BIFURCATION ANALYSIS OF A GENERALIZED GAUSE MODEL WITH PREY HARVESTING AND A GENERALIZED HOLLING RESPONSE FUNCTION OF TYPE III*

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Abstract. In this paper we study a generalized Gause model with prey harvesting and a generalized Holling response function of type III: $p(x) = \frac{mx^2}{ax^2+bx+1}$. The goal of our study is to give the bifurcation diagram of the model. For this we need to study saddle-node bifurcations, Hopf bifurcation of codimension 1 and 2, heteroclinic bifurcation, and nilpotent saddle bifurcation of codimension 2 and 3. The nilpotent saddle of codimension 3 is the organizing center for the bifurcation diagram. The Hopf bifurcation is studied by means of a generalized Liénard system, and for $b = 0$ we discuss the potential integrability of the system. The nilpotent point of multiplicity 3 occurs with an invariant line and can have a codimension up to 4. But because it occurs with an invariant line, the effective highest codimension is 3. We develop normal forms (in which the invariant line is preserved) for studying of the nilpotent saddle bifurcation. For $b = 0$, the reversibility of the nilpotent saddle is discussed. We study the type of the heteroclinic loop and its cyclicity. The phase portraits of the bifurcations diagram (partially conjectured via the results obtained) allow us to give a biological interpretation of the behavior of the two species.

Keywords. generalized Gause model with prey harvesting, generalized Holling response function of type III, saddle-node bifurcation, Hopf bifurcation, heteroclinic bifurcation, nilpotent saddle bifurcation.

1. Introduction

The first predator-prey model has been suggested independently by A. Lotka(1925) [28] and V. Volterra(1926) [34]. Since that time, the models are refined so as to better reflect the specific characteristics of the different populations. The proposed models usually depend on parameters and are studied through bifurcation theory. The evolution of a population x submitted to regular harvesting is modeled by (see [3])

$$\dot{x} = F(x) - S(x, h),$$

where $F(x)$ describes the dynamics of the population without harvesting, and $S(x, h)$ is the harvesting rate; the parameter h is called the *intensity of harvesting*. There exists two standard harvesting strategies ([3], [4]): the first one consists in harvesting a constant number of individuals per unit of time, modeled by a constant

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rate, $S(x,h) = h$; in the second strategy, the number of individuals harvested per unit of time is proportional to the population, $S(x,h) = hx$. More sophisticated strategies, as periodic harvesting, etc., are also studied. In this paper we choose the first strategy.
The study of the dynamics of a harvested population is a topic studied in mathematical bioeconomics ([4], [15], [36]), inside a larger chapter dealing with optimal management of renewable resources (see Clark [8]). The exploitation of biological resources and the harvesting of interacting species is applied in fisheries, forestry and fauna management ([4], [15], [36]). According to Clark [8], the management of renewable resources is based on the notion of *maximum sustainable yield* (MSY) of harvesting; the MSY is the maximum harvesting compatible with survival. Hence, if the harvesting of a population exceeds its MSY (i.e. the population is over-exploited [36]), then this population will become extinct.
Qualitatively, the study of a predator-prey model with harvesting of preys is more involved than that of a mere predator-prey model. From the point of view of renewable resources, we must determine the MSY for the harvesting (when harvesting is allowed) of each population and give conditions assuring the preservation of species ([4], [15] and [36]).
The system that we consider is called in the literature *generalized Gause model with harvesting of prey* ([23], [19], [21], [35], [2]). It has the form

$$\begin{cases} \dot{x} = g(x) - yp(x) - h_1, \\ \dot{y} = y[-d + cp(x)], \end{cases} \tag{1.1}$$

where

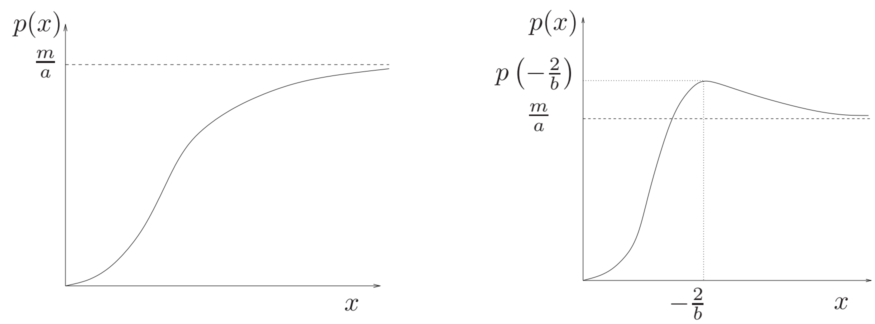
- x represents the population of preys;
- y represents the population of predators;
- d is the natural mortality rate of predators,
- the function $g(x) = rx(1 - \frac{x}{k})$ models the behavior of preys in absence of predators: r is the growth rate of preys when x is small, while k is the capacity of the environment to support the preys;
- the constant h_1 is the rate of harvesting of preys.

The function

$$p(x) = \frac{mx^2}{ax^2 + bx + 1} \tag{1.2}$$

(where m and a are positive constant, and b is an arbitrary constant), called in the literature *generalized Holling response function of type III* [3], is one of the potential *response functions* of predators to preys, modeling the consumption of preys by predators. It reflects very small predation when the number of preys is small ($p'(0) = 0$), and a group advantage for the preys when the number of preys is high ($p(x)$ tends to $\frac{m}{a}$ when x tends to infinity). There exists several types of generalized Holling response function of type III, depending whether the group advantage is weak or strong (Figure 1.1): if b is negative, the group advantage is stronger than when b is positive. We also note that, when b is negative, this function increases to a maximum and then decreases, approaching $\frac{m}{a}$ as x approaches infinity. Thus, when b is negative, $p(x)$ models the situation where the prey can better defend or disguise themselves when their population become large enough: this phenomenon is called *group defense* ([38], [37], [35], [21]). In this paper we limit ourselves to the case $b \geq 0$. Recently, in the same spirit, but without harvesting of populations, Broer-Naudot-Roussarie-Saleh (see [5]) and Coutu-Lamontagne-Rousseau (see [29]) respectively studied a predator-prey system with Holling response function of type IV ($p(x) = \frac{mx}{ax^2+bx+1}$) and a predator-prey system with generalized Holling response function of type III (with $b > 0$, $b = 0$, or $b < 0$). Note that, for the Holling response function of type IV, the response of predators goes to zero when the population of preys is very large, thus modeling a very large group advantage for the preys.

From the biological point of view, it is interesting to determine how the harvesting of preys affects the sub-system with no harvesting $(1.1)|_{h_1=0}$, when $p(x)$ is given in (1.2).



(a) for $b \geq 0$ (b) for $b < 0$

FIGURE 1.1. Generalized Holling function of type III.

Hence, we study the following system:

$$\begin{cases} \dot{x} = rx(1 - \frac{x}{k}) - \frac{mx^2y}{ax^2+bx+1} - h_1, \\ \dot{y} = y(-d + \frac{cmx^2}{ax^2+bx+1}), \\ x \geq 0, y \geq 0, \end{cases} \tag{1.3}$$

where the eight parameters: r, k, m, a, c, d, h_1 are strictly positive and $b \geq 0$. Through the following linear transformation and time scaling

$$(X, Y, T) = \left(\frac{1}{k}x, \frac{1}{ck}y, cmk^2t\right),$$

we can reduce the number of parameters to five: The simplified system that we consider is the following

$$\begin{cases} \dot{x} = \rho x(1 - x) - yp(x) - \lambda, \\ \dot{y} = y(-\delta + p(x)), \\ x \geq 0, y \geq 0, \quad \textit{where} \end{cases} \tag{1.4}$$

$$p(x) = \frac{x^2}{\alpha x^2 + \beta x + 1} \tag{1.5}$$

and the parameters are

$$(\rho, \alpha, \beta, \delta, \lambda) = \left(\frac{r}{cmk^2}, ak^2, bk, \frac{d}{cmk^2}, \frac{h_1}{cmk^3}\right). \tag{1.6}$$

The bifurcation diagram will reveal surprising biological consequences and highlight that one must be very careful with a constant rate harvesting strategy. Even with a very small harvesting, this strategy leads systematically to the extinction of both species when the mortality rate of predators is small. Even with parameter values compatible with the survival of both species, the region of initial conditions leading to survival is not necessarily very large. Another surprising conclusion: if we start harvesting when the number of preys is high, this automatically leads to extinction, even with small harvesting. On the contrary, if the number of preys is small we can have survival for the same initial number of predators. Hence, our study highlights the need to study new harvesting strategies or models with simultaneous harvesting of predators and preys, so as to identify the strategies preventing the extinction of species.

The paper is organized as follows. Section 2 contains a summary of results. In Section 3 we show that all trajectories remaining in the first quadrant are attracted in a finite region of the plane. In Section 4 we study the number of singular points, their type and the saddle-node bifurcations. The Hopf bifurcation of codimension 1 and 2 is studied in Section 5. In Section 6, we discuss the bifurcation of nilpotent saddle of codimension 2 and 3 (this nilpotent saddle of codimension 3 is the organizing center of our bifurcation diagram). Finally, in Section 7, we give the global bifurcation diagram (the small conjectural part of it is clearly identified). In Section 8, we deduce the biological interpretation of potential behaviors depending on the parameter values and of the initial conditions.

Remark 1.1. *(i) When λ is small, the system (1.4) is a perturbation of the sub-system $(1.4)|_{\lambda=0}$. The bifurcation diagram of the sub-system $(1.4)|_{\lambda=0}$ (determined in* [29]*) is necessary to understand the bifurcation diagram of the system (1.4) when λ is small (see Figure 2.2).*
(ii) The parameter $\lambda_1 := \frac{\lambda}{\rho}$ is important since all equations giving the locus of bifurcation surfaces are homogeneous in λ and ρ.
(iii) Our computations were done with MAPLE.

2. Summary of the results

The x-axis of the system (1.3) is invariant. The system has 2 singular points, C and D, on the positive x-axis for $\rho > 4\lambda$ and none for $\rho < 4\lambda$, the two points merging in a saddle-node for $\rho = 4\lambda$. In the first quadrant, there is at most one singular point E which is always of anti-saddle type, (i.e., a node, focus, weak focus or center). The singular point E disappears from the first quadrant by a saddle-node bifurcation while merging, with either C, or D. The point E can undergo a Hopf bifurcation of order at most two. When the order is two, the second Lyapunov coefficient is positive (the weak focus is repelling). Thus, when the system has two limit cycles, the attracting cycle is surrounded by a repelling cycle.

Theorem A. *If $\rho = 4\lambda$ and $\delta = \frac{1}{\alpha+2\beta+4}$, the three singular points C, D and E merge in $B = (\frac{1}{2}, 0)$. This point is a nilpotent saddle. If $\alpha = \frac{\beta^2+8\beta+24}{\beta+6}$ with*

$\beta > 0$, this nilpotent saddle is of codimension 4. However, since the horizontal axis is invariant, the codimension of one less. This point is the organizing center of the bifurcation diagram.

We succeeded in highlighting the three principal parameters of the system, that is to say λ, α and δ. This allows to give the bifurcation diagram in the (α, δ)-plane for various values of λ.

Theorem B. *The bifurcation diagram with phase portraits of the model (1.4) is, according to the values of the parameter $\lambda > 0$, presented to the Figures 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 and 2.8 by using the notations of Table 2.1. It is the simplest bifurcation diagram compatible with all the constraints of the system. The Hopf and saddle-nodes bifurcations curves are exact. Are conjectured:*

(1) *the exact position of the heteroclinic loop bifurcation curve, but this curve cuts any line $\delta = constant$ exactly once;*
(2) *the uniqueness of the locus of the bifurcations of codimension 2 (namely H_2, $\mathcal{C}$ and HL_2) and, consequently, the triangle $\mathcal{C} - H_2 - HL_2$ which, for $\beta > 0$, moves towards the line $\delta = 0$ when λ decreases and, for $\beta = 0$, is reduced to the point $\mathcal{C} = (\alpha, \delta)$ defined by $\alpha = \frac{\rho}{\lambda}$ and $\delta = \frac{\lambda}{2\rho}$.*

H_a:	attracting Hopf bifurcation
H_r:	repelling Hopf bifurcation
H_2:	Hopf bifurcation of codimension two
HL_a:	attracting heteroclinic loop bifurcation
HL_r:	repelling heteroclinic loop bifurcation
HL_2:	heteroclinic loop bifurcation of codimension two
DC:	double limit cycle
C:	intersection of (H) and (HL)
B_+:	nilpotent saddle bifurcation with positive X^2Y coefficient
B_-:	nilpotent saddle bifurcation with negative X^2Y coefficient
SN_{a_i}:	internal attracting saddle-node
SN_{r_i}:	internal repelling saddle-node
SN_{a_e}:	external attracting saddle-node
SN_{r_e}:	external repelling saddle-node

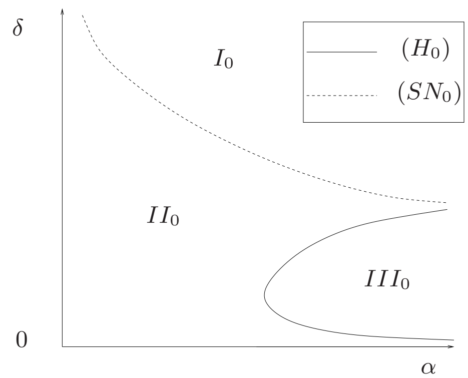
TABLE 2.1. Description of the bifurcation curves of Figure 6.3.

When $\beta = 0$, we conjecture that the system has a center as soon as the order of the bifurcation is greater or equal to two, the bifurcation diagram being that of Figure 2.8.

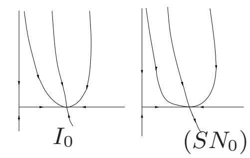
3. Behaviour of trajectories at infinity

In this section we show that all trajectories of (1.4) remaining in the first quadrant are attracted to a finite region of the plane.

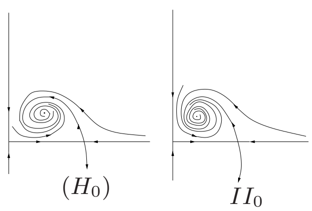
Theorem 3.1. *For all $\alpha, \beta, \delta, \rho, \lambda$ defined in (1.4), there exists a rectangle $R = [0, 1] \times [0, l]$, where $l = l(\alpha, \beta, \delta, \rho, \lambda)$, with the following property: For any trajectory γ with initial condition in the first quadrant,*



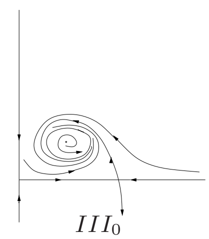
(a) Bifurcation diagram for $\lambda = 0$



(b)



(c)



(d)

FIGURE 2.1. Bifurcation diagram and phase portraits for $\lambda = 0$ ([29] or [17]).

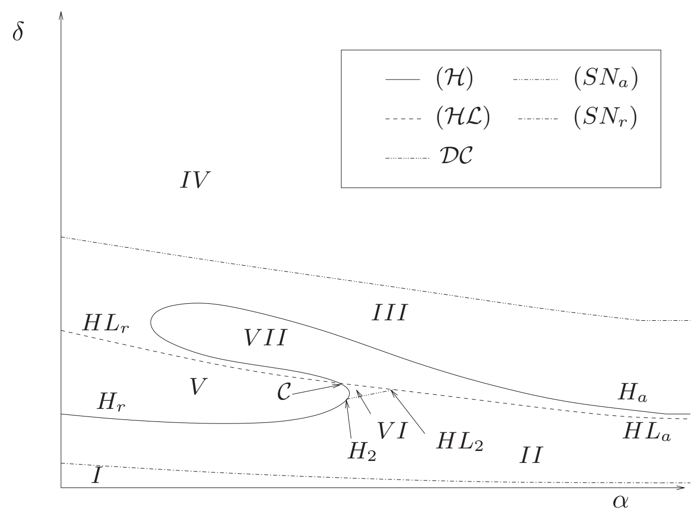


FIGURE 2.2. Bifurcation diagram when $\beta > 0$ and λ is small.

: *(i) either γ escapes from the first quadrant by crossing the positive y-axis;*
: *(ii) or γ has its ω-limit set inside R.*

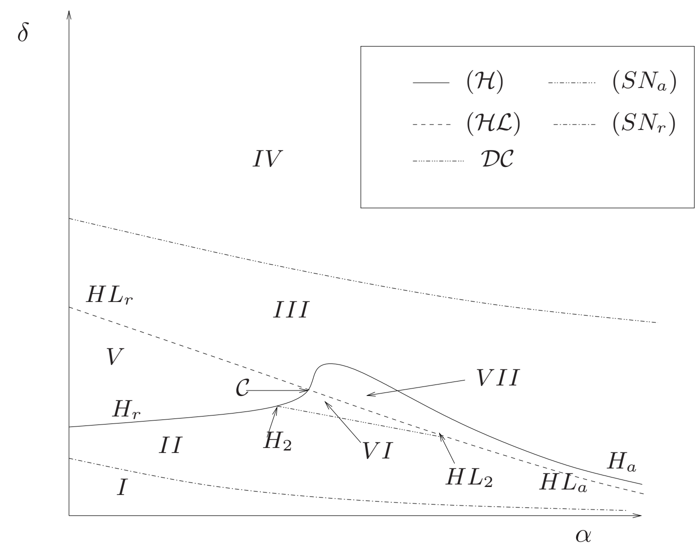


FIGURE 2.3. Bifurcation diagram when $\beta > 0$ and λ is neither small, nor close to $\frac{\varrho}{4}$.

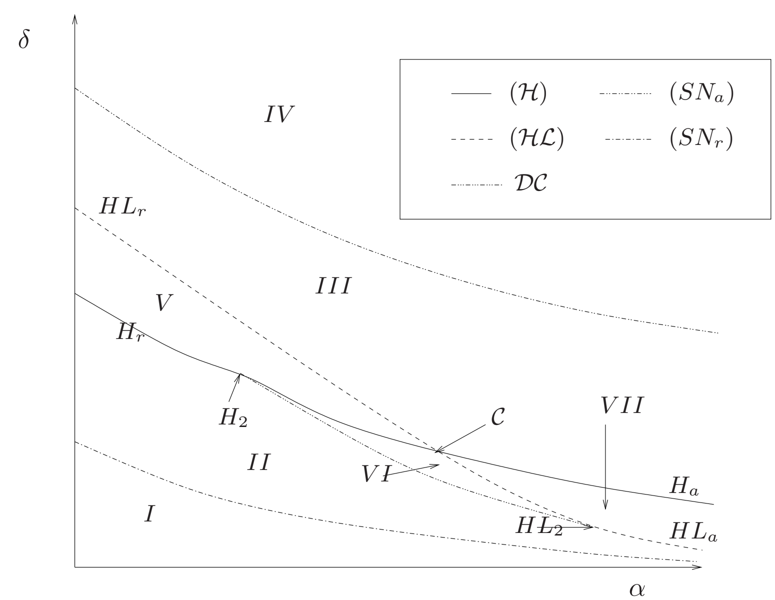


FIGURE 2.4. Bifurcation diagram when $\beta > 0$ and λ is close to $\frac{\varrho}{4}$.

Only, case (ii) can happen when $\lambda = 0$ *since the axes are invariant.*

Proof. We treat the case $\lambda > 0$, since the case $\lambda = 0$ is easy. Indeed, all trajectories of (1.4) passing through the y-axis must leave the first quadrant. For $x > 1$, one has that $\dot{x} < 0$ and $\frac{dy}{dx}$ is bounded for large y; then all the trajectories enter the region "$x \leq 1$ ". If $\delta > \frac{1}{\alpha}$, then $\dot{y} < -\eta$ for some positive η and, for any $l > 0$,

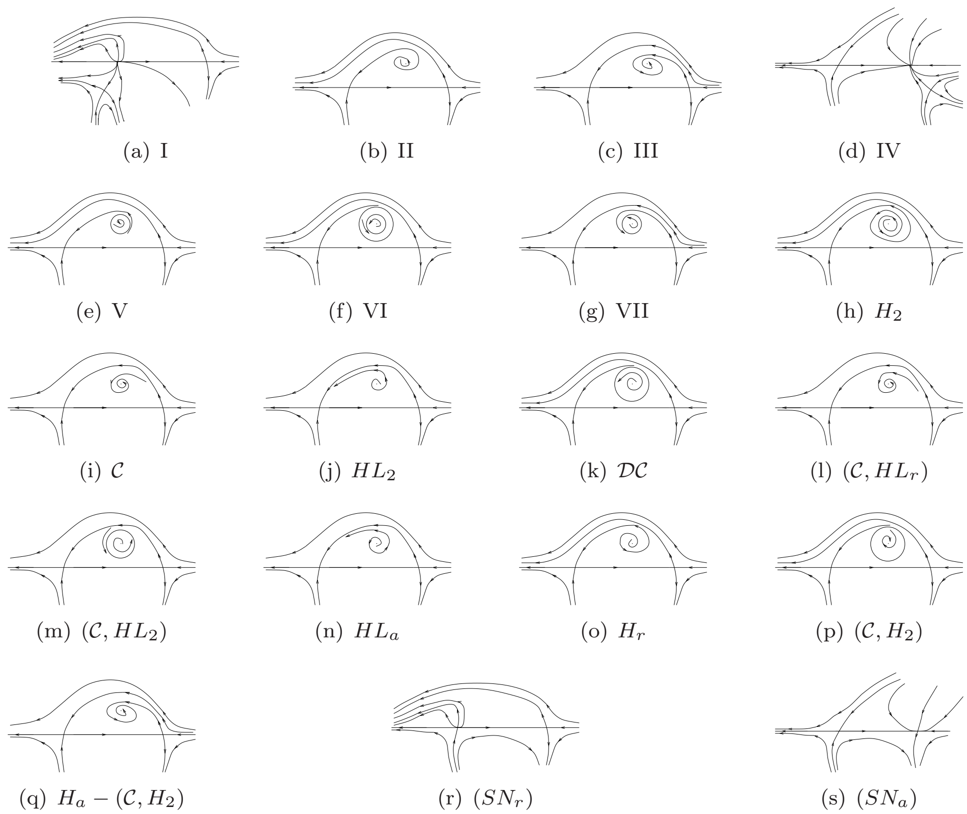


FIGURE 2.5. Phase portraits of the bifurcation diagrams in Figure 2.2, Figure 2.3 and Figure 2.4.

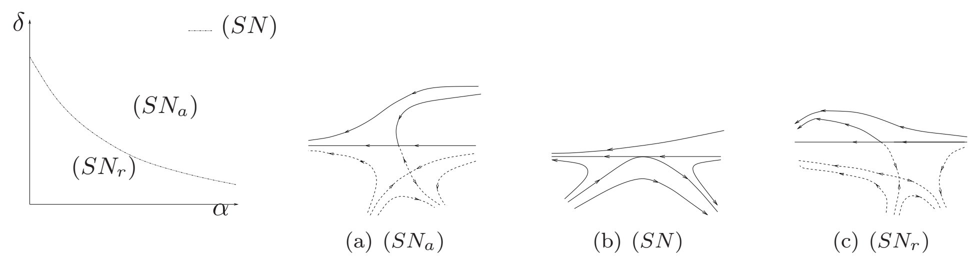


FIGURE 2.6. Bifurcation diagram and phase portraits when $\lambda = \frac{\rho}{4}$.

all trajectories enter the region "$y \leq l$ " or leave the first quadrant. If $\delta \leq \frac{1}{\alpha}$, let $x_p(\alpha, \beta, \delta)$ the solution of $p(x) = \delta$. Then, $\dot{y} < 0$ if, and only if $x \in [0, x_p(\alpha, \beta, \delta)]$. Let $\epsilon \in]0, x_p(\alpha, \beta, \delta)[$ smaller than the x-coordinate of the leftmost singular point on the x-axis. Then $\dot{x}, \dot{y} < 0$ in the strip "$x \in [0, \epsilon]$ " and all trajectories escape through the y-axis. Moreover, in the region "$x \in [0, 1]$ ", we have that $\dot{x} < 0$ if, and

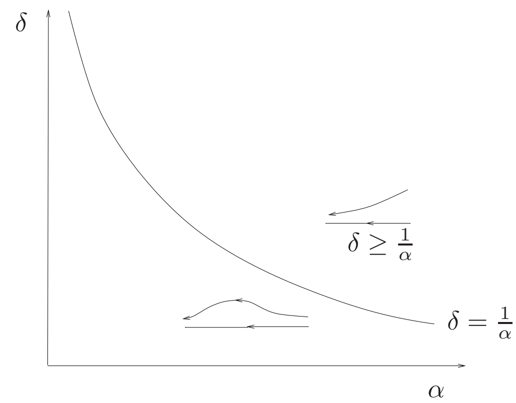


FIGURE 2.7. Bifurcation diagram and phase portraits when $\lambda > \frac{\rho}{4}$.

only if $y > \frac{\rho x(1-x)-\lambda}{p(x)}$. Let

$$N := \max_{x\in[\epsilon,1]} \frac{\rho x(1-x)-\lambda}{p(x)}.$$

The slope of the field is given by

$$\frac{dy}{dx} = \frac{\delta - p(x)}{p(x) - \frac{\rho x(1-x)-\lambda}{y}}. \tag{3.1}$$

However, one has that

$$\lim_{y\to+\infty} \frac{dy}{dx} = \frac{\delta - p(x)}{p(x)} \quad \textit{is bounded on} \quad [\epsilon, 1]. \tag{3.2}$$

Thus, there exists at least a trajectory $(x(t), y(t)), t \in [0, T]$ such that $x(0) = 1$, $x(T) = \epsilon$ and, for all t, $y(t) \geq N$. Let $(x_1(t), y_1(t)), t \in [0, T_1]$ be the lowest trajectory verifying this property and let

$$l := \max_{t\in[0,T_1]} y_1(t).$$

By (3.2), it comes that the trajectories cannot go at infinity in the half-strip "$x \in [\epsilon, 1], y > N$" and must thus enter the strip "$x \in [0, \epsilon]$" (where $\dot{x}, \dot{y} < 0$). Hence, it will exit the first quadrant through the y-axis. So, the only trajectories that may have their ω limit set in the first quadrant enter R through either the right side or the top side. □

4. Bifurcations and type of the singular points

4.1. Number of singular points.

Theorem 4.1. *Let*

$$x_{01} = \frac{1}{2} - \frac{\sqrt{\rho(\rho-4\lambda)}}{2\rho}, x_{02} = \frac{1}{2} + \frac{\sqrt{\rho(\rho-4\lambda)}}{2\rho}. \tag{4.1}$$

The number of singular points of the system (1.4), according to the values of the parameters, is given to Table 4.1.

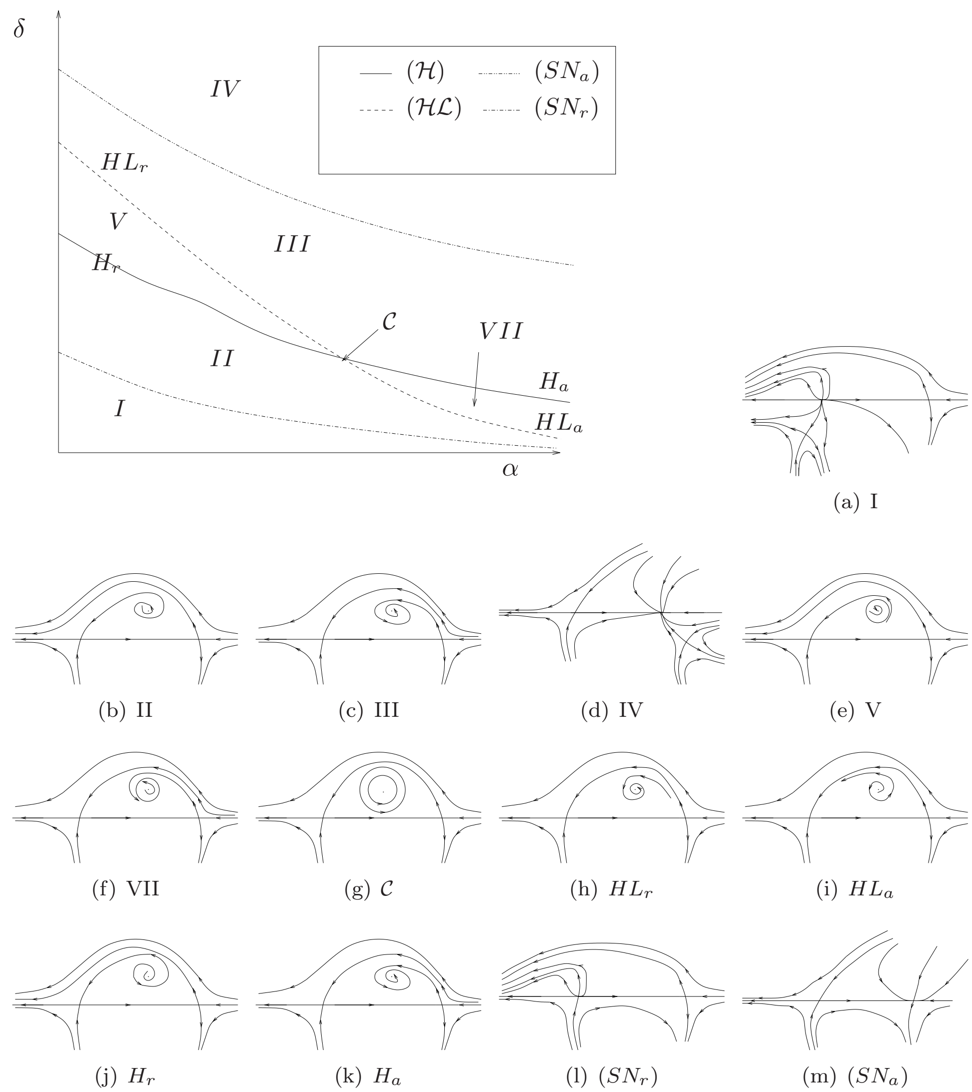


FIGURE 2.8. Bifurcation diagram and phase portraits when $\beta = 0$ and $\lambda \in]0, \frac{\rho}{4}[$.

Proof. The singular points of (1.4) have coordinates (x_0, y_0), where x_0, y_0 are solutions of the system with unknown (x, y)

$$\begin{cases} \rho x(1-x) - yp(x) - \lambda = 0, \\ y(-\delta + p(x)) = 0, \end{cases} \tag{4.2}$$

such that $x_0 \geq 0, y_0 \geq 0$.
By the second equation of (4.2), one has $y = 0$ or $p(x) = \delta$. Then:

(1) For $y = 0$, the first equation of (4.2) gives:

$$\rho x^2 - \rho x + \lambda = 0 \tag{4.3}$$

Region	Singular points
$\rho < 4\lambda$	none
$\rho = 4\lambda$	$(\frac{1}{2}, 0)$: *double point if* $\delta \neq \frac{1}{\alpha+2\beta+4}$ and, *triple point if* $\delta = \frac{1}{\alpha+2\beta+4}$
$\rho > 4\lambda$ and $x_0 \in]x_{01}, x_{02}[$	$(x_{01}, 0)$, $(x_{02}, 0)$ and (x_0, y_0) where $p(x_0) = \delta$ and $y_0 = \frac{\rho x_0(1-x_0)-\lambda}{\delta}$
$\rho > 4\lambda$ and $x_0 = x_{01}$	$(x_{01}, 0)$ double point and $(x_{02}, 0)$
$\rho > 4\lambda$ and $x_0 = x_{02}$	$(x_{01}, 0)$ and $(x_{02}, 0)$ double point
$\rho > 4\lambda$ and $x_0 \in]0, x_{01}[\cup]x_{02}, +\infty[$	$(x_{01}, 0)$ and $(x_{02}, 0)$

TABLE 4.1. Number of singular points of the system (1.4)

whose discriminant is $\Delta_1 := \rho(\rho - 4\lambda)$. It follows that:
• When $\rho < 4\lambda$, there is no singular point on $y = 0$.
• When $\rho = 4\lambda$, then $(\frac{1}{2}, 0)$ is a double singular point.
• When $\rho > 4\lambda$, then (4.3) has two solutions x_{01} and x_{02} given in (4.1) and corresponding to singular points $(x_{01}, 0)$ and $(x_{02}, 0)$.

(2) For $p(x) = \delta$, one looks for $x_0 \geq 0$ such that $p(x_0) = \delta$ and

$$y_0 = \frac{1}{\delta}[\rho x_0(1 - x_0) - \lambda]. \tag{4.4}$$

However, $p(x) = \delta$ if and only if

$$f(x) = (\alpha\delta - 1)x^2 + \beta\delta x + \delta = 0. \tag{4.5}$$

Hence:
a) If $\alpha\delta - 1 = 0$, then the only real solution of (4.5) is $x_{01} = \frac{-1}{\beta} < 0$.
b) If $\alpha\delta - 1 \neq 0$, then the discriminant of (4.5) is

$$\Delta_2 := (\beta\delta)^2 - 4\delta(\alpha\delta - 1) :$$

b-1) If $\alpha\delta - 1 > 0$ and $\Delta_2 > 0$ (resp. $\Delta_2 < 0$) then (4.5) has two solutions whose product is $\frac{\delta}{\alpha\delta-1} > 0$ and whose sum is $-\frac{\beta\delta}{\alpha\delta-1} < 0$ (resp. (4.5) does not have any solution): therefore, there is no admissible singular point when $\alpha\delta - 1 > 0$.
b-2) If $\alpha\delta - 1 < 0$, then $\Delta_2 > 0$. Thus, the product of the solutions of (4.5) is negative and the positive solution is

$$x_0 = \frac{\beta\delta + \sqrt{\delta[\delta(\beta^2 - 4\alpha) + 4]}}{-2(\alpha\delta - 1)}. \tag{4.6}$$

Therefore, one obtains at most one singular point (x_0, y_0) where x_0, defined by $p(x_0) = \delta$, verifies (4.6). We also need $y_0 > 0$. Since y_0 is defined by (4.4), its sign is exactly that of $-\rho x_0^2 + \rho x_0 - \lambda$ whose discriminant is $\Delta_1 := \rho(\rho - 4\lambda)$. Consequently:
• If $\rho < 4\lambda$, then there are no singular points in the first quadrant because any solution (x_0, y_0) with x_0 defined in (4.6) satisfies $y_0 < 0$.
• If $\rho = 4\lambda$ and x_0 is defined in (4.6), we have $y_0 = -\rho(x_0 - \frac{1}{2})^2 \leq 0$. In particular, by (4.4), (4.1) and (4.5), one has the singular point $(x_0 = \frac{1}{2}, y_0 = 0)$ if, and only if $\delta = \frac{1}{\alpha+2\beta+4}$.
• If $\rho > 4\lambda$, then, by (4.4), $y_0 \geq 0 \Leftrightarrow x_0 \in [x_{01}, x_{02}] \subset]0; +\infty[$, where x_{01} and x_{02} are defined in (4.1).

□

Remark 4.2 (Interesting cases of b-2)). ***b-2-1)*** *If $\rho = 4\lambda$ and $\delta = \frac{1}{\alpha+2\beta+4}$ then $\alpha\delta - 1 < 0$, and the singular point $(\frac{1}{2}, 0)$ becomes triple because $x_0 = \frac{1}{2}$ is a double solution of (4.3) and a simple solution of (4.5).*
b-2-2) *If $\rho > 4\lambda$ and $x_0 = x_{01}$ or $x_0 = x_{02}$ such that $p(x_0) = \delta$, then the singular point $(x_0, 0)$ is double, because x_0 is solution of (4.3) and (4.5). The equations $x_0 = x_{01}$ or $x_0 = x_{02}$ represent two surfaces in the product of x-space by the parameter space: we will determine their equation in the parameter space when we study the saddle-node bifurcations.*

4.2. **Type of singular points.** The Jacobian matrix of (1.4) in (x, y) is given by

$$\mathbf{Jac(x,y)} = \begin{pmatrix} \rho - 2\rho x - \frac{xy(\beta x+2)}{(\alpha x^2+\beta x+1)^2} & -p(x) \\ \frac{xy(\beta x+2)}{(\alpha x^2+\beta x+1)^2} & -\delta + p(x) \end{pmatrix}. \tag{4.7}$$

By Table 4.1, we need to study it in two cases: $\rho = 4\lambda$ and $\rho > 4\lambda$. In this section we limit ourselves to the case $\rho > 4\lambda$ and leave the case $\rho = 4\lambda$ for the Sections 4.3 and 6.

Theorem 4.3. *For $\rho > 4\lambda$, the type of the singular points $C = (x_{01}, 0)$, $D = (x_{02}, 0)$ and $E = (x_0, y_0)$ is, according to values of the parameters, given in Table 4.2.*

Region	Singular points	Type
$\delta < p(\frac{1}{2} - \eta)$	C, D, E	C is a repelling node D and E are hyperbolic saddle E is nonadmissible
$\delta = p(\frac{1}{2} - \eta)$	C, D	C is a repelling saddle-node (studied below) D is hyperbolic saddle
$p(\frac{1}{2} - \eta) < \delta < p(\frac{1}{2})$	C, D, E	C and D are hyperbolic saddles E is a anti-saddle
$p(\frac{1}{2}) \le \delta < p(\frac{1}{2} + \eta)$	C, D, E	C and D are hyperbolic saddles E is an attracting (focus/node)
$\delta = p(\frac{1}{2} + \eta)$	C, D	C is hyperbolic saddle D is an attracting saddle-node (studied below)
$\delta > p(\frac{1}{2} + \eta)$	C, D, E	D is an attracting node C and E are hyperbolic saddles E is nonadmissible

TABLE 4.2. Types of the singular points when $\rho > 4\lambda$

Proof. (1) *Type of the singular points $C = (x_{01}, 0)$ and $D = (x_{02}, 0)$*
We have that $x_{01} := \frac{1}{2} - \eta$, $x_{02} := \frac{1}{2} + \eta$, where

$$\eta := \frac{\sqrt{\rho(\rho - 4\lambda)}}{2\rho}. \tag{4.8}$$

Moreover, $x_{01}x_{02} = \frac{\lambda}{\rho}$ and $x_{01} + x_{02} = 1$. By (4.7), we have respectively

(4.9)
$$\mathbf{Jac(C)} = \begin{pmatrix} 2\rho\eta & -p(\frac{1}{2}-\eta) \\ 0 & -\delta + p(\frac{1}{2}-\eta) \end{pmatrix} \quad and \quad \mathbf{Jac(D)} = \begin{pmatrix} -2\rho\eta & -p(\frac{1}{2}+\eta) \\ 0 & -\delta + p(\frac{1}{2}+\eta) \end{pmatrix}.$$

Moreover, $p(\frac{1}{2} \pm \eta) > 0$ and $p^{'}(x) = \frac{\beta x^2+2x}{(\alpha x^2+\beta x+1)^2} > 0$ for all $x > 0$; i.e. p is strictly increasing on $]0, +\infty[$. Since $0 < \frac{1}{2} - \eta < \frac{1}{2} + \eta$, we have

$$p\left(\frac{1}{2}-\eta\right) < p\left(\frac{1}{2}+\eta\right). \tag{4.10}$$

The type of the points C and D is thus that of Table 4.2.

(2) *Type of the singular point $E = (x_0, y_0)$*
The Jacobian matrix, (4.7), evaluated at E is

$$\mathbf{Jac(x_0, y_0)} = \begin{pmatrix} \rho(1-2x_0) - y_0 p^{'}(x_0) & -\delta \\ y_0 p^{'}(x_0) & 0 \end{pmatrix}$$

whose trace and determinant are respectively

$$Tr(\mathbf{Jac(x_0, y_0)}) = \rho(1-2x_0) - y_0 p'(x_0),$$
$$Det(\mathbf{Jac(x_0, y_0)}) = \delta y_0 p^{'}(x_0) > 0.$$

Moreover, $x_{01} < \frac{1}{2} < x_{02}$. Hence,

i) If $x_0 \in [\frac{1}{2}, x_{02}[$, corresponding to the region of parameter space $p(\frac{1}{2}) \leq \delta < p(\frac{1}{2}+\eta)$, then $1-2x_0 < 0$ and $y_0 p^{'}(x_0) > 0$; thus, $Tr(\mathbf{Jac(x_0, y_0)}) < 0$, and E is thus an attracting (node/focus).

ii) If $x_0 \in]x_{01}, \frac{1}{2}[$, corresponding to the region of parameter space $p(\frac{1}{2} - \eta) < \delta < p(\frac{1}{2})$, then $\rho(1-2x_0) > 0$ and $y_0 p^{'}(x_0) > 0$. Therefore, $Tr(\mathbf{Jac(x_0, y_0)}) := \rho(1-2x_0) - y_0 p^{'}(x_0)$ may vanish. Consequently, since $Det(\mathbf{Jac(x_0, y_0)}) := \delta y_0 p^{'}(x_0) > 0$, $E = (x_0, y_0)$ can undergo a *Hopf bifurcation.*

iii) The trace of the saddle-node when $x_0 = x_{01}$ (resp. $x_0 = x_{02}$) is positive (resp. negative), yielding that the saddle-node is repelling (resp. attracting).

iv) In order to better understand the bifurcation diagram, it is interesting (though $y_0 < 0$) to see that:
If $x_0 < x_{01}$ or $x_{02} < x_0$, corresponding to the region of parameter space $\delta < p(\frac{1}{2} - \eta)$ or $\delta > p(\frac{1}{2} + \eta)$, then: $y_0 < 0$ and $Det(\mathbf{Jac(x_0, y_0)}) := \delta y_0 p^{'}(x_0) < 0$. Therefore, $E = (x_0, y_0)$ is a hyperbolic saddle there.

□

4.3. **Saddle-node Bifurcations.**

Theorem 4.4. *The double point $B = (\frac{1}{2}, 0)$ is a saddle-node when $\rho = 4\lambda$ and $\delta \neq \frac{1}{\alpha+2\beta+4}$. It is attracting if $\delta > \frac{1}{\alpha+2\beta+4}$ and repelling if $\delta < \frac{1}{\alpha+2\beta+4}$.*

Proof. Indeed, the translation $(x_1, y_1) = (x - \frac{1}{2}, y)$ brings the singularity $B = (\frac{1}{2}, 0)$, to the origin. In the neighborhood of $x_1 = 0$ and since $\rho = 4\lambda$, the system (1.4)

becomes
(4.11)
$$\begin{cases}\dot{x}_1 = \left(8\,\frac{(\alpha\,\beta+6\,\alpha-8)y_1}{(\alpha+2\,\beta+4)^3}-\rho\right)x_1{}^2-4\,\frac{(\beta+4)y_1x_1}{(\alpha+2\,\beta+4)^2}-\frac{y_1}{\alpha+2\,\beta+4}+O(|(x_1,y_1)|^4),\\ \dot{y}_1 = -8\,\frac{(\alpha\,\beta+6\,\alpha-8)y_1x_1{}^2}{(\alpha+2\,\beta+4)^3}+4\,\frac{(\beta+4)y_1x_1}{(\alpha+2\,\beta+4)^2}-\frac{(-1+\delta\,\alpha+2\,\delta\,\beta+4\,\delta)y_1}{\alpha+2\,\beta+4}+O(|(x_1,y_1)|^4).\end{cases}$$

The Jacobian matrix $\mathbf{M_B}$ of (4.11) is diagonalizable with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -\delta + \frac{1}{\alpha+2\beta+4} \neq 0$, and respective eigenvectors $v_1 = (1,0)$ and $v_2 = (\frac{1}{\delta(\alpha+2\beta+4)-1}, 1)$. By the transformation $\binom{X}{Y} = \begin{pmatrix} 1 & -\frac{1}{\delta(\alpha+2\beta+4)-1} \\ 0 & 1 \end{pmatrix}\binom{x_1}{y_1}$, the system becomes:
(4.12)
$$\begin{cases}\dot{X} = -2\,\frac{(\rho\,\alpha+8\,\delta+4\,\rho+2\,\rho\,\beta+2\,\delta\,\beta)XY}{(-1+\delta\,\alpha+2\,\delta\,\beta+4\,\delta)(\alpha+2\,\beta+4)}-\rho\,X^2-\frac{(\rho\,\alpha+4\,\delta\,\beta+4\,\rho+2\,\rho\,\beta+16\,\delta)Y^2}{(\alpha+2\,\beta+4)(-1+\delta\,\alpha+2\,\delta\,\beta+4\,\delta)^2}+O(|(X,Y)|^3),\\ \dot{Y} = -\frac{(-1+\delta\,\alpha+2\,\delta\,\beta+4\,\delta)Y}{\alpha+2\,\beta+4}+4\,\frac{(\beta+4)XY}{(\alpha+2\,\beta+4)^2}+4\,\frac{(\beta+4)Y^2}{(\alpha+2\,\beta+4)^2(-1+\delta\,\alpha+2\,\delta\,\beta+4\,\delta)}+O(|(X,Y)|^3).\end{cases}$$

It is not necessary to calculate the center manifold. The theorem *of Chochitaïchvili*[1] yields directly that the system (4.12) is topologically equivalent to the system:

$$\begin{cases}\dot{X} = -\rho X^2 + O(|X|^4),\\ \dot{Y} = (\frac{1}{\alpha+2\beta+4}-\delta)Y.\end{cases} \tag{4.13}$$

Consequently, since the coefficient of X^2 is $-\rho \neq 0$, then the point $B = (\frac{1}{2}, 0)$ is a saddle-node. It is *attracting* (resp. *repelling*) if $\delta > \frac{1}{\alpha+2\beta+4}$ (resp. $\delta < \frac{1}{\alpha+2\beta+4}$). $\square$

Theorem 4.5. *Let* $\eta := \frac{\sqrt{\rho(\rho-4\lambda)}}{2\rho}$. *When* $\rho > 4\lambda$, *the singular point* $C := (\frac{1}{2}-\eta, 0)$ *(resp.* $D := (\frac{1}{2}+\eta, 0)$*) is a repelling (resp. attracting) saddle-node on the surface* (SN_r) *(resp.* (SN_a)*) of equation* $\delta = p(\frac{1}{2}-\eta)$ *(resp.* $\delta = p(\frac{1}{2}+\eta)$*).*
If $\lambda \in]0, \frac{\rho}{4}[$, *then the union,* (SN_g), *of the two bifurcation surfaces* (SN_r) *and* (SN_a), *is defined by the equation:*

$$\begin{aligned}(SN_g) : &\left(\lambda^2\alpha^2 + \rho\,\left(-2\,\alpha + \beta^2 + \beta\,\alpha\right)\lambda + \rho^2\,(1+\beta+\alpha)\right)\delta^2\\ &+ \left(-2\,\lambda^2\alpha - \rho\,(-2+\beta)\,\lambda - \rho^2\right)\delta + \lambda^2 = 0.\end{aligned} \tag{4.14}$$

(4.14) is a polynomial equation of degree 2 in δ, *with polynomial coefficients in* α, β, ρ *and* λ. *At the limit, when* $\lambda = 0$, (SN_r) *merge with* $\delta = 0$.

Proof. Indeed, for $a := \frac{1}{2} \pm \eta$, the two double points are both of the form $M = (a, 0)$, where $\rho a^2 - \rho a + \lambda = 0$ and $p(a) = \delta$. The Jacobian matrix is given in (4.9). It has eigenvalues $\lambda_1 = \pm 2\rho\eta$ and 0. Moreover, $\lambda_1 > 0$ for C and $\lambda_1 < 0$ for D. A calculation allows to verify that $M = (a, 0)$ is exactly of multiplicity 2. If $\lambda \in]0, \frac{\rho}{4}[$, then (SN_r) and (SN_a) correspond to the loci where a singular point of the first open quadrant coallesces with a singular point located on the x-axis, i.e. when the resultant of $f(x) := (\alpha\delta - 1)x^2 + \beta\delta x + \delta$ (which gives the x-coordinate of the singular points in the first quadrant) and $g(x) := \rho x^2 - \rho x + \lambda$ (which gives the x-coordinate of the singular points on the x-axis) vanishes. The resultant R of f and g is given in (4.14). The discriminant of R is given by $\rho\,(\rho - 4\,\lambda)\,(\lambda\,\beta + \rho)^2 > 0$; from which the result follows.

$\square$

5. Hopf bifurcation

To study the Hopf bifurcation of our system, we transform it to a generalized Liénard system, with the weak focus at the origin, because the calculation of the Lyapunov coefficients of such a system can be done very easily. The transformation to the generalized Liénard system will be global, but it will not preserve the coordinate axes. Because of the global character of the transformation, the generalized Liénard system has several singular points, but we will only study the Hopf bifurcation at the origin.

5.1. **Calculation of the Lyapunov coefficients of a generalized Liénard system.** Let us consider a generalized Liénard system,

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = g(x) + yf(x), \quad \textit{where} \end{cases} \tag{5.1}$$

$$g(x) := \sum_{i=2}^{+\infty} a_i\, x^i, \quad f(x) := \sum_{j=1}^{+\infty} b_j\, x^j. \tag{5.2}$$

One knows by [33] that, for (5.1), there exists a power series

$$F := \frac{1}{2}(x^2+y^2) + \sum_{p=3}^{\infty} F_p(x,y) \quad \textit{where} \quad F_p(x,y) = \sum_{i=0}^{p} a_{i,p-i} x^i y^{p-i} \tag{5.3}$$

such that

$$\dot{F} = \sum_{k=1}^{\infty} L_k (x^2+y^2)^{k+1}. \tag{5.4}$$

The L_k are called *"Lyapunov coefficients"* or *"Hopf bifurcation coefficients"* of (5.1). They are found by solving (5.4) iteratively, degree per degree:

Theorem 5.1. *The first two Lyapunov coefficients of a generalized Liénard system, (5.1) , the second coefficient simplified under the condition that the first coefficient is zero, are:*

$$L_1 = \frac{1}{8}(b_2 - a_2 b_1), \tag{5.5}$$

$$L_2 = \frac{1}{16}(\frac{5}{3}a_2 b_1 a_3 - \frac{5}{3}a_2 b_3 + b_4 - a_4 b_1). \tag{5.6}$$

If we rather suppose that

$$g(x) := \sum_{i=1}^{+\infty} A_i\, x^i, \quad f(x) := \sum_{j=1}^{+\infty} B_j\, x^j, \tag{5.7}$$

with $A_1 > 0$, then the formulas take the useful form

$$L_1 = \frac{1}{8A_1^{\frac{3}{2}}}(B_2 A_1 - A_2 B_1), \tag{5.8}$$

$$L_2 = \frac{1}{16A_1^{\frac{5}{2}}}(\frac{5}{3}A_2 A_3 B_1 - \frac{5}{3}A_1 A_2 B_3 + B_4 A_1^2 - A_1 A_4 B_1). \tag{5.9}$$

Remark 5.2. *We also calculated L_3, L_4 and L_5 to validate our conjecture that $E = (x_0, y_0)$ is a center when $\beta = 0$. The formulas can be found in the thesis* [17].

5.2. **Existence and order of the Hopf bifurcation.**

Theorem 5.3. *When $\beta > 0$, the order of the Hopf bifurcation at $E = (x_0, y_0)$ is less than or equal to two (*[11],[27]*). The Hopf bifurcation occurs when $x_0 \in]x_{01}, \frac{1}{2}[$). When the order of the bifurcation is 2, the coefficient L_2 is strictly positive and the global dynamics of the model is given at the (h) of the Figure 2.5.*

Proof. Let us recall that

(5.10)
$$\rho > 4\lambda, 0 < \frac{1}{2} - \frac{\sqrt{\rho(\rho-4\lambda)}}{2\rho} < x_0 < \frac{1}{2}, p(x_0) = \delta < \frac{1}{\delta}, y_0 = \frac{\rho x_0(1-x_0) - \lambda}{\delta},$$
$$Det(\mathbf{Jac}(\mathbf{x_0}, \mathbf{y_0})) = \delta y_0 p'(x_0) > 0.$$

Then:

• We divide (1.4) by $p(x) > 0$, and bring back E to the origin by the translation $(x_1, y_1) = (x - x_0, y - y_0)$. The system then has the form

$$\begin{cases} \dot{x}_1 = & h_1(x_1) - y_1, \\ \dot{y}_1 = & h_2(x_1) + y_1 h_3(x_1). \end{cases} \tag{5.11}$$

The generalized Liénard system is simply obtained by letting $(X, Y) = (x_1, y_1 - h_1(x_1)$:

$$\begin{cases} \dot{X} = & -Y, \\ \dot{Y} = & h_2(X) + h_1(X)h_3(X) + Y(h_3(X) + h_1'(X)). \end{cases} \tag{5.12}$$

Note that $h_3(0) + h_1'(0) = 0$ is equivalent to the fact that the system has a Hopf bifurcation at the origin (and to the vanishing of the trace of the jacobian matrix at the origin). The Hopf bifurcation occurs when

$$\lambda = \frac{\rho(2\alpha x_0^3 + \beta x_0^2 - \alpha x_0^2 + 1)x_0}{\beta x_0 + 2}. \tag{5.13}$$

(This is well defined because, for all $\alpha > 0, \beta \geq 0, \rho > 0 : 0 < \frac{1}{2} - \frac{\sqrt{\rho(\rho-4\lambda)}}{2\rho} < x_0 < \frac{1}{2} \Rightarrow 2\alpha x_0^3 + \beta x_0^2 - \alpha x_0^2 + 1 > 0$) . We also have $Det(\mathbf{Jac}(\mathbf{x_1} = \mathbf{0}, \mathbf{y_1} = \mathbf{0})) = \rho(1 - 2x_0)(\alpha x_0^2 + \beta x_0 + 1)x_0^2 > 0$ since $0 < x_0 < \frac{1}{2}$.

However the expression of the transformed system of the form (5.1) under condition (5.7) is simpler if we postpone the replacement of λ by its value (5.13). We get

$$A_1 = -\frac{\rho(2x_0 - 1)}{\delta} \tag{5.14}$$

(5.15)
$$A_2 = \frac{1}{x_0^6}\left[-\alpha\rho x_0^6 + \alpha^2\delta\rho x_0^6 - \beta\lambda x_0^3 + 2\alpha\beta\delta\lambda x_0^3 - \beta^2\delta\rho x_0^3 - 2\alpha\delta\rho x_0^3 + 2\beta\delta\rho x_0^3 + \rho x_0^3 \right.$$
$$\left. +3\beta^2\delta\lambda x_0^2 + 6\alpha\delta\lambda x_0^2 - 3\lambda x_0^2 - 6\beta\delta\rho x_0^2 + 3\delta\rho x_0^2 + 12\beta\delta\lambda x_0 - 6\delta\rho x_0 + 10\delta\lambda\right]$$

(5.16)
$$A_3 = -\frac{1}{x_0^7}\left[-\beta\lambda x_0^3 + 2\alpha\beta\delta\lambda x_0^3 - \beta^2\delta\rho x_0^3 - 2\alpha\delta\rho x_0^3 + 2\beta\delta\rho x_0^3 + \rho x_0^3 + 4\beta^2\delta\lambda x_0^2 + 8\alpha\delta\lambda x_0^2 - 4\lambda x_0^2 - 8\beta\delta\rho x_0^2 + 4\delta\rho x_0^2 + 20\beta\delta\lambda x_0 - 10\delta\rho x_0 + 20\delta\lambda\right]$$

(5.17)
$$A_4 = \frac{1}{x_0^8}\left[-\beta\lambda x_0^3 + 2\alpha\beta\delta\lambda x_0^3 - \beta^2\delta\rho x_0^3 - 2\alpha\delta\rho x_0^3 + 2\beta\delta\rho x_0^3 + \rho x_0^3 + 5\beta^2\delta\lambda x_0^2 + 10\alpha\delta\lambda x_0^2 - 5\lambda x_0^2 - 10\beta\delta\rho x_0^2 + 5\delta\rho x_0^2 + 30\beta\delta\lambda x_0 - 15\delta\rho x_0 + 35\delta\lambda\right]$$

(5.18)
$$A_5 = -\frac{1}{x_0^9}\left[-\beta\lambda x_0^3 + 2\alpha\beta\delta\lambda x_0^3 - \beta^2\delta\rho x_0^3 - 2\alpha\delta\rho x_0^3 + 2\beta\delta\rho x_0^3 + \rho x_0^3 + 6\beta^2\delta\lambda x_0^2 + 12\alpha\delta\lambda x_0^2 - 6\lambda x_0^2 - 12\beta\delta\rho x_0^2 + 6\delta\rho x_0^2 + 42\beta\delta\lambda x_0 - 21\delta\rho x_0 + 56\delta\lambda\right]$$

$$B_1 = -\frac{2\alpha\rho x_0^4 - \beta\delta x_0^2 - 2\delta x_0 + 2\beta\lambda x_0 - 2\rho x_0 + 6\lambda}{x_0^4} \tag{5.19}$$

$$B_2 = -\frac{\beta\delta x_0^2 + 3\delta x_0 - 3\beta\lambda x_0 + 3\rho x_0 - 12\lambda}{x_0^5} \tag{5.20}$$

$$B_3 = \frac{\beta\delta x_0^2 + 4\delta x_0 - 4\beta\lambda x_0 + 4\rho x_0 - 20\lambda}{x_0^6} \tag{5.21}$$

$$B_4 = -\frac{\beta\delta x_0^2 + 5\delta x_0 - 5\beta\lambda x_0 + 5\rho x_0 - 30\lambda}{x_0^7} \tag{5.22}$$

By the formulas of the Lyapunov coefficients obtained in Theorem 5.1, (5.13) and using that $\delta = p(x_0)$, we have that the sign of L_1 is also that of:

$$\begin{aligned} L_1(x_0) :=&\rho^2 {x_0}^2 \left(1 - 2\,x_0\right) \left(\alpha\,{x_0}^2 + \beta\,x_0 + 1\right)^2 \left[\left(\beta^3 + 2\,\alpha\,\beta - \alpha\,\beta^2\right){x_0}^4 \right.\\ &\left. + \left(6\,\beta^2 - 6\,\alpha\,\beta\right){x_0}^3 + \left(6\,\beta - 6\,\alpha\right){x_0}^2 + 4\,\beta\,x_0 + 6\right] \end{aligned} \tag{5.23}$$

which vanishes for

$$\alpha = \frac{\beta^3 {x_0}^4 + 6\,\beta^2 {x_0}^3 + 6\,\beta\,{x_0}^2 + 4\,\beta\,x_0 + 6}{{x_0}^2\left(-2\,\beta\,{x_0}^2 + 6\,\beta\,x_0 + 6 + \beta^2 {x_0}^2\right)} \tag{5.24}$$

(well defined because, for all $\beta > 0$: $0 < x_0 < \frac{1}{2} \Rightarrow -2\,\beta\,{x_0}^2 + 6\,\beta\,x_0 + 6 + \beta^2 {x_0}^2 = \beta x_0(1 - 2x_0) + 5\beta x_0 + 6 + \beta^2 x_0^2 > 0$).
It is noticed that $L_1(x_0)$ can also be written (using (5.13)) like

(5.25)
$$l_1(\lambda, x_0) := -2\,\beta^2\rho\,{x_0}^5 - 6\,\rho\,\beta\,{x_0}^4 + \left(\lambda\,\beta^2 - 2\,\beta\,\lambda - \rho\,\beta - 6\,\rho\right){x_0}^2 + 6\,\lambda\,\beta\,x_0 + 6\,\lambda.$$

Therefore, for α given in (5.24), one has that the sign of L_2 is also that of

$$L_2(x_0) = \frac{2\,(-2\,x_0+1)\,x_0^2\beta\,\rho^2\,\left(\beta\,{x_0}^2+\beta\,x_0+3\right)^3(\beta\,x_0+2)^7}{\left(-2\,\beta\,{x_0}^2+6\,\beta\,x_0+6+\beta^2{x_0}^2\right)^4} \left[3+(4\,\beta+18)\,x_0+4\,\beta\,{x_0}^2+18\,\beta\,{x_0}^3+11\,\beta^2{x_0}^4+\beta^3{x_0}^5\right] \tag{5.26}$$

which is strictly positive for all $\beta > 0$; from which the result follows. (Note that L_2 vanishes for $\beta = 0$.) □

5.3. **Case $\beta = 0$.** Recall that the results on the saddle-node bifurcations and Hopf bifurcation of order 1 are the same for the cases $\beta > 0$ and $\beta = 0$. For the other bifurcations, the cases $\beta > 0$ and $\beta = 0$ do not function similarly and we are led to the following conjecture.

Conjecture 5.4. *If $\beta = 0$ and $L_1 = 0$, then the singular point $E = (x_0, y_0)$ is a center (i.e. there exists a neighborhood U of E such that all orbits inside $U \setminus \{E\}$ are periodic). The annulus of periodic solutions ends in a heteroclinic loop through the two saddle points on the x-axis.*

Rationale. It is well known since Poincaré that a singular point with two imaginary eigenvalues is of center type if and only if all its Lyapunov coefficients vanish. (An explicit reference is for instance **Corollary 11** of [9]). The reference [17] contains an explicit calculation of L_3, L_4, L_5. They all have the form

$$L_i = (-1)^{i+1} C_i \frac{\left(\alpha\,{x_0}^2-1\right)}{\rho^{\frac{1}{2}}{x_0}^{2i+1}\,(1-2\,x_0)^{i-\frac{1}{2}}\,\left(\alpha\,{x_0}^2+1\right)^{2i-\frac{1}{2}}} Q_i(x_0,\alpha,\rho) \tag{5.27}$$

where $C_i \in \mathbb{N}^*$ and the $Q_i(x_0, \alpha, \rho)$ are polynomials with integer coefficients. Then, for $\beta = 0$ and $L_1 = 0$ the (5.24) yields $\alpha = \frac{1}{x_0^2}$ and hence, $L_2 = L_3 = L_4 = L_5 = 0$.

6. Nilpotent saddle bifurcation

6.1. Normal form at the nilpotent point.

Theorem 6.1. *If $\rho = 4\lambda$ and $\delta = \frac{1}{\alpha+2\beta+4}$, then there exists a nilpotent saddle bifurcation in the neighborhood of the singular point $B = (\frac{1}{2}, 0)$: the system localized at the singular point of multiplicity 3 is, for $k > 5$, C^k-equivalent with the system*

$$\begin{cases} \dot{X} = Y + aX^2, \\ \dot{Y} = Y\left(X + \alpha_2 X^2 + \alpha_3 X^3 + \alpha_4 X^4 + O(|X|^5)\right), \end{cases} \tag{6.1}$$

where

$$\alpha_2 = -\frac{\alpha\,\beta + 6\,\alpha - \beta^2 - 8\,\beta - 24}{2\,(\beta+4)^2}.$$

For $\alpha_2 \neq 0$, the point is of codimension 3.
For all $\beta \geq 0$, α_2 vanishes in

$$\alpha = \alpha_\beta := \frac{\beta^2 + 8\,\beta + 24}{\beta + 6} > 0. \tag{6.2}$$

If $\alpha = \alpha_\beta$ and $\beta > 0$, then for $i = 3, 4$, the coefficients of X^iY in the normal form (6.1) become $\tilde{\alpha}_i$, where:

$$\tilde{\alpha}_3 = -\frac{3\left(\beta^2+12\,\beta+48\right)}{4\,(\beta+6)^2} < 0 \quad \textit{and} \quad \tilde{\alpha}_4 = \frac{3\beta\left(\beta^2+18\,\beta+96\right)}{8\,(\beta+6)^3} > 0.$$

Remark 6.2. *If $\beta = 0$ and $\alpha_2 = 0$, then $\alpha_4 = \alpha_6 = 0$. We thus conjecture that:*

Conjecture 6.3. *If $\beta = 0$ and $\alpha_2 = 0$, then the normal form (6.1) is invariant under the change $X \mapsto -X, t \mapsto -t$, i.e. the system is reversible and thus of infinite codimension.*

Remark 6.4.
(1) *In the case $\alpha_2 = 0$ and $\tilde{\alpha}_4 \neq 0$, we conjecture that the codimension is 4 because the coefficient $\tilde{\alpha}_3$ does not seem to play any role in the structure of the bifurcation diagram. Indeed, it does not play any role for the local bifurcations, and it remains to check that this is also the case for the global bifurcations.*
(2) *In our system, the constraint given by the fact that the line of equation $Y = 0$ is invariant decreases the effective codimension by one.*

Proof of Theorem 6.1. It will be done in several steps:

(1) Localization of the initial system: We bring back the singularity $B = (\frac{1}{2}, 0)$ to the origin by the translation $(x_1, y_1) = (x - \frac{1}{2}, y)$, we use the fact that $\delta = \frac{1}{\alpha+2\beta+4}$ and $\rho = 4\lambda$, and we multiply the system by $\frac{1}{p\left(x_1+\frac{1}{2}\right)} > 0$.

Let $K := \frac{4(\beta+4)}{\alpha+2\,\beta+4}$. Then, by the transformation $(x_2, y_2) = (Kx_1, -Ky_1)$ the preceding system becomes:

(6.3)
$$\begin{cases} \dot{x}_2 = \frac{(\alpha+2\,\beta+4)^4 \rho\,(\beta+8) {x_2}^5}{16(\beta+4)^4} - \frac{(\alpha+2\,\beta+4)^3 \rho\,(\beta+6) {x_2}^4}{8(\beta+4)^3} - a\,{x_2}^3 + a\,{x_2}^2 + y_2 + O(|x_2|^6), \\ \dot{y}_2 = y_2 \left[-\frac{(\alpha+2\,\beta+4)^3 (\beta+10) {x_2}^4}{8(\beta+4)^4} + \frac{(\alpha+2\,\beta+4)^2 (\beta+8) {x_2}^3}{4(\beta+4)^3} - \frac{(\alpha+2\,\beta+4)(\beta+6) {x_2}^2}{2(\beta+4)^2} + x_2 + O(|x_2|^5) \right], \end{cases}$$

where

$$a := -\frac{\rho\,(\alpha + 2\,\beta + 4)^2}{4(\beta + 4)} < 0, \tag{6.4}$$

whose Jacobian matrix, evaluated in the origin, is $\mathbf{J} := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

(2) Normalizing change of variables and scaling of time: There exists a change of variables preserving the invariant line $y_2 = 0$ and a scaling of time bringing the system (6.3) to

$$\begin{cases} \dot{X} = Y + aX^2, \\ \dot{Y} = Y\left(X + O(|X|^2)\right). \end{cases} \tag{6.5}$$

Indeed, the system (6.3) is of the form

$$\begin{cases} \dot{x}_2 := y_2 + a x_2^2 h(x_2), \\ \dot{y}_2 := y_2 g(x_2), \quad \textit{where} \end{cases}$$

$$h(x_2) = 1 - x_2 + O(|x_2|^2) \quad \textit{and} \quad g(x_2) = x_2 + O(|x_2|^2). \tag{6.6}$$

Now let us consider the following changes of variables and scaling of time

$$X = x_2\sqrt{h(x_2)} = x_2(1 + O(|x_2|)) := H(x_2), \quad t = k(X)T \tag{6.7}$$

where

$$k(X) := \left(H^{-1}\right)'(X) = 1 + O(|x_2|).$$

One has

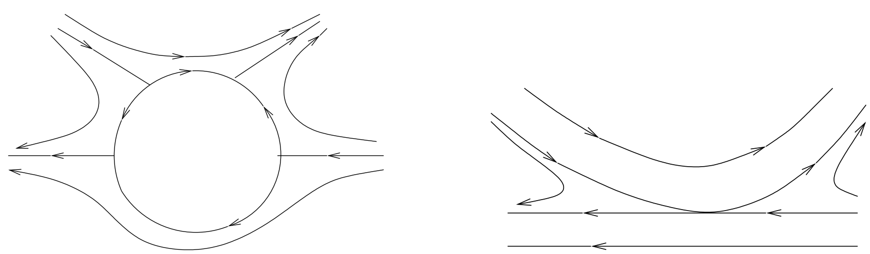
$$\begin{aligned}\frac{dX}{dT} =& \frac{dX}{dx_2}\frac{dx_2}{dt}\frac{dt}{dT}\\ =& H^{'}(x_2)\left[y_2 + ax_2^2h(x_2)\right]k(X)\\ =& \left[\left(H^{-1}\right)^{'}(X)\right]^{-1}(Y + aX^2)\left(H^{-1}\right)^{'}(X)\\ =& Y + aX^2 \quad \textit{and} \qquad (6.8)\end{aligned}$$

$$\begin{aligned}\frac{dY}{dT} =& \frac{dY}{dy_2}\frac{dy_2}{dt}\frac{dt}{dT}\\ =& Y\left(H^{-1}(X) + \sum_{i=2}^{6}\alpha_{0i}\left(H^{-1}(X)\right)^{i} + ...\right)\left(H^{-1}\right)^{'}(X) = YG(X) \qquad (6.9)\end{aligned}$$

where $G(X) = X + o(X)$ is calculated easily. This is exactly (6.1).

The details of the calculations and simplifications of the α_i were omitted. □

Topological type of the singular point $B = (\frac{1}{2}, 0)$. Since $a < 0$, then the topological type of B is that of a nilpotent saddle (Figure 6.1). This is easily seen by a weighted blow-up (see also [39]).



(a) complete blow-up

(b) Topological type of the singular point $B = (\frac{1}{2}, 0)$

FIGURE 6.1. *Nilpotent saddle*

6.2. **Normal form for the family unfolding the nilpotent saddle.** Let

$$\nu_1 := \rho - 4\lambda, \quad \nu_2 := \delta - \frac{1}{\alpha + 2\beta + 4} \quad \textit{and} \quad \nu := (\nu_1, \nu_2). \qquad (6.10)$$

We study the system (1.4) when the parameter $\nu := (\nu_1, \nu_2)$ is in a neighborhood of $(0, 0)$ and (x, y) is in a neighborhood of $(\frac{1}{2}, 0)$.

Proposition 6.5. *If (x, y, ν_1, ν_2) is in a neighborhood of $(\frac{1}{2}, 0, 0, 0)$, then the system (1.4) is topologically orbitally equivalent to*

$$\begin{cases}\dot{\overline{X}} = \overline{Y} + q(\overline{X})r(\overline{X}),\\ \dot{\overline{Y}} = \overline{Y}f(\overline{X}), \quad \textit{where}\end{cases} \qquad (6.11)$$

$$\begin{cases} q(\overline{X}) := a\overline{X}^2 + \mu_0(\nu), \\ r(\overline{X}) := 1 + O(|\nu|)\left(1 + O(|\overline{X}|)\right), \\ f(\overline{X}) := O(|\nu|) + (1 + O(|\nu|))\,\overline{X} + \sum_{i=2}^{4}(\alpha_i + O(|\nu|))\overline{X}^i + O(|\overline{X}|^5), \end{cases} \tag{6.12}$$

the α_i being defined in (6.1).

Proof. In order to reduce the text and avoid repetitions, we rather explain the method and, when it is useful, we present certain expressions. Indeed:

• The translation $(x_1, y_1) = (x - \frac{1}{2}, y)$ is applied, and the system is divided by $p\left(x_1 + \frac{1}{2}\right)$. Then one obtains a system in which the coefficient of the x_1y_1-term in $\dot{y}_1$ is given by

$$K_{\nu_2} = \frac{4(\beta+4)}{\alpha + 2\,\beta + 4}[1 + \nu_2(\alpha + 2\,\beta + 4)]. \tag{6.13}$$

• The transformation $(x_2, y_2) = (K_{\nu_2}x_1, -K_{\nu_2}y_1)$ is applied, where K_{ν_2} is given in (6.13): one obtains

$$\begin{cases} \dot{x}_2 = f_2(x_2, \nu_1, \nu_2) + y_2, \\ \dot{y}_2 = y_2 g_2(x_2, \nu_1, \nu_2), \quad \textit{where} \end{cases} \tag{6.14}$$

$$f_2(0,0,0) = 0, \frac{\partial f_2}{\partial x_2}(0,0,0) = 0 \quad \textit{and} \quad \frac{\partial^2 f_2}{\partial x_2^2}(0,0,0) = a < 0.$$

Then, by the Weierstrass preparation theorem [10], there exists an analytical function u in a neighborhood of $(0,0,0) \in \mathbf{R}\times\mathbf{R}^2$ and two analytical functions $\epsilon_0(\nu_1, \nu_2)$ and $\epsilon_1(\nu_1, \nu_2)$ in a neighborhood of $(0,0) \in \mathbf{R}^2$ such that $u(0,0,0) = a < 0$, $\epsilon_0(0,0) = 0$, $\epsilon_1(0,0) = 0$ and

$$f_2(x_2, \nu_1, \nu_2) = \left[\epsilon_0(\nu_1, \nu_2) + \epsilon_1(\nu_1, \nu_2)x_2 + x_2^2\right] u(x_2, \nu_1, \nu_2).$$

Thus

$$\dot{x}_2 = y_2 + \left[\left(x_2 + \frac{\epsilon_1(\nu)}{2}\right)^2 - \frac{\epsilon_1(\nu)^2 - 4\epsilon_0(\nu)}{4}\right] u(x_2, \nu_1, \nu_2), \tag{6.15}$$

where $u(x_2, 0, 0) = ah(x_2)$ and $h(x_2)$ is defined in (6.6).

• The result follows by applying the translation $(X, Y) = (x_2 + \frac{\epsilon_1(\nu)}{2}, y_2)$ and letting $\mu_0(\nu_1, \nu_2) := -\frac{a\left[\epsilon_1(\nu_1,\nu_2)^2 - 4\epsilon_0(\nu_1,\nu_2)\right]}{4}$. □

Theorem 6.6. (1) *If $\alpha_2 \neq 0$, then the system (1.4) is topologically orbitally equivalent to*

$$\begin{cases} \dot{X} = Y + a(\nu)X^2 + \mu_2, \\ \dot{Y} = Y\left(\mu_3 + X + \epsilon_2 X^2 + O(X^3)\right) + Y^2 Q_1(X, \nu), \end{cases} \tag{6.16}$$

where $a(\nu) < 0$, $\epsilon_2 = \mp 1$, $Q_1(X, 0) = 0$.

(2) *If $\alpha_2 = 0$ and $\beta > 0$, then the system (1.4) is topologically orbitally equivalent to*

$$\begin{cases} \dot{X} = Y + a(\nu)X^2 + \mu_2, \\ \dot{Y} = Y\left[\mu_3 + X + \mu_4 X^2 + \hat{\alpha}_3 X^3 + X^4 + O(X^5)\right] + Y^2 Q_2(X, \nu), \end{cases} \tag{6.17}$$

where $a(\nu) < 0$, $\hat{\alpha}_3 := (\tilde{\alpha}_3 + O(|\nu|))\,(\tilde{\alpha}_4 + O(|\nu|))^{-\frac{2}{3}}$, $Q_2(X, 0) = 0$.

Proof. One starts by applying to the full system (depending on the parameters) the transformations which bring the system evaluated at $\nu=0$ to the form (6.1). One then applies Proposition 6.5 to it. One now takes the following changes of variable and scaling of time

$$\overline{X}=X,\overline{Y}=Yr(\overline{X}),\overline{t}=\frac{t}{r(\overline{X})}. \tag{6.18}$$

Then

$$\dot{Y}:=\frac{dY}{dt}=Y\left[\frac{f(X)}{r(X)}-\frac{r^{'}(X)q(X)}{r(X)}\right]+Y^2\left[-\frac{r^{'}(X)}{r(X)}\right], \tag{6.19}$$

where $f(X)$, $r(X)$ and $q(X)$ are defined in (6.12). Thus
(6.20)

$$\begin{cases}\dot{X}=Y+q(X),\\ \dot{Y}=Y\Big[(\alpha+2\,\beta+4)\nu_2-\frac{\epsilon_1(\nu)}{2}+o(|\nu|)+(1+O(|\nu|))X+(\alpha_2+O(|\nu|))X^2\\ \qquad +O(|\nu|)X^3+(\alpha_4+O(|\nu|))X^4+O(|X|^5)\Big]+Y^2Q_1(X,\nu),\end{cases}$$

where $Q_1(X,0)=0$ since one started from a system which had already this form for $\nu=0$, and since the transformation is the identity for $\nu=0$. Let $K_1:=1+O(|\nu|)$. Two cases are essential:

- If $\alpha_2\neq 0$, then a transformation $(\tilde{X},\tilde{Y})=(K_1X,K_1Y)$ transforms to the case $\alpha_2=\pm1$ by renaming a and $\mu_0(\nu)$.
- If $\alpha_2=0$ and $\beta>0$, then $\alpha_2+O(|\nu|)$ (the coefficient of YX^2 in (6.20)) becomes $O(|\nu|)$ (independent of the first two). Since $\overline{\alpha}_4>0$ by Theorem 6.1, a transformation $(\tilde{X}_1,\tilde{Y}_1)=((\overline{\alpha}_4)^{\frac{1}{3}}\,X,(\overline{\alpha}_4)^{\frac{2}{3}}\,Y)$ allows to bring to the case $\alpha_4=1$. □

6.3. **Bifurcation diagram of the families (6.16) and (6.17).** As expected in this kind of problems, the bifurcation diagram of the family (6.16) will be the same (topologically) as that of the standard family

$$\begin{cases}\dot{X}=Y+a(\nu)X^2+\mu_2,\\ \dot{Y}=Y\left(\mu_3+X+\epsilon_2X^2\right).\end{cases} \tag{6.21}$$

Similarly, we expect that the bifurcation diagram of the family (6.17) will be the same as that of the standard family

$$\begin{cases}\dot{X}=Y+a(\nu)X^2+\mu_2,\\ \dot{Y}=Y\left(\mu_3+X+\mu_4X^2+\tilde{\alpha}_3X^3+X^4\right),\end{cases} \tag{6.22}$$

where $\epsilon_2=\pm1$, $a(\nu)<0$ for $\nu:=(\nu_1,\nu_2)$ sufficiently small and $\mu_i:=\mu_i(\nu)$.

This will be discussed in detail in [18] and much details can be found in [17]. For the sake of completeness, we give a summary of the proof here: to help with interpretation, one use the same letters C, D and E as for the corresponding singular points of (1.4). But it should be noticed that the region $y>0$ of (1.4) corresponds here to $Y<0$.

Theorem 6.7. *The bifurcation diagram of the system (6.16) appears in Figure 6.2. The figure presents the case $\epsilon_2=+1$ and the case $\epsilon_2=-1$ is obtained by the transformation $(X,Y,\mu_2,\mu_3,t)\longmapsto(-X,Y,\mu_2,-\mu_3,-t)$.*

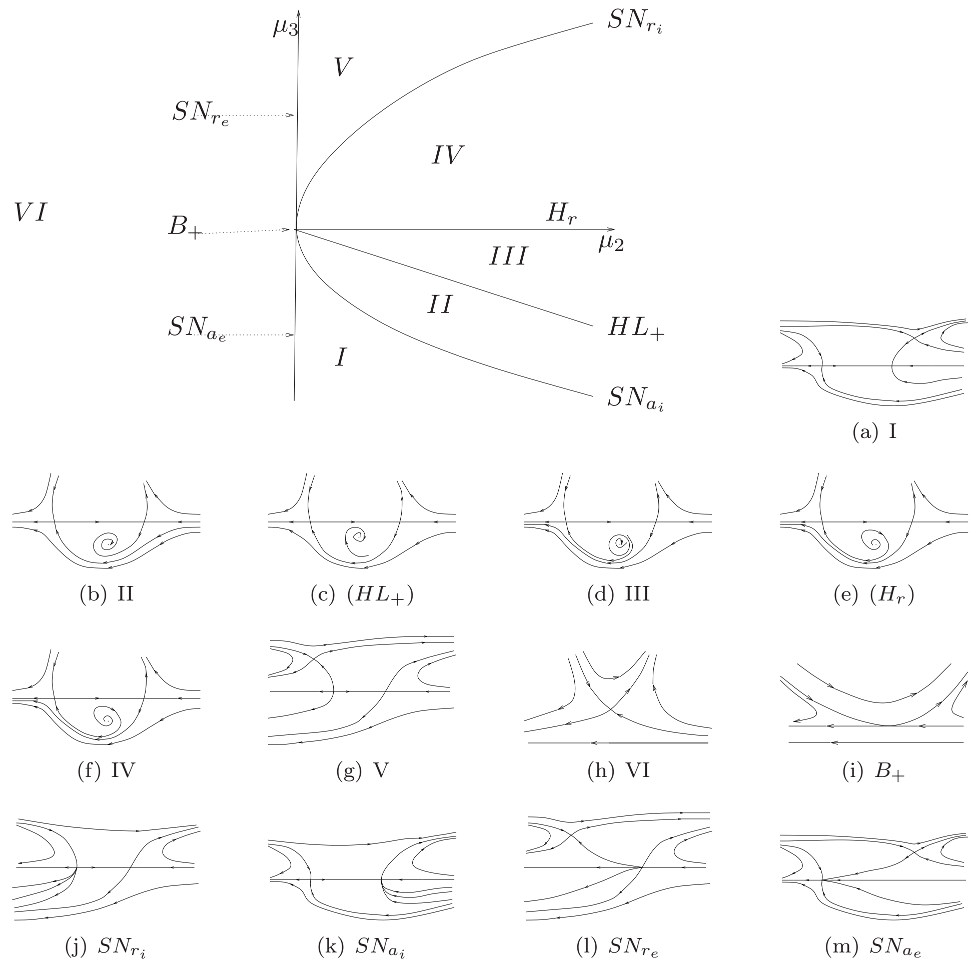


FIGURE 6.2. Bifurcation diagram and phase portraits of the system (6.21) for $\epsilon_2 = +1$.

Proof. The system has two singular points C and D on the X-axis when μ_2 is positive and a saddle-node bifurcation for $\mu_2 = 0$. By the implicit functions theorem, there is another singular point $E = (x_0, y_0)$ which is one anti-saddle (resp. saddle) when it is located below (resp. above) $Y = 0$. At the time when E crosses the X-axis in C (resp. D), one has a attracting (resp. repelling) saddle-node bifurcation. In the region where E is an anti-saddle, there is a Hopf bifurcation of order 1, whose first Lyapunov coefficient has the sign of ϵ_2. The limit cycle created around E must disappear before E crosses the X-axis. This can occur only in a heteroclinic loop bifurcation. The hyperbolicity ratio of a saddle point is the absolute value of the quotient of its negative eigenvalue by its positive eigenvalue. Let r_C (resp. r_D) be the hyperbolicity ratio of C (resp. D). By calculating the product $r_C r_D$ at the time of heteroclinic loop, one checks that $r_C r_D > 1$ (resp. $r_C r_D < 1$), i.e. this loop is repelling (resp. attracting) for $\epsilon_2 > 0$ (resp. $\epsilon_2 < 0$). Since the system is rotational in μ_3 for μ_2 is fixed, this yields that the locus of the heteroclinic loop is of the form $\mu_3 = g(\mu_2)$ for $\mu_2 > 0$. □

Theorem 6.8. *The bifurcation diagram of the system (6.17) has the structure of a cone. Its intersection with a sphere S_ϵ minus a point is presented in Figure 6.3 (see also Table 2.1), where $S_\epsilon = \{(\mu_2, \mu_3, \mu_4) \in \mathbf{R}^3 \quad \text{such that} \quad \mu_2^2 + \mu_3^2 + \mu_4^2 = \epsilon^2\}$ with $\epsilon > 0$ sufficiently small. The following bifurcations are exact: saddle-node bifurcations, Hopf bifurcations of order 1 and 2, and the qualitative position of the heteroclinic loop bifurcation. Are conjectured:*

- *the conic structure;*
- *the uniqueness of the points corresponding to a heteroclinic loop of order 2;*
- *the exact order of the heteroclinic bifurcation when it is higher than 1;*
- *the uniqueness of the point of intersection of the Hopf bifurcation curve with the heteroclinic loop bifurcation curve;*
- *the fact that there are at most two limit cycles.*

In the particular case where $\hat{\alpha}_3 = O(\nu)$ and the term in XY^2 has a coefficient $O(\nu)$, these conjectures are proved.

Proof. The study of the singular points is similar to the preceding case. In the case of the Hopf bifurcation of order 2, it is easy to verify that the term in X^3Y of coefficient $\hat{\alpha}_3$ does not play an essential role and cannot destroy the tendency given by the term in X^4Y. For μ_2, μ_3 fixed (resp. μ_2, μ_4 fixed), the system is rotational in μ_4 (resp. μ_3). This shows that the surface of heteroclinic loop bifurcation cuts each line corresponding to μ_2, μ_3 constant (resp. μ_2, μ_4 constant) at most once. We thus obtain the qualitative position of the surface. It is clear that the heteroclinic bifurcation is repelling (resp. attracting) in the neighborhood of $\mu_2 = \mu_3 = 0$, $\mu_4 > 0$ (resp. $\mu_4 < 0$) and that the quantity $r_C r_D - 1$ must thus vanish between the two. One way of making a complete study of the heteroclinic bifurcation when the codimension is higher than 2 is to make a blow-up which brings back the system to a perturbation of a Hamiltonian system. But, the method can only be applied under the condition $\hat{\alpha}_3 = O(\nu)$, and the hypothesis that the term in XY^2 has a coefficient $O(\nu)$. On the other hand, in this last case, we can prove all the details ([18]). □

We continue to work to fill the holes in the proof of the Theorem 6.8. A better normal form for a system with an invariant line would allow to make our argument completely rigorous.

7. Bifurcation diagram

7.1. **Presentation of the section.** Recall that the system (1.4) has 5 real parameters: $\rho, \alpha, \delta, \lambda$ are strictly positive, $\beta \geq 0$; and we use the parameter η defined in (4.8).

The (preceding) study show that the bifurcations of (1.4) occur only under the following basic conditions:

$$\rho \geq 4\lambda \quad \text{or} \quad \delta < \frac{1}{\alpha}. \tag{7.1}$$

We choose to present the bifurcation diagram in the space $(\alpha, \delta, \lambda)$ for the different values of (β, ρ). Most of the time, we will give the slices of the bifurcation diagram in the (α, δ)-plane for different $\lambda \geq 0$.

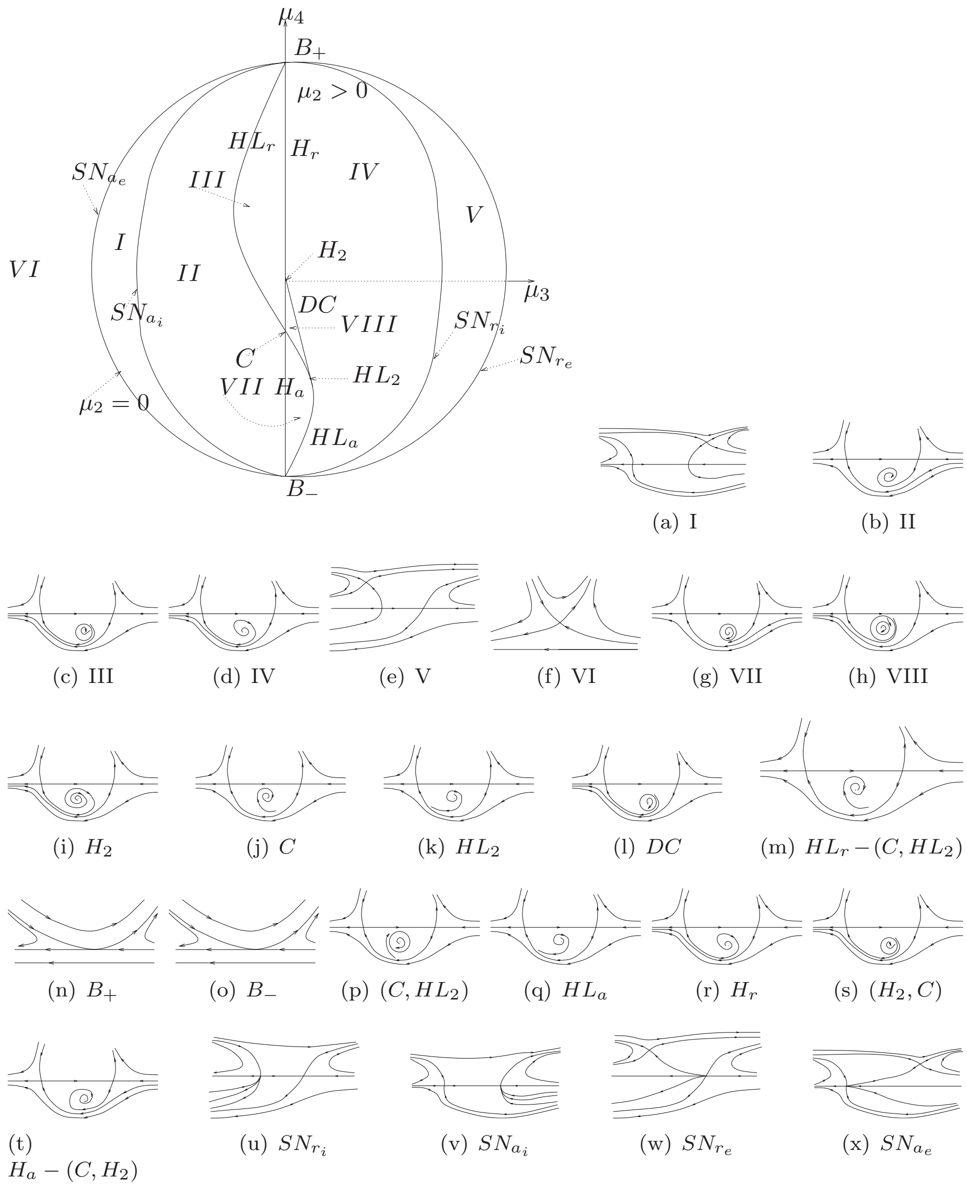


FIGURE 6.3. Trace of the bifurcation diagram of (6.22) on S_ϵ minus a point, and phase portraits (see also Table 2.1).

Let us start with the small remark.

Remark 7.1. *The line of equation $y = 0$ remains invariant under the flow of the system (1.4), whereas the y-axis is no more invariant as soon as $\lambda \neq 0$.*

7.2. **Position of the saddle-node bifurcations.**

Theorem 7.2. (1) *In the (α, δ)-plane, let*

$$(SN) : \delta = p\left(\frac{1}{2}\right) = \frac{1}{\alpha + 2\beta + 4} : \tag{7.2}$$

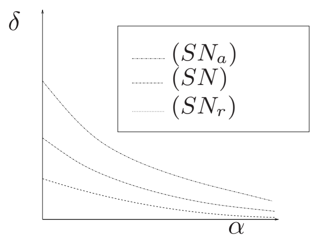
(SN_r), (SN) and (SN_a) are branches of hyperboles represented on Figure 7.1(a).
More precisely:

(2) *If $\lambda = \frac{\rho}{4}$ then:*
• *the union of (SN_r) and (SN_a) merges exactly with (SN), which corresponds to a triple singular point on the x-axis (i.e $B = (\frac{1}{2}, 0)$, nilpotent saddle);*
• *there exists a repelling saddle-node, (SN_r), (resp. attracting saddle-node, (SN_a)) if $\delta < \frac{1}{\alpha+2\beta+4}$ (resp. if $\delta > \frac{1}{\alpha+2\beta+4}$): see Figure 7.1(b).*

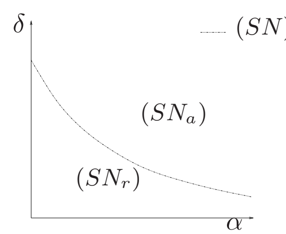
(3) *(SN_a) is below (SN) which, in turn, is below (SN_r).*

(4) *The line $\delta = 0$ is a horizontal asymptote of (SN_r), (SN_a) and with (SN).*

(5) *The line $\alpha = 0$ intersects (SN_r), (SN_a) and (SN) respectively at the points $(0, f_r(0))$, $(0, f_a(0))$ and $(0, f_n(0))$ where $f_r(0) := \frac{(\frac{1}{2}-\eta)^2}{\beta(\frac{1}{2}-\eta)+1}$, $f_a(0) := \frac{(\frac{1}{2}+\eta)^2}{\beta(\frac{1}{2}+\eta)+1}$ and $f_n(0) = \frac{1}{2(\beta+2)}$.*



(a) for $\rho > 4\lambda$



(b) for $\rho = 4\lambda$

FIGURE 7.1. Sections, parallel to the plan (α, δ), of the saddle-node bifurcation surfaces (SN_r), (SN_a) and (SN).

Proof. Indeed:

(1) It is known that (SN_r) is given by $\delta = p(\frac{1}{2} - \eta)$ and (SN_a) by $\delta = p(\frac{1}{2} + \eta)$. The figure 7.1(a) thus derives from the study of the functions $p\left(\frac{1}{2} - \eta\right)$, $p\left(\frac{1}{2}\right)$ and $p\left(\frac{1}{2} + \eta\right)$ which all are of the form $\frac{c_1}{\alpha c_2 + c_3}$ where $c_i > 0$.

(2) If $\lambda = \frac{\rho}{4}$, then $\eta = 0$ and $g(x) := \rho x^2 - \rho x + \lambda$ has a double root; thus:
• (SN_r) and (SN_a) merge exactly along the curve (SN) which corresponds to the triple singular point $B = (\frac{1}{2}, 0)$ of nilpotent saddle type;
• from Section 4.3, the two singular points on the x-axis merge in a repelling saddle-node, (SN_r), (resp. attracting, (SN_a)) if $\delta < \frac{1}{\alpha+2\beta+4}$ (resp. if $\delta > \frac{1}{\alpha+2\beta+4}$).

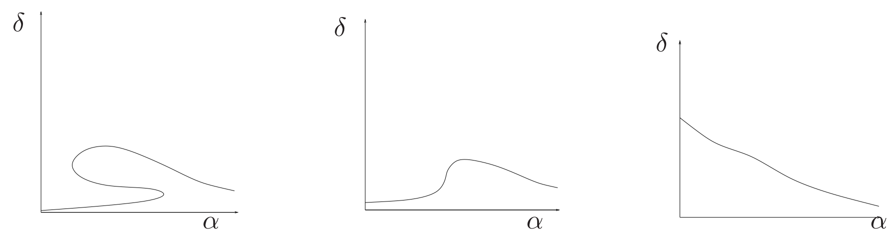
(3) It comes from $\frac{1}{2} - \eta \le \frac{1}{2} \le \frac{1}{2} + \eta$, since p is increasing.

(4) It comes from the shape of the curves $p\left(\frac{1}{2} \pm \eta\right)$ and $p\left(\frac{1}{2}\right)$.

□

7.3. **Hopf bifurcation.** Recall that here, one has (5.10) for η defined in (4.8).

Theorem 7.3. *The sections of the surface $(\mathcal{H})$ of Hopf bifurcation by planes parallel with the (α,δ)-plane in the first quadrant are represented in Figure 7.2.*



(a) λ is small. (b) λ is neither small, nor close to $\frac{\rho}{4}$. (c) λ is close to $\frac{\rho}{4}$.

FIGURE 7.2. Sections, parallel to the (α,δ)-plane in the first quadrant, of the Hopf bifurcation $(\mathcal{H})$.

Proposition 7.4. *For all $\lambda\in]0,\frac{\rho}{4}[$, the Hopf bifurcation surface (in (α,δ,λ)-space)*
(i) *strictly lies between $(SN_r):\delta=p(\frac{1}{2}-\eta)$ and $(SN):\delta=p(\frac{1}{2})$;*
(ii) *is included in the surface of equation*

$$P(\alpha,\delta,\lambda):=A(\alpha,\lambda)\delta^4+B(\alpha,\lambda)\delta^3+C(\alpha,\lambda)\delta^2+D(\alpha,\lambda)\delta+E(\alpha,\lambda)=0, \tag{7.3}$$

where
(7.4)

$$\begin{aligned}
A(\alpha,\lambda)=&\quad \alpha\left(-\beta^2+4\,\alpha\right)\left(\lambda^2\alpha^2+\rho\left(\rho-2\lambda+\beta\lambda\right)\alpha+\rho\lambda\beta^2+\rho^2\left(1+\beta\right)\right),\\
B(\alpha,\lambda)=&\quad -\alpha^2\left(16\,\alpha-3\,\beta^2\right)\lambda^2-\rho\left(2\,\alpha\,\beta^2-16\,\alpha^2+\beta^4-\alpha\,\beta^3+8\,\alpha^2\beta\right)\lambda\\
&\quad -\rho^2\left(\beta^3+\beta^2-\alpha\,\beta^2+8\,\alpha^2\right),\\
C(\alpha,\lambda)=&\quad 3\,\alpha\left(-\beta^2+8\,\alpha\right)\lambda^2+4\,\rho\left(-\beta^2-2\,\alpha+\beta\,\alpha\right)\lambda+\rho^2\left(-2\,\beta+5\,\alpha\right),\\
D(\alpha,\lambda)=&\quad \left(\beta^2-16\,\alpha\right)\lambda^2-\rho^2\quad \textit{and}\quad E(\alpha,\lambda)=4\,\lambda^2.
\end{aligned}$$

Proof. Let $\lambda\in]0,\frac{\rho}{4}[$:
(i) It follows from (5.10) and the fact that p is strictly increasing in $]0,+\infty[$.
(ii) The determinant of the Jacobian matrix of (1.4) in (x_0,y_0) being strictly positive, a Hopf bifurcation occurs when the trace is zero, i.e. when the resultant of $f(x):=(\alpha\delta-1)x^2+\beta\delta x+\delta$ and

$$u(x):=-2\,\rho\,\alpha\,x^4+\rho\,\left(\alpha-\beta\right)x^3+\left(-\rho+\lambda\,\beta\right)x+2\,\lambda \tag{7.5}$$

vanishes. □

Remark 7.5. *For all $\rho\geq 4\lambda$, the factor*

$$A_0(\alpha,\lambda):=\lambda^2\alpha^2+\rho\left(\rho-2\lambda+\beta\lambda\right)\alpha+\rho\lambda\beta^2+\rho^2\left(1+\beta\right)$$

of $A(\alpha,\lambda)$ is strictly positive.

We will study in detail the curve $P(\alpha,\delta,\lambda)=0$ in the (α,δ)-plane for fixed values of λ and we will determine which branch corresponds to a Hopf bifurcation.

Remark 7.6. *For α large, the coefficients of $P(\alpha,\delta,\lambda)$ are of alternate sign. Hence, by the criterion of Descartes, there are zero, or 2, or 4 branches at infinity in the* *first quadrant. Below, we will prove that there are exactly 2 branches at infinity as soon as $\lambda > 0$.*

In order to help with the analysis of $P(\alpha, \delta, \lambda)$, we recall some classical results on the number of roots of a polynomial of degree 4.

Theorem 7.7 (Part of Theorem 5.3.2 of [17], see also [25])**.** *Let $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ where $a_4 \neq 0$, $P(x) \in \mathbf{R}[x]$, and Δ, its discriminant.*
Let $P_1(X) = a_4\left(X^4 + pX^2 + qX + r\right)$ the image of $P(x)$ by the translation $X = x + \frac{a_3}{4a_4}$, whose discriminant is noted by Δ_1. Then $\Delta = \Delta_1$.
When $q \neq 0$, one has

$$\begin{aligned}\Delta =& \frac{a_4^6}{27}\left[4\left(p^2+12r\right)^3 - \left(8p^3+27q^2-6p(p^2+12r)\right)^2\right]\\ =& \frac{a_4^6}{27}\left[-729\,q^4 + 108\,p\left(-p^2+36\,r\right)q^2 + 432r\left(p^2-4\,r\right)^2\right].\end{aligned} \tag{7.6}$$

(i) *If $\Delta < 0$, then $P(x)$ has two real simple roots and two complex simple roots.*
(ii) *If $\Delta = 0$, then:*
- *for $p^2 + 12r = 0$, $P(x)$ has a real triple root and a real simple root;*
- *for $p^2 + 12r \neq 0$ and ($p^2 - 4r \leq 0$ or $p \geq 0$), $P(x)$ has a real double root and two complex simple roots;*
- *for $p^2 + 12r \neq 0$ and $p^2 - 4r > 0$ and $p < 0$, $P(x)$ has a real double root and two real simple roots.*

(iii) *If $\Delta > 0$, then:*
- *for $p^2 - 4r \leq 0$ or $p \geq 0$, $P(x)$ has four complex simple roots;*
- *for $p^2 - 4r > 0$ and $p < 0$, $P(x)$ has four real simple roots.*

Theorem 7.8. *The restriction of the curve $P(\alpha, \delta, \lambda) = 0$ to the first quadrant in (α, δ)-plane is represented in Figure 7.3. More precisely:*
(i) *The line $\alpha = \frac{\beta^2}{4}$ (resp. $\delta = 0$) is a vertical (resp. horizontal) asymptote.*
(ii) *For λ small, the curve $P(\alpha, \delta, \lambda) = 0$ admits a single point of turning-back (coming from infinity when $\lambda = 0$), with coordinates $(\alpha_r = \alpha(\lambda), \delta_r = \delta(\lambda))$ and such that $\alpha_r \to +\infty, \delta_r \to 0^+$ when $\lambda \to 0$.*
(iii) *The restriction of the curve $P(\alpha, \delta, \lambda) = 0$ to the first quadrant in (α, δ)-plane always admits a point of self-intersection.*

Proof. **(i)** Considering $P(\alpha, \delta, \lambda)$ as a polynomial in δ, $\left(4\,\alpha - \beta^2\right)$ is a factor of $A(\alpha, \lambda)$, the coefficient of δ^4. Considering $P(\alpha, \delta, \lambda)$ as a polynomial in α, δ^4 is a factor of the coefficient of α^4; the result follows.

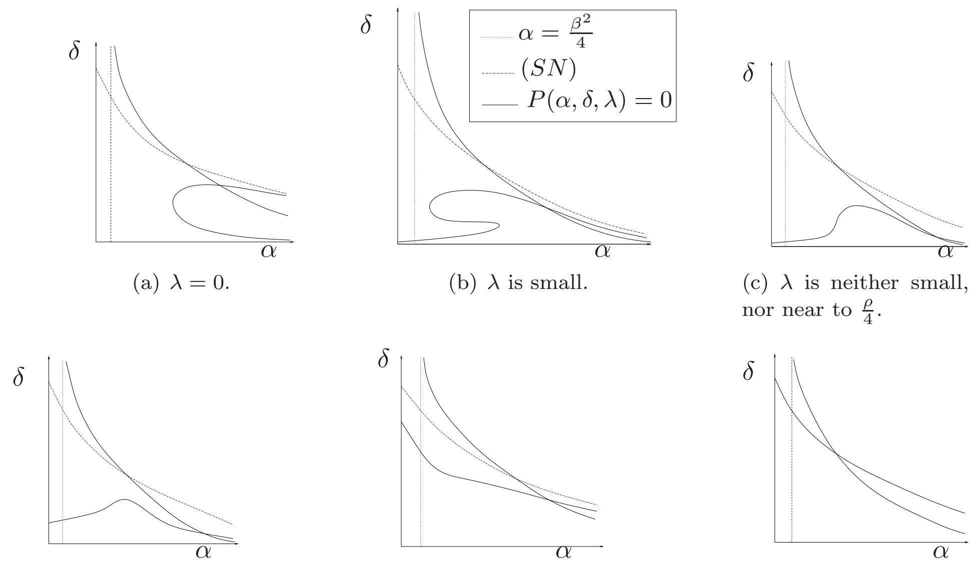
(ii) Let λ be small. Let $\tau := \frac{1}{\alpha}$, $P(\alpha, \delta, \lambda) := P_1(\tau, \delta, \lambda)$ and $\frac{\partial P}{\partial \delta}(\alpha, \delta, \lambda) := P_2(\tau, \delta, \lambda)$. Let $D_\lambda(\tau)$ be the resultant of $P_1(\tau, \delta, \lambda)$ and $P_2(\tau, \delta, \lambda)$ in δ. Then,

$$\begin{aligned}D_\lambda(\tau) = \;& \tau^2\left[\left(-24+3\,\beta^2\tau\right)\lambda^2 + \left(4\,\rho\,\tau^2\beta^2 + \left(-4\,\rho\,\beta + 8\,\rho\right)\tau\right)\lambda + 2\,\rho^2\tau^2\beta - 5\,\rho^2\tau\right]\\ & \left[\left(-128+\tau^2\beta^4+16\,\beta^2\tau\right)\lambda^4 + \left(64\,\rho\,\tau^2\beta^2 - 64\,\rho\,\left(\beta-2\right)\tau\right)\lambda^3\right.\\ & \left. + \left(-2\,\beta\,\rho^2\left(-16+\beta\right)\tau^2 - 48\,\rho^2\tau\right)\lambda^2 + \tau^2\rho^4\right] := \tau^2 D_1(\tau, \lambda) D_2(\tau, \lambda).\end{aligned}$$

One has that

$$D_1(0,0) = 0 \quad \textit{and} \quad \frac{\partial D_1}{\partial \tau}(0,0) = -5\rho^2. \tag{7.7}$$

Therefore, by the implicit function theorem, for λ small, there exists only one solution $\tau = \tau(\lambda)$ in the neighborhood of 0 (and thus only one $\alpha(\lambda) := \alpha_r \to +\infty$) such



(a) $\lambda = 0$.

(b) λ is small.

(c) λ is neither small, nor near to $\frac{\rho}{4}$.

(d) λ is close to $\frac{\rho}{4}$.

(e) λ is very close to $\frac{\rho}{4}$.

(f) $\lambda = \frac{\rho}{4}$.

FIGURE 7.3. Branches of the curve $P(\alpha, \delta, \lambda) = 0$ in the first quadrant of (α, δ)-plane.

that $D_1(\tau(\lambda), \lambda) = 0$. However, by the Newton diagram [10] of $D_1(\tau, \lambda)$, it is seen that the dominant terms of $D_1(\tau, \lambda)$ are $-24\lambda^2$ and $-5\,\rho^2\tau$; then $D_1(\tau, \lambda)$ is approximated, for (λ, τ) small, by $-24\lambda^2 - 5\,\rho^2\tau$. Similarly, $D_2(\tau, \lambda)$ is approximated, for (λ, τ) small, by $\tau^2\rho^4 - 48\,\rho^2\tau\lambda^2 - 128\lambda^4$. For the approximation of $D_1(\tau, \lambda)$, the corresponding root in τ is strictly negative: it is excluded! For the approximation of $D_2(\tau, \lambda)$, there are two roots in τ of opposite sign: there thus exists only one strictly positive root in τ. Consequently, there exists $\delta_r = \delta(\lambda)$, a common root of $P_1(\tau, \delta)$ and $P_2(\tau, \delta)$. Lastly, it is clear that $\delta_r \to 0^+$ since the line of equation $\delta = 0$ is, for λ small (resp. $\lambda = 0$) a horizontal asymptote of all the branches (resp. a component) of the curve $P(\alpha, \delta, \lambda) = 0$.

(iii) Indeed, the discriminant of $P(\alpha, \delta, \lambda)$ in δ is of the form $\Delta(\alpha, \beta, \lambda) := 4\rho^2\Delta_1(\alpha, \beta, \lambda)[\Delta_2(\alpha, \beta, \lambda)]^2$, where:

$$\begin{aligned}\Delta_2(\alpha, \beta, \lambda) = &-2\,\rho^2\lambda\,\alpha^2 + \rho\left(18\,\beta^2\lambda^2 + \beta^3\lambda^2 + 2\,\rho^2 + 16\,\rho\,\beta\,\lambda - \rho^2\beta\right)\alpha \\ &+ \beta^2\left(\rho + \rho\,\beta + \lambda\,\beta^2\right)\left(\beta^2\lambda^2 - 2\,\rho\,\lambda + \rho^2 + 2\,\rho\,\beta\,\lambda\right)\end{aligned} \tag{7.8}$$

and $\Delta_1(\alpha, \beta, \lambda)$ is a polynomial of degree 4 in α.

However, since $\rho \geq 4\lambda$, then $\Delta_2(\alpha, \beta, \lambda)$ (which is a polynomial of degree 2 in α) admits two roots of opposite sign. Thus, there exists only one positive root $\alpha := \alpha^* > 0$ such that $\Delta_2(\alpha^*, \beta, \lambda) = 0$ (and hence $\Delta(\alpha^*, \beta, \lambda) = 0$). Consequently, the equation curve $P(\alpha, \delta, \lambda) = 0$ admits a point of self-intersection. This point, (either double or triple), is in the first quadrant in (α, δ)-plane: indeed, it is located there for $\lambda = \frac{\rho}{4}$ and for $\lambda = 0$, and it does not cross the axes; therefore, by continuity, it remains there. Otherwise, one would have more than one intersection point with the δ-axis; this is not the case according to Proposition 7.9 below. Or, if it passes at infinity, there would be only one infinite branch in the direction α; this is not the case according to Proposition 7.13 below. However, as will be seen in Proposition 7.17 below, the highest branch of the curve $P(\alpha,\delta,\lambda)=0$ in the first quadrant in (α,δ)-plane is nonadmissible for the Hopf bifurcation. Thus this point of intersection between a relevant branch and a nonrelevant branch of the equation curve $P(\alpha,\delta,\lambda)=0$ is not relevant for our bifurcation diagram. □

There are five special cases where the study of the equation $P(\alpha,\delta,\lambda)=0$ is easier:

$$\alpha=0,\quad \alpha=\frac{\beta^2}{4},\quad \delta=\frac{1}{\alpha+2\beta+4},\quad \lambda=0,\quad \lambda=\frac{\rho}{4}.$$

We will study each of the five cases. Indeed, let us note

$$P_{(\alpha,\lambda)}(\delta):=P(\alpha,\delta,\lambda)=0 \tag{7.9}$$

the equation with unknown δ.

Proposition 7.9. *If $\alpha=0$ and $\rho>4\lambda$, then the equation (7.9) admits only one positive solution, $\delta_1\in]0,\frac{1}{2(\beta+2)}[$.*

Proof. One has $P_{(0,\lambda)}(0)=4\,\lambda^2>0$ and $P_{(0,\lambda)}\left(\frac{1}{2(\beta+2)}\right)<0$; thus, there exists $\delta_1\in]0,\frac{1}{2(\beta+2)}[$ such as $P(0,\delta_1,\lambda)=0$.
Let us show the uniqueness of δ_1. $P_{(0,\lambda)}(\delta)$ is a polynomial of degree 3 in δ, $P_{(0,\lambda)}(0)>0$ and $\lim_{\delta\to+\infty}P_{(0,\lambda)}(\delta)=-\infty$. Then, $P_{(0,\lambda)}(\delta)$ has an odd number of positive roots, that is to say 1 or 3. However, if there were 3 positive roots, then the coefficients of $P_{(0,\lambda)}(\delta)$ must have alternate signs; but this is not the case because the coefficients of δ^3 and δ^2 are both negative. Thus $P_{(0,\lambda)}(\delta)$ has exactly one positive root. □

Remark 7.10. *When $\alpha=\frac{\beta^2}{4}$ and $\rho>4\lambda$, the number of solutions of the equation (7.9) is not relevant to obtain Figure 7.3. Nevertheless, this case is studied in detail in* [17].

Proposition 7.11. (1) *For $\lambda\neq\frac{\rho}{4}$ $(SN):\delta=\frac{1}{\alpha+2\beta+4}$ intersects the curve of equation $P(\alpha,\delta,\lambda)=0$ in a single point of coordinates*

$$(\alpha_0,\delta_0)=\left(\frac{(\beta+2)\left(\lambda\,\beta^2+(\rho+4\lambda)\,(\beta+1)\right)}{\rho},\frac{\rho}{(\beta+2)\,(\lambda\,\beta^2+(\rho+4\lambda)\,\beta+3\,\rho+4\,\lambda)}\right). \tag{7.10}$$

(2) *For $\lambda=\frac{\rho}{4}$, $(SN):\delta=\frac{1}{\alpha+2\beta+4}$ is a branch of the curve $P(\alpha,\delta,\lambda)=0$.*

Proof. By substituting $\delta=\frac{1}{\alpha+2\beta+4}$ in $P(\alpha,\delta,\lambda)$, we get

$$P_\lambda(\alpha):=\frac{2}{(\alpha+2\beta+4)^4}\,(\beta+4)^2\,(\rho-4\,\lambda)\left(\alpha\rho-(\beta+2)\left(\lambda\,\beta^2+(\rho+4\lambda)\,(\beta+1)\right)\right). \tag{7.11}$$

Hence, $P_\lambda(\alpha)=0$ if, and only if $\rho=4\lambda$ or $\alpha=\alpha_0:=\frac{(\beta+2)\left(\lambda\,\beta^2+(\rho+4\lambda)(\beta+1)\right)}{\rho}$; in substituting $\alpha=\alpha_0$ in (SN), one gets $\delta=\delta_0:=\frac{\rho}{(\beta+2)(\lambda\,\beta^2+(\rho+4\lambda)\beta+3\,\rho+4\,\lambda)}$. □

Remark 7.12. *When $\lambda=0$, the curve $P(\alpha,\delta,\lambda)=0$ is represented in Figure 7.3(a). More precisely, (see Proposition 1.5.2 of chapter 1 of* [17]*), the aforementioned curve has:*

• *four branches at infinity in the direction α: one is $\delta = 0$ and the 3 others are given by the roots of a polynomial of degree 3 in δ. Indeed, for large α, it is easily seen that the discriminant of $P(\alpha, \delta, 0)$ with respect to δ is strictly positive and that the second condition of Theorem 7.7, case **(iii)**, is satisfied.*
• *only two branches at infinity in the direction δ since, for large δ, it is clear that the discriminant of $P(\alpha, \delta, 0)$ with respect to α is strictly negative and we conclude by case **(i)** of Theorem 7.7.*

Proposition 7.13. *When $\lambda > 0$, the curve $P(\alpha, \delta, \lambda) = 0$ has two branches at infinity in each of the directions α and δ.*

Proof. Indeed:
• for large α , the discriminant of $P(\alpha, \delta, \lambda)$ in δ has the sign of

$$4\rho^2(-27\rho^2\lambda^2\alpha^4)\left(-2\rho^2\lambda\alpha^2\right)^2, \tag{7.12}$$

which is negative.
• for large δ, the discriminant of $P(\alpha, \delta, \lambda)$ in α has the sign of

$$\begin{aligned}&-\delta^{16}\rho^2\left[\beta^2\rho(-\rho+4\lambda)(\beta^2\lambda+4\rho+2\beta\rho)^2\delta^4\right]\\ &\left[\beta(\beta^2\lambda+\rho+\beta\rho)(\lambda\beta+\rho)(\beta^2\lambda+4\rho+2\beta\rho)\delta^2\right]^2,\end{aligned} \tag{7.13}$$

which is negative. One concludes by **(i)** of Theorem 7.7. □

Proposition 7.14. *When $\lambda = \frac{\rho}{4}$, the curve $P(\alpha, \delta, \lambda) = 0$ is represented in Figure 7.3(f); more precisely:*

(1) ***(i)*** *$P(\alpha, \delta, \lambda) = 0$ if, and only if $(SN) : \delta = \frac{1}{\alpha+2\beta+4}$ or*

$$\begin{aligned}Q_{(\alpha,\beta)}(\delta) :=&\alpha\,(\alpha+2\,\beta+4)\left(-4\,\alpha+\beta^2\right)\delta^3+\left(2\,\beta^3+12\,\alpha^2+4\,\beta^2-2\,\alpha\,\beta^2\right)\delta^2\\ &+\left(\beta^2+8\,\beta-12\,\alpha\right)\delta+4=0.\end{aligned} \tag{7.14}$$

(ii) *(SN) intersects the curve of equation (7.14) at the point with coordinates $(\alpha_0(\beta), \delta_0(\beta)) = \left(\frac{(\beta+2)\left(\beta^2+8\beta+8\right)}{4}, \frac{4}{(\beta+2)(\beta^2+8\beta+16)}\right)$.*

(2) *$Q_{(\alpha,\beta)}(\delta) = 0$ does not intersect $\alpha = \frac{\beta^2}{4}$ in the first quadrant.*

(3) *If $\alpha \neq \frac{\beta^2}{4}$, then the equation (7.14):*
(i) *admits only one positive solution if $\alpha > \frac{\beta^2}{4}$;*
(ii) *does not admit any positive solution if $\alpha < \frac{\beta^2}{4}$.*

Proof. (1) **(i)** Indeed,

$$\begin{aligned}P_\lambda(\alpha,\delta)|_{\rho=4\lambda} :=& -\lambda^2\left(\delta(\alpha+2\beta+4)-1\right)[\alpha\,(\alpha+2\beta+4)\left(-4\,\alpha+\beta^2\right)\delta^3\\ &+\left(12\alpha^2+(4-2\alpha)\beta^2+2\beta^3\right)\delta^2+\left(\beta^2+8\beta-12\alpha\right)\delta+4],\end{aligned} \tag{7.15}$$

from which the result follows.
(ii) is an immediate consequence of Proposition 7.11.

(2) When $\alpha = \frac{\beta^2}{4}$, $Q_{(\alpha,\beta)}(\delta)$ becomes

$$Q_\beta(\delta) := \left(4\,\beta^2+2\,\beta^3+\frac{1}{4}\,\beta^4\right)\delta^2+\left(8\,\beta-2\,\beta^2\right)\delta+4, \tag{7.16}$$

whose discriminant in δ is $-64\,\beta^3 < 0$. Then, $Q_\beta(\delta) > 0$.

(3) Let $\alpha \neq \frac{\beta^2}{4}$, the discriminant of $Q_{(\alpha,\beta)}(\delta)$ in δ is

$$D(\beta,\alpha) := -4\,\left(432\,\alpha^2 - 72\,\alpha\,\beta^2 + 16\,\beta^3 + 4\alpha\,\beta^3 - \beta^4\right)\left(2\,\beta^2 + \beta^3 + 8\,\alpha\right)^2, \tag{7.17}$$

whose sign is that of

$$d_\beta(\alpha) := -\left(432\,\alpha^2 - 72\,\alpha\,\beta^2 + 16\,\beta^3 + 4\alpha\,\beta^3 - \beta^4\right). \tag{7.18}$$

(i) If $\alpha > \frac{\beta^2}{4}$, then the coefficients $\alpha\,(\alpha + 2\,\beta + 4)\left(-4\,\alpha + \beta^2\right)$ and 4 of $Q_{(\alpha,\beta)}(\delta)$ are of opposite sign. Since $Q_{(\alpha,\beta)}(\delta)$ is of degree 3 in δ, then (7.14) has 1 or 3 positive solutions. However, for $\alpha > \frac{\beta^2}{4}$ one has that $432\,\alpha^2 > 108\alpha\,\beta^2$ and $4\alpha\,\beta^3 > \beta^5$; from which

$$\begin{aligned} 432\,\alpha^2 - 72\,\alpha\,\beta^2 + 16\,\beta^3 + 4\alpha\,\beta^3 - \beta^4 &> 108\alpha\,\beta^2 - 72\,\alpha\,\beta^2 + 16\,\beta^3 + \beta^5 - \beta^4 \\ &= 36\alpha\,\beta^2 + 16\,\beta^3 + \beta^5 - \beta^4 \\ &> 8\beta^4 + 16\,\beta^3 + \beta^5 > 0. \end{aligned} \tag{7.19}$$

Thus, $D(\beta,\alpha) < 0$. Therefore, (7.14) has only one real solution which is positive.

(ii) If $\alpha < \frac{\beta^2}{4}$, then the coefficients $\alpha\,(\alpha + 2\,\beta + 4)\left(-4\,\alpha + \beta^2\right)$ and 4 of $Q_{(\alpha,\beta)}(\delta)$ have the same sign. Therefore, (7.14) has 0 or 2 positive solutions. However, the discriminant of $d_\beta(\alpha)$ in α is $16\,\beta^3\,(\beta - 12)^3$. Then,
• for $\beta < 12$, (7.14) has only one real solution which is negative;
• for $\beta > 12$, then $d_\beta(\alpha)$ admits two distinct roots $\alpha_1 < \alpha_2$:
- if ($\alpha < \alpha_1$ or $\alpha > \alpha_2$), then (7.14) has only one negative real solution;
- if $\alpha \in [\alpha_1, \alpha_2]$, then (7.14) has three negative real solutions. Indeed, roots of $Q_{(\alpha,\beta)}(\delta)$ cannot disappear, nor pass at infinity (see (7.14)); then, by continuity, they will never change sign!

□

Proposition 7.15. *The curve $P(\alpha,\delta,\lambda) = 0$ in (α,δ)-plane admits, for $\lambda = \lambda^\star$, a contact point of order 2, $(\alpha^\star,\delta^\star)$, with the vertical direction, where $\lambda^\star \in]0, \frac{\rho}{4}[$, $\alpha := \alpha^\star > 0$ and $\delta = \delta^\star \in]0, \frac{1}{\alpha+2\beta+4}[$ are such as*

$$\begin{cases} P(\alpha,\delta^\star,\lambda^\star) = 0, \\ P_\delta^{'}(\alpha,\delta^\star,\lambda^\star) = 0, \\ P_\delta^{''}(\alpha,\delta^\star,\lambda^\star) = 0, \\ P_\delta^{'''}(\alpha,\delta^\star,\lambda^\star) \neq 0. \end{cases} \tag{7.20}$$

Proof. Indeed:
- for λ small, there are two points where the tangent to the curve $P(\alpha,\delta,\lambda) = 0$ is vertical in the first quadrant in (α,δ)-plane;
- for λ close to $\frac{\rho}{4}$, there is no point with vertical tangent with the equation curve $P(\alpha,\delta,\lambda) = 0$.
However, *these two points cannot leave the first quadrant* (otherwise, there would be 2 or 3 points of intersection with the δ-axis; this is not the case according to Proposition 7.9!) *nor go at infinity* (otherwise, one would have 4 branches at infinity in the direction α; this is not the case according to Proposition 7.13!): hence, by continuity, *they merge!* □

This ends the proof of Theorem 7.8.

Remark 7.16. *Let us return to Figure 7.3. Intuitively:*

(1) *When λ is small, case (b) of Figure 7.3 appears:*
- from the perturbation of two curves when $\lambda = 0$: $\delta = 0$ (nonadmissible) and the Hopf bifurcation curve;
*- from the existence of a point of turning-back coming from infinity (see **(iii)** of Theorem 7.8).*
(2) *Figure 7.3 (c) is the necessary passage to go from (b) to (d).*
(3) *When λ tends to $\frac{\rho}{4}$, the lower branch of the curve $P(\alpha, \delta, \lambda) = 0$ tends to (SN) in the first quadrant of (α, δ)-plane.*

Proposition 7.17. (1) *The branch of $P(\alpha, \delta, \lambda) = 0$ which intersects (SN) is not related to a singular point located in the first quadrant.*
(2) *The branch of $P(\alpha, \delta, \lambda) = 0$ which intersects the lines of equations $\alpha = \frac{\beta^2}{4}$ and $\alpha = 0$ is admissible for the Hopf bifurcation.*

Proof. Indeed:

(1) $P(\alpha, \delta, \lambda) = 0$ intersects (SN) in (α_0, δ_0), defined by (7.10). If one evaluates $f(x) = (\alpha\delta - 1)x^2 + \beta\delta x + \delta$ in (α_0, δ_0), one obtains

$$F(x) := \frac{\rho\,(1 - 2\,x)\,((\beta + 2)\,x + 1)}{(\beta + 2)\,(\lambda\,\beta^2 + (\rho + 4\lambda)\,\beta + 3\,\rho + 4\,\lambda)} \tag{7.21}$$

with roots in x given by $x = \frac{1}{2}$ and $x = -\frac{1}{\beta+2}$. But, from the expression of the trace (see in (7.5)), one has that

$$u\left(\frac{1}{2}\right) = \frac{1}{8\,(\beta + 4)\,(4\,\lambda - \rho)} < 0 \quad \textit{and} \quad u\left(-\frac{1}{\beta + 2}\right) = 0, \tag{7.22}$$

i.e. the trace vanishes only at the singular point whose x-coordinate is negative; this singular point is thus not first quadrant. Therefore, by continuity, the branch of solution of $P(\alpha, \delta, \lambda) = 0$ which intersects (SN) is not related to a singular point of the first quadrant (because, since the x-coordinate of the singular point can pass neither by 0, nor at infinity, then it will never change sign!).

(2) Indeed, by substituting $\alpha = \frac{\beta^2}{4}$ in $P(\alpha, \delta, \lambda)$, $f(x)$ (characterizing the x-coordinate of the singular point) and $u(x)$ (characterizing the trace in this singular point), one obtains respectively:

$$\begin{aligned} P_{(\beta,\lambda)}(\delta) := &-\frac{1}{16}\beta^2\left(\lambda\,\beta^2 + 4\,\rho + 2\,\beta\,\rho\right)^2\delta^3 \\ &+ \frac{1}{4}\beta\left[3\,\lambda^2\beta^3 + 4\,\lambda\,\rho\,\beta^2 + \rho\,(5\,\rho - 24\,\lambda)\,\beta - 8\,\rho^2\right]\delta^2 \\ &+ \left(-3\,\beta^2\lambda^2 - \rho^2\right)\delta + 4\,\lambda^2, \end{aligned} \tag{7.23}$$

$$f_1(\delta, x) := \frac{1}{4}\,(x\beta + 2)^2\,\delta - x^2 \quad \textit{and} \tag{7.24}$$

$$u_1(x) := -\frac{1}{2}\rho\,\beta^2 x^4 + \rho\,\left(\frac{1}{4}\beta^2 - \beta\right)x^3 + (-\rho + \lambda\,\beta)\,x + 2\,\lambda. \tag{7.25}$$

However $P_\beta(\delta,\lambda)$ and $f_1(\delta,x)$ admit a common root in δ if, and only if their resultant in δ, $R_0(x)$, vanishes. Since

$$
\begin{aligned}
R_0(x) :=& -\frac{1}{4}\left(-4\lambda+2\rho\,x-\rho x^2\beta+2x^3\rho\beta\right)[(3\rho\beta^2+2\rho\beta+4\lambda\beta^3)x^3\\
&+(5\rho\beta+12\lambda\beta^2)x^2+(2\rho+12\lambda\beta)x+4\lambda],
\end{aligned}
\tag{7.26}
$$

then $R_0(x)$ vanishes at $x>0$ if, and only if

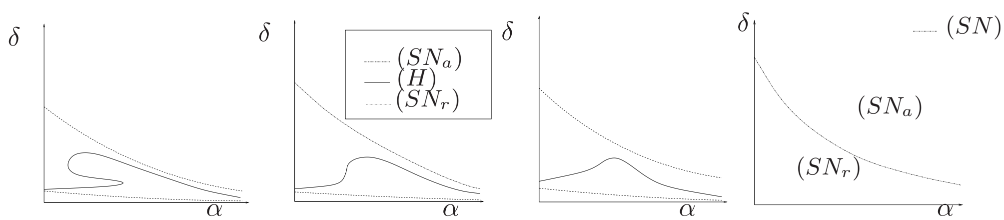
$$
R_1(x) := x^3-\frac{1}{2}x^2+\frac{x}{\beta}-2\,\frac{\lambda}{\beta\,\rho}=0. \tag{7.27}
$$

But at this singular point, the trace vanishes if, and only if the resultant in x of $R_1(x)$ and $u_1(x)$ is zero. A calculation shows that this resultant indeed vanishes identically. Also, $R_1(x)$ has only positive root(s), either 1 or 3; moreover, since the branch of solution of $P(\alpha,\delta,\lambda)=0$ which intersects $\alpha=\frac{\beta^2}{4}$ strictly lies between (SN_r) and (SN), one thus has that $x_0\in]\frac{1}{2}-\eta,\frac{1}{2}[$. Consequently, the y-coordinate of the singular point and the determinant of the linearized system are strictly positive. The result follows by continuity.

□

This finishes the proof of Theorem 7.3. As a consequence of Theorems 7.3, 7.2 and Proposition 7.4, we have the following result.

Corollaire 7.18. *The local bifurcation diagram of codimension* 1 *of the system (1.4) in the first quadrant is given in Figure 7.4.*



(a) λ is small. (b) λ is neither small, nor near to $\frac{\rho}{4}$. (c) λ is close to $\frac{\rho}{4}$. (d) $\lambda=\frac{\rho}{4}$.

FIGURE 7.4. Surfaces of the local bifurcations of the system (1.4) in the first quadrant of the plan (α,δ).

The bifurcations we are missing are global: the heteroclinic loop bifurcation and the double cycle bifurcation.

7.4. Position of the separatrices of the singular points of the x-axis for the system (1.4).

Theorem 7.19. *Let $C=(x_{01},0)$ and $D=(x_{02},0)$ be the singular points on the x-axis. As soon as C or D is a saddle point, its separatrix cannot have a vertical asymptote: the separatrix comes from the right side for C, and from the left for D as illustrated on the Figure 7.6. In particular, when C and D are saddle points, the three possible positions of their separatrices are presented in Figure 7.7.*

Proof. We have $x_{01} = \frac{1}{2} - \eta$ and $x_{02} = \frac{1}{2} + \eta$ for $\eta := \frac{\sqrt{\rho(\rho-4\lambda)}}{2\rho}$.
• The separatrix of C and D cannot go at infinity. Indeed,

$$\begin{aligned}\lim_{y\to+\infty}\left(\frac{dy}{dx}\right) &= \lim_{y\to+\infty}\left(\frac{\dot{y}}{\dot{x}}\right)\\ &= \lim_{y\to+\infty}\frac{y(-\delta+p(x))}{-yp(x)+\rho x(1-x)-\lambda}\\ &= \frac{\delta-p(x)}{p(x)} := L(x). \end{aligned} \tag{7.28}$$

However, L is bounded since continuous on the compact $[x_1, x_2]$. Since the slope of the field is bounded between the lines of equation $x = x_1$ and $x = x_2$, the separatrices of C and D cannot go at infinity.
Moreover, $\dot{x} < 0$ on $x = x_1$ and $x = x_2$, which ensures that, in the neighborhood of the singular points, the portion of the separatrix contained in the first quadrant lies inside the strip $\{x \in]x_1, x_2[\}$ (see Figure 7.5).

There are thus two possible positions for the left separatrix and two possible

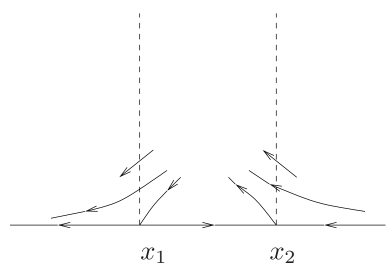


FIGURE 7.5. Behavior of the trajectories at the neighborhoods of C and D (when those are saddle points).

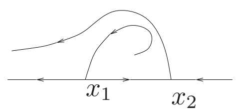
positions for the right separatrix (see Figure 7.6).
• If C and D are saddle points, then the coordinates of the third singular point

(a) possible positions of the separatrix of C

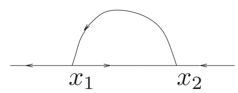
(b) possible positions of the separatrix of D

FIGURE 7.6. Possible positions of the separatrices of the singular points on the x-axis for the system (1.4)

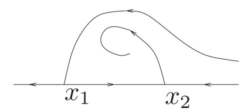
$E = (x_0, y_0)$ satisfy $x_1 < x_0 < x_2$ and $y_0 > 0$.
Hence, when C and D are saddle points, the three possible configurations of their separatrices are presented in Figure 7.7. □



(a) 1^{st} case



(b) 2^{nd} case



(c) 3^{rd} case

FIGURE 7.7. Possible positions of the separatrices of the singular points C and D when they are saddle points.

Proposition 7.20. *For $\lambda, \alpha, \beta, \rho$ fixed such that C and D are saddle points, the portion of the separatrix of C (resp. D) inside the strip $\{x \in]x_1, x_0]\}$ (resp. $\{x \in [x_0, x_2[\}$) lies above the isocline $\dot{x} = 0$. In this region, corresponding to $\dot{x} < 0$, the vector field is rotational with respect to δ and the separatrices move in a monotonous way when δ increases: the separatrix of C (resp. D) moves up (resp. down) when δ increases. Hence, there is at most one value of δ for which a heteroclinic loop bifurcation occurs.*

Proof. For $\lambda, \alpha, \beta, \rho$ fixed, one easily checks that the vector product of the vector field evaluated at $\delta = \delta_1$ with the vector field evaluated at $\delta = \delta_2$ does not vanish in the region $\dot{x} < 0$ if $\delta_1 \neq \delta_2$. The monotonous movement of the separatrices is shown for example in [30]. $\square$

7.5. **Heteroclinic loop bifurcation.** In figure 7.4, let us consider the following case

$$\lambda \in]0, \frac{\rho}{4}[\quad and \quad p\left(\frac{1}{2} - \eta\right) < \delta < p\left(\frac{1}{2} + \eta\right). \tag{7.29}$$

There are exactly three singular points, C, D and E where $C = (\frac{1}{2} - \eta, 0)$ and $D = (\frac{1}{2} + \eta, 0)$ are saddle points located on the x-axis and $E = (x_0, y_0)$ is in the first quadrant.

Proposition 7.21. (1) *A heteroclinic loop bifurcation occurs in the parameter region limited by (SN_r) and (SN_a). More precisely, on each line $\alpha =$ constant, there exist a unique point of heteroclinic loop bifurcation located between the intersection points of this line with (SN_r) and (SN_a).*

(2) *The heteroclinic loop bifurcation surface, $(\mathcal{HL})$, tends to $\delta = 0$ when $\lambda \to 0$.*

Proof. (1) *Existence of the heteroclinic loop bifurcation*

In the region limited by (SN_r) and $(\mathcal{H})$, the position of the separatrices (see Theorem 7.19) and phase portraits of (1.4) (see Figures 2.2, 2.3 and 2.4) in the neighborhood of (SN_r) (where there is no limit cycle because (x_0, y_0) merges with a singular point of the x-axis) and in the neighborhood of H_r (where there is a repelling limit cycle) show that there is necessarily a heteroclinic loop allowing the limit cycle to disappear before the saddle-node bifurcation. Indeed, the limit cycle cannot disappear at infinity (see Theorem 7.19). The uniqueness of the value of δ comes from Proposition 7.20.

(2) For λ small, one has that (SN_r) and the lower branch of $(\mathcal{H})$ tend to $\delta = 0$ (see the equations of (SN_r) and of $(\mathcal{H})$). Since $(\mathcal{HL})$ does not exist any more when $\lambda = 0$, the only possibility is that $(\mathcal{HL})$ tends to $\delta = 0$ when λ tends to 0.

□

The type and the codimension of the heteroclinic loop (for which the connection $\overline{CD}$ is fixed) are given by the following proposition:

Proposition 7.22. *Let*

$$H(\alpha) := \frac{1}{2}\left[p\left(\frac{1}{2}-\eta\right)+p\left(\frac{1}{2}+\eta\right)\right]. \tag{7.30}$$

The heteroclinic loop is:

- *of codimension greater or equal to two, i.e.* $r_C r_D = 1$, *if*

$$(\mathcal{R}) : \delta = H(\alpha); \tag{7.31}$$

- *of codimension* 1 *and repelling (resp. attracting) (i.e.* $r_C r_D < 1$ *(resp.* $r_C r_D > 1$*)) if* $\delta < H(\alpha)$ *(resp.* $\delta > H(\alpha)$*).*

Proof. The criteria $r_C r_D = 1$, $r_C r_D < 1$ and $r_C r_D > 1$ are well-known in the literature (see for example [13]). Recall that, for $p\left(\frac{1}{2}-\eta\right) < \delta < p\left(\frac{1}{2}+\eta\right)$, the points C and D are hyperbolic saddles; their hyperbolicity ratios are given respectively by

$$r_C := \frac{\delta - p(\frac{1}{2}-\eta)}{2\rho\eta} \quad \text{and} \quad r_D := \frac{2\rho\eta}{p(\frac{1}{2}+\eta)-\delta}, \tag{7.32}$$

so that

$$r_C r_D - 1 = \frac{2\delta - [p(\frac{1}{2}-\eta)+p(\frac{1}{2}+\eta)]}{p(\frac{1}{2}+\eta)-\delta}.$$

By (7.29), the sign of $r_C r_D - 1$ are exactly that of

$$N(\delta,\eta) := 2\delta - \left[p\left(\frac{1}{2}-\eta\right)+p\left(\frac{1}{2}+\eta\right)\right].$$

It is clear that $N(\delta,\eta) = 0$ if and only if (7.31). □

Conjecture 7.23. *In the region limited by* (SN_r) *and* (SN_a)*, the type (attracting or repelling) of the single branch of heteroclinic loop,* $(\mathcal{HL})$*, is determined by the organizing center, i.e. at the time of the nilpotent saddle bifurcation for* $B = (\frac{1}{2}, 0)$ *(triple point for* $\lambda = \frac{\rho}{4}$ *and* $\delta = \frac{1}{\alpha+2\beta+4}$*). This point is of codimension* 2 *except for* $\alpha = \alpha_\beta := \frac{\beta^2+8\beta+24}{\beta+6} > 0$ *where it is of codimension* 3 *when* $\beta > 0$ *and conjectured to be of infinite codimension when* $\beta = 0$*.*

1. *When* $\beta > 0$*, the system has a unique point of heteroclinic loop of codimension* 2 *in the neighborhood of the organizing center of codimension* 3 *(corresponding to* $\rho = 4\lambda$, $\delta = \frac{1}{\alpha+2\beta+4}$ *and* (6.2)*) located at* $HL_2 := (\mathcal{HL}) \cap (\mathcal{R})$ *where* $(\mathcal{R})$ *is given in (7.31) and (7.30). This point is the endpoint of the curve of double limit cycle. All other points along the heteroclinic loop bifurcation curve have codimension* 1*. The conjecture is that this is also the situation for all smaller values of* λ*.*
2. *When* $\beta = 0$ *and we are located at* $(\mathcal{HL}) \cap (\mathcal{R})$*, then it is conjectured that the system (1.4) is integrable with an annulus of periodic solutions surrounded by a center and ending in a heteroclinic loop.*

7.6. **Place of the Hopf bifurcation of order** 2**.** From the study (in Section 5) of the first two Lyapunov coefficients, the Hopf bifurcation of order 2 exists for all the values of $\lambda \in]0, \frac{\rho}{4}[$. We now make a deeper stuy of the locus where the first

Lyapunov coefficient vanishes in the parameter space. Indeed, we saw in Section 5 via (5.10), that the first Lyapunov coefficient is given, modulo a positive factor multiplicative, by $L_1(x)$ in (5.23) where x represents the x-coordinate of E, i.e. the positive root of $f(x)$ given in (4.5). $L_1(x)$ vanishes at the singular point of x-coordinate $x \neq \frac{1}{2}$ if, and only if its last factor vanishes, namely

(7.33)
$$l_1(x,\alpha) := \left(\beta^3 + 2\,\alpha\,\beta - \alpha\,\beta^2\right)x^4 + \left(6\,\beta^2 - 6\,\alpha\,\beta\right)x^3 + \left(6\,\beta - 6\,\alpha\right)x^2 + 4\,\beta\,x + 6 = 0$$

or, according to a equivalent formula to $L_1(x)$ (obtained from $L_1(x)$ by using that $u(x) := -2\,\rho\,\alpha\,x^4 + \rho\,(\alpha-\beta)\,x^3 + (-\rho+\lambda\,\beta)\,x + 2\,\lambda = 0$, expressing the fact that the trace is zero at the singular point),

(7.34)
$$m_1(x,\lambda) := -2\,\beta^2\rho\,x^5 - 6\,\rho\,\beta\,x^4 + \left(\lambda\,\beta^2 - 2\,\beta\,\lambda - \rho\,\beta - 6\,\rho\right)x^2 + 6\,\lambda\,\beta\,x + 6\,\lambda = 0.$$

It is observed that:
- $\lim_{\alpha\to 0} l_1(x,\alpha) = \beta^3x^4 + 6\,\beta^2x^3 + 6\,\beta\,x^2 + 4\,x\beta + 6 > 0$ for all $x \in]\frac{1}{2}-\eta, \frac{1}{2}[$;
- $\lim_{\alpha\to+\infty} l_1(x,\alpha) = \lim_{\alpha\to+\infty}[-\alpha x^2\left(-2\,x\beta\,(x-3) + x^2\beta^2 + 6\right)] < 0$ for all $x \in]\frac{1}{2}-\eta, \frac{1}{2}[$;
- $\lim_{\lambda\to 0} m_1(x,\lambda) = -\rho\,x^2\left(2\,\beta^2x^3 + 6\,\beta\,x^2 + \beta + 6\right) < 0$ for all $x \in]\frac{1}{2}-\eta, \frac{1}{2}[$;
- When $\lambda \to \left(\frac{\rho}{4}\right)^-$, $(\mathcal{H})$ tends to (SN) and $x \to \left(\frac{1}{2}\right)^-$. Then,

$$\begin{aligned}\lim_{\lambda\to\left(\frac{\rho}{4}\right)^-} m_1(x,\lambda) = &-\frac{1}{4}\,\rho\,(-1+2\,x)\,[4\,x^4\beta^2 + 12\,\beta\,x^3 + 2\,\beta^2x^3 \\ &+ 6\,\beta\,x^2 + x^2\beta^2 + 12\,x + 6\,x\beta + 6] \to 0^+.\end{aligned}$$

(7.35)

- In the parameter space, $L_1(x)$ and $f(x)$ vanish at the singular point with x-coordinate given by x if and only if the resultant of $l_1(x,\alpha)$ and $f(x)$ with respect to x, which we call $L(\alpha,\delta)$, vanishes. We have that

$$\begin{aligned}L(\alpha,\delta) = &\left(2\,\beta^3 + \beta^2 - 2\,\alpha\,\beta^2 - 6\,\alpha\,\beta + 9\,\alpha^2\right)\left(\beta^2 - 4\,\alpha\right)^2\delta^4 \\ &+ 12\,\left(\beta^2 - 4\,\alpha\right)\left(2\,\beta^3 + \beta^2 - 2\,\alpha\,\beta^2 - 6\,\alpha\,\beta + 9\,\alpha^2\right)\delta^3 \\ &+ \left(468\,\alpha^2 - 264\,\alpha\,\beta + 36\,\beta^2 - 108\,\alpha\,\beta^2 + 84\,\beta^3\right)\delta^2 \\ &+ \left(72\,\beta - 216\,\alpha + 8\,\beta^2\right)\delta + 36.\end{aligned} \tag{7.36}$$

Let $(\mathcal{L}) : L(\alpha,\delta) = 0$ and $H_2 := (\mathcal{H}) \cap (\mathcal{L})$ (the locus of the Hopf bifurcation of order 2). Let us determine H_2.
As for $(\mathcal{H})$, we will be interested in what occurs when λ is small, λ is close to $\frac{\rho}{4}$ and when λ is neither small, nor close to $\frac{\rho}{4}$:

7.6.1. *Locus of the Hopf bifurcation of order* 2 *when* λ *tends to* 0.

Proposition 7.24. *When λ is small, the Hopf bifurcation of order 2 is located very far on the right along the Hopf curve (i.e. for α very large) and passes at infinity when $\lambda = 0$.*

Proof. For $\lambda = 0$, there occurs a supercritical Hopf bifurcation of order one (see [29]). Then, by structural stability, it comes that [27]: for each compact K in the space $(\alpha,\beta,\rho,\delta)$, there exists $\lambda_K > 0$ such that the supercritical Hopf bifurcation persists for $(\alpha,\beta,\rho,\delta) \in K$ and $\lambda < \lambda_K$. But, since the space $(\alpha,\beta,\rho,\delta)$ is not compact, it is not possible to find a uniform $\lambda > 0$ for this space (indeed, the larger K, the smaller λ_K is). However, for β, ρ, δ fixed inside K, the smaller λ, the larger the corresponding α is. Therefore, for $\lambda \to 0$, the locus of the Hopf bifurcation of order 2, denoted $H_2 := (\alpha_p, \delta_p)$, corresponds to $\alpha_p = \alpha(\lambda)$ with $\alpha(\lambda) \to +\infty$ when $\lambda \to 0$. It remains to show that H_2 cannot escape from the bifurcation diagram for positive λ. Indeed, by (5.23), one has that

$$\begin{aligned}\lim_{\alpha\to+\infty} L_1(x) &= \lim_{\alpha\to+\infty} l_1(x,\alpha)\\ &= \lim_{\alpha\to+\infty}[-\alpha x^2\left(-2\,x\beta\,(x-3)+x^2\beta^2+6\right)] < 0\end{aligned} \tag{7.37}$$

for all $x \in]\frac{1}{2}-\eta, \frac{1}{2}[$ (region of the Hopf bifurcation). □

7.6.2. *Locus of the Hopf bifurcation of order 2 when λ tends to $\left(\frac{\rho}{4}\right)^-$.*

Proposition 7.25. *When λ tends to $\left(\frac{\rho}{4}\right)^-$, one has that:*

(1) *$(\mathcal{H})$ tends to (SN).*
(2) *H_2 tends to the point of coordinates (α_n, δ_n), defined by*

$$\alpha_n := \frac{\beta^2+8\,\beta+24}{\beta+6} \quad \text{and} \quad \delta_n := \frac{\beta+6}{3\,(\beta+4)^2}, \tag{7.38}$$

which corresponds to the nilpotent point of codimension 3.

Proof. (1) This comes from Proposition 7.11 (2).

(2) Indeed:
(i) Let $\sigma := \rho - 4\lambda$. Then, $m_1(x,\lambda)$ is linear in σ. Thus, $m_1(z,\sigma) = 0$ has a unique solution $\sigma = \sigma(x)$ such that $\sigma(\frac{1}{2}) = 0$.
(ii) $l_1(x,\alpha)$ is linear in α and thus has a unique zero $\alpha = \alpha(x)$ such that $\alpha(\frac{1}{2}) = \frac{\beta^2+8\beta+24}{\beta+6}$.
(iii) The result follows by substituting $\alpha = \frac{\beta^2+8\beta+24}{\beta+6}$ in the equation (SN) : $\delta = \frac{1}{\alpha+2\beta+4}$, yielding $\delta = \frac{1}{3}\,\frac{\beta+6}{(\beta+4)^2}$.

□

7.6.3. *Locus of the Hopf bifurcation of order 2 when λ is neither small, nor close to $\frac{\rho}{4}$.*

Conjecture 7.26. *$(\mathcal{H})$ and $(\mathcal{L})$ have only one intersection point in the first quadrant.*

Illustration. Indeed, if there were more than one intersection point, then the passage from one intersection point (when λ is close to $\frac{\rho}{4}$) to two or three intersection points (when λ moves away from $\frac{\rho}{4}$ without being small) would require the existence of a contact point between $(\mathcal{H})$ and $(\mathcal{L})$, i.e. for β, ρ and $\lambda \in]0, \frac{\rho}{4}[$ fixed, there would exist $\alpha = \alpha_c$ and $\delta = \delta_c$ such that

$$\begin{cases} L(\alpha_c,\delta_c) = 0,\\ P(\alpha_c,\delta_c,\lambda) = 0,\\ \nabla L(\alpha_c,\delta_c)//\nabla P(\alpha_c,\delta_c,\lambda) \quad i.e \quad \left(\frac{\partial P}{\partial\alpha}\frac{\partial L}{\partial\delta} - \frac{\partial P}{\partial\delta}\frac{\partial L}{\partial\alpha}\right)(\alpha_c,\delta_c) = 0.\end{cases} \tag{7.39}$$

The Conjecture 7.26 follows from the following conjecture.

Conjecture 7.27. *There is no "admissible" solution of (7.39).*

Such a conjecture can be validated in particular cases with the use Gröbner bases. But, the calculations were quite involved and we limited ourselves to some isolated tests (see [17]).

7.6.4. *Numerical validation of the position of H_2.* Let $\lambda_1 := \frac{\lambda}{\rho}$. For β fixed, some numerical tests highlight that:
• If λ_1 is very small, then H_2 is located not far from the lower point of turning-back (i.e. on the lower branch);
• When $(\mathcal{H})$ has a contact point of order 2 with the vertical direction, then H_2 is located slightly left of this point;
• If λ_1 is close to $\frac{1}{4}$, then H_2 (which, by Proposition 7.25, tends to the point (α_n, δ_n) defined in (7.38)) is located to the left of $\mathcal{C}$ on the curve $(\mathcal{H})$.

7.7. **Bifurcation diagram and phase portraits.** Corollary 7.18, Conjecture 7.23 and Figure 6.3, "prove" Theorem B (of course modulo the conjecture!).

Let us recall that the important parameter is $\lambda_1 := \frac{\lambda}{\rho}$, rather than the two independent parameters λ and ρ. There are thus three essential parameters: α, δ and λ_1. The parameter β seems to be a measure of the non-integrability of the system. Indeed, in the case of two limit cycles or a heteroclinic loop, the larger β, the more hyperbolic the limit cycles or the loop are.

8. A Biological Interpretation

We give a biological interpretation of the system when its parameters are in an open region of the bifurcation diagram (because the vector field is structurally stable there); and in each one of these regions, we make the assumption that the initial conditions are realistic biologically.

Remark 8.1. *Let us note immediately that the model is not realistic in a narrow strip along the y-axis: $x \in [0, \epsilon]$ because the y-axis is not invariant and the trajectories would cross to the region $x < 0$. Therefore, it is necessary to interpret the model for $x > \epsilon$ for some threshold $\epsilon > 0$.*

The regimes which we will define are stable for initial conditions outside the stable or unstable manifolds of singular points, and not on an unstable limit cycle; this implies that the final regime will be the same after a small change in the initial conditions. There are three types of stable regimes:
• REP(**R**egime with **E**xtinction of the **P**redators): it is a regime where there exists an open set of initial conditions for which there is extinction of predators, and the population of preys reaches a stable equilibrum.
• RME(**R**egime of **M**ixted **E**quilibrum): a regime where there exist an open set of initial conditions for which predators and preys co-exist while tending to a stable equilibrum(*attracting singular point in the first open quadrant*).
• OR(**O**scillatory **R**egime): a regime where there exist an open set of initial conditions for which predators and preys tend to a stable oscillatory regime (*stable limit cycle*).
Except for the open regions **I** and **II** (where there is *extinction of the preys*), each generic vector field (i.e. whose parameter values are in an open region of the bifurcation diagram) has one of the stable regimes described above. More precisely:
- The phase portrait of the open region **III** corresponds to the stable regime **RME** under the separatrix of the left singular point and to the extinction of the preys

elsewhere.
- The phase portrait of the open region **V** corresponds to the stable regime **RME** in a small region corresponding to the interior of the limit cycle.
- For parameter values inside the open region **IV** and good initial conditions we obtain the stable regime **REP**.
- For parameter values inside the open region **VI** and **VII** we get the stable regime **OR**, but only for a small open set of initial conditions.

What is striking when we observe these phase portraits, is that the two species become quite vulnerable when one introduces prey harvesting of the type considered here without corresponding harvesting of predators, and this, even if λ is small. A very large set of initial positions leads to the extinction of the two species. For example, it is the case of all the initial conditions in the regions **I** and **II**, even for λ very small. The regions **I** and **II** correspond to δ small, i.e. a low level of mortality of the predators. In the other regions, a very important role is played by the separatrices of the points C and D. Thus, in the regions **V** and **VI**, any initial condition $(x(0), y(0))$, where $x(0)$ is large and $y(0) > 0$, leads to the extinction of the species. It is absolutely necessary to take parameter values above the curve of heteroclinic loop (regions **III**, **IV** and **VII**) to ensure the existence of initial conditions $(x(0), y(0))$, where $x(0)$ is large and $y(0) > 0$, which allow survival of the population of preys: $(x(0), y(0))$ must be located under the separatrix of the point C, i.e. only initial conditions with $y(0)$ small for the predators are allowed for the preys to survive. But, if the number of predators is somewhat large, there is an increasing risk of extinction of the preys. As a general conclusion, for $\lambda = 0$, one observes the survival of the preys for all the values of the remaining parameters and all initial conditions $(x(0), y(0))$, with $x(0), y(0) > 0$. In our model, as soon as λ is positive, for any value of the remaining parameters, there exist initial conditions $(x(0), y(0))$, with $x(0), y(0) > 0$ leading to the extinction of species.

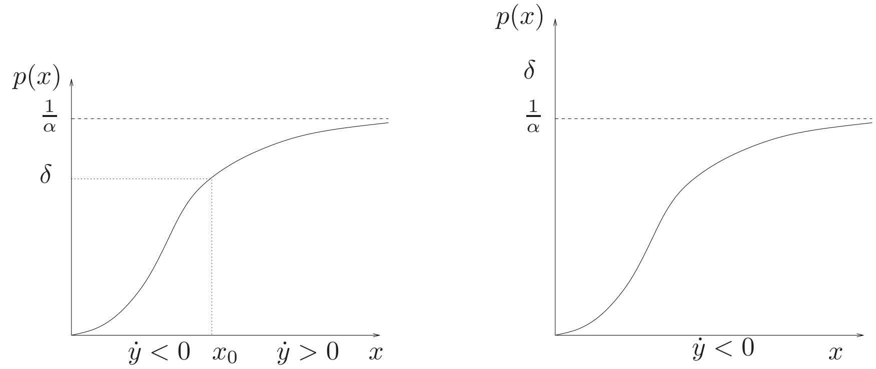
Our results thus suggest several avenues for further research:

- to make a quantitative analysis of the results described above for determining the approximate position of the separatrices of the points C and D, while concentrating on the parameter values which are realistic biologically. This analysis will allow to determine quantitatively the initial conditions leading to the survival of the species;
- to see whether other strategies of harvesting are less "dangerous" ecologically. It is already the case of the rate $S(x, h) = hx$ since the model with such a rate is equivalent, after scaling, to our model (1.4) for $\lambda = 0$ and new values of the remaining parameters;
- to combine the strategy of prey harvesting studied here with a strategy of predator harvesting in order to determine whether the simultaneous harvesting of predators and preys increases the chances of survival of the two species.

Remark 8.2. (1) *When $\lambda = 0$ and $\beta \geq 0$, the phase portraits of the open regions I_0, II_0 and III_0 (see Figure 2.1) correspond respectively to the stable regimes **REP**, **RME** and **OR**.*

(2) *The absolute maximum sustainable yield (MSY) of the prey harvesting is* $\lambda_{MSY} = \frac{\rho}{4}$*: indeed, if* $\lambda \geq \frac{\rho}{4}$*, then* $\dot{x} < 0$ *and there is extinction of the preys.*

(3) *Quantitative analysis:*
Let us recall that $\dot{y} = y(-\delta + p(x))$*.*
a) If $\delta < \frac{1}{\alpha}$*, then there exist* x_0 *such as* $p(x_0) = \delta$ *(see Figure 8.1(a)): if*



(a) $\dot{y}$ for $\delta < \frac{1}{\alpha}$ (b) $\dot{y}$ for $\delta \geq \frac{1}{\alpha}$

FIGURE 8.1. The sign of $\dot{y}$.

$x < x_0$*, then* $\dot{y} < 0$*; if* $x > x_0$*, then* $\dot{y} > 0$*; and if* $x = x_0$*, then* $\dot{y} = 0$*. Thus* y *grows if* x *is large and* y *decrease if* x *is small.*
b) If $\delta \geq \frac{1}{\alpha}$*, then* $\dot{y} < 0$ *for all* $x \geq 0$ *(see Figure 8.1(b)).*

(4) *Extreme cases:*
By (2) and (3), we observe the two following extreme cases:
- *There is extinction of the predators when* δ *is large.*
- *For* $\lambda \geq \frac{\rho}{4}$*, no population can survive.*

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