x_1,0) = F(x,0).$$

This induces

$$(x_1 - x)(x_1 + x + b) = 0,$$

implying that $f(x,b) + x + b < 0$ as claimed.

□

We can also observe that

$$f''(x,b) = \frac{\partial^2 f}{\partial x^2}(x,b) = \frac{((f(x,b))^2 + bf(x,b) + 1)}{(x^2+bx+1)^2} \cdot \frac{(x^2 - (f(x,b))^2)}{(f(x,b))^3} < 0, \tag{A.17}$$

implying that the graph of $f(x,b)$, for each $b > 0$ separately, is a concave curve. It intersects $\{x = 0\}$ at $f(0,b) < -b < 0$, and stays below the straight line with equation

$$f = -x - b$$

in a (x,f)-plane.
One can show that

$$\lim_{x\to\infty} \frac{f(x,b)}{x} = -1.$$

References

[ADL] J.C. Artés, F. Dumortier, J. Llibre, Limit cycles near hyperbolas in quadratic systems, preprint.

[ALS] J.C. Artés, J. Llibre, D. Schlomiuk, Quadratic differential systems with a weak focus of second order, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **16** (2006), 3127–3194.

[DD1] P. De Maesschalck, F. Dumortier, Time analysis and entry-exit relation near planar turning points, *Journal of Differential Equations*, **215** (2005), 225–267.

[DD2] P. De Maesschalck, F. Dumortier, Canard solutions at non-generic turning points, *Transactions of the American Mathematical Society*, **358** (2006), 2291–2334.

[DD3] P. De Maesschalck, F. Dumortier, Canard cycles in the presence of slow dynamics with singularities, *Proc. Royal Society of Edinburgh-A*, to appear.

[DPR] F. Dumortier, D. Panazzolo and R. Roussarie, More limit cycles than expected in Liénard equations, *Proc. Amer. Math. Soc.* 135 (2007), 1895–1904.

[DR1] F. Dumortier, R. Roussarie, Multiple canard cycles in generalized Liénard equations, *Journal of Differential Equations*, **174** (2001), 1–29.

[DR2] F. Dumortier, R. Roussarie, Canard cycles with two breaking parameters, *Discrete and Continuous Dynamical Systems, Series A* **17**, nr4 (2007), 787–806.

[DRR] F. Dumortier, R. Roussarie, and C. Rousseau, Hilbert's 16-th problem for quadratic vector fields, *J. Differential Equations* **110** (1994), 86–133.

[R] C. Rousseau, Normal forms, bifurcations and finiteness properties of vector fields, in *Normal forms, bifurcations and finiteness properties of vector fields*, NATO Sci. Ser. II Math. Phys. Chem., **137**, Kluwer Acad. Publ., Dordrecht, 2004, 431–470.

[RS] R. Roussarie and D. Schlomiuk, On the geometric structure of the class of quadratic systems, in *Qual. Theory Dyn. Syst.* **3** (2002), 93–121.

[T] F. Takens, Partially hyperbolic fixed points, *Topology* **10** (1974), 133–147.