COMPLETE SYSTEM OF ANALYTIC INVARIANTS FOR UNFOLDED DIFFERENTIAL LINEAR SYSTEMS WITH AN IRREGULAR SINGULARITY OF POINCARÉ RANK k

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ABSTRACT. In this, paper, we give a complete modulus for germs of generic unfoldings of nonresonant linear differential systems with an irregular singularity of Poincaré rank k at the origin, under analytic equivalence. The modulus comprises a formal part depending analytically on the parameters which, for generic values of the parameters, is equivalent to the set of eigenvalues of the residue matrices of the system at the Fuchsian singular points. The analytic part of the modulus is given by unfoldings of the Stokes matrices. For that purpose, we cover a fixed neighbourhood of the origin in the variable with sectors on which we have an almost unique linear transformation to a (diagonal) formal normal form. The comparison of the corresponding fundamental matrix solutions yields the unfolding of the Stokes matrices. The construction is carried on sectoral domains in the parameter space covering the generic values of the parameters corresponding to Fuchsian singular points.

1. INTRODUCTION

In this paper, we provide an analytic classification of germs of generic unfoldings of the systems

$$x^{k+1}y' = A(x)y, \tag{1.1}$$

where A is a matrix of germs of analytic functions in x at the origin such that $A(0)$ has distinct eigenvalues, $x \in (\mathbb{C}, 0)$, $y \in \mathbb{C}^n$, and k is a strictly positive integer called the Poincaré rank. The case $k = 1$ has been completely investigated in [8].

The analytic classification of the nonresonant linear differential systems with an irregular singular point of Poincaré rank k (i.e. the systems (1.1) when the irregular singularity is at the origin) is an important chapter of mathematics, with contributions of many mathematicians. A complete statement in an essentially final form can be found in W. Balser, W.B. Jurkat and D.A. Lutz [1]. The modulus (or complete system of analytic invariants) is given by the formal normal form and equivalence classes of Stokes matrices (a concise presentation can be found in [6] pp. 351–372), and the modulus space has been identified.

The unfolding of the systems (1.1) that we pursue replaces the x^{k+1} in (1.1) by the family of polynomials $p_\epsilon(x)$ of degree $k+1$, turning the system into a generic family of systems parameterized by a multi-parameter $\epsilon \in \mathbb{C}^k$ whose generic element has only Fuchsian singular points. The aim is to relate the analytic invariants of

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the original system to the ones of the generic unfolded system and, from this, to understand the meaning of the Stokes data.

When unfolding the systems (1.1), the modulus is composed of formal and analytic invariants. We prove that the formal invariants are simply given through the formal normal form, which is a diagonal system with the matrix being a polynomial of degree k in x: the formal invariants consist in $n(k+1)$ germs of analytic functions in the parameter. The formal normal form is the polynomial part of degree k of an analytic prenormal form obtained in [8].

The analytic part of the modulus measures the obstruction to the existence of an analytic equivalence between the system and its formal normal form over a fixed neighbourhood $\mathbb{D}_r$ of the origin in x-space. The first aim is in some sense to generalize the Stokes sectors. To do this, we cover a neighbourhood of the origin in x-space minus the singular points with $2k$ sectors over which there exist linear (in y) changes of coordinates (i.e., gauge transformations) to the formal normal form. These linear changes of coordinates are unique up to diagonal changes of coordinates, thus providing natural fundamental matrix solutions over the sectors. The analytic invariants measure the mismatch between the fundamental matrix solutions. They are equivalence classes of *unfolded Stokes matrices*, which converge when $\epsilon \to 0$ to the known Stokes matrices.

The construction of these sectors in x cannot be performed continuously in the parameters ϵ. We will partition the ϵ-space into *sectoral domains*. While this partition can be done for all values of the parameters in a neighbourhood of the origin, it will suffice and be simpler to give it for the open set Σ_0 of generic values of the parameters corresponding to Fuchsian singular points. There will C_k such domains providing a covering of Σ_0, where C_k is the k-th Catalan number. Each of these sectoral domains is invariant under the action of the positive reals given by rescaling of the roots of the polynomial p_ϵ, and so has a natural cone structure with a vertex at the origin; all constructions will converge to a unique construction when $\epsilon \to 0$ along the rescaling. For each sectoral domain in the parameter space, one has a partition of the x-plane into $2k$ sectors, which indeed converge to Stokes sectors.

The construction of the sectors and sectoral domains comes from [10] and [9] and were inspired by [5] and [7]. The closure of the $2k$ sectors in x-space provides a covering of $\mathbb{D}_r$. For a generic parameter value, each sector is adherent to two distinct singular points. Also, each sector intersects its two neighbouring sectors (they are the same for $k = 1$) along two smaller sectors ending on $\partial\mathbb{D}_r$, each adherent to one singular point, and called *intersection sectors*. The sectors are chosen so that, on the intersection sectors, the space of solutions of the system has a natural flag structure provided by the asymptotic behavior of solutions near the singular point. The flags corresponding to the two intersection sectors associated to a sector are transversal if the deformation parameter ϵ is small. Thus, appropriate intersections of the flag subspaces of solutions provide elements of a basis of solutions with good asymptotic behavior at the two singular points; each element of this basis is determined up to a constant (in x) scalar, and so the basis is determined up to the action of the diagonal matrices. The flag structure guarantees that the comparison of the fundamental matrix solutions over an intersection sector is a triangular matrix. When $k = 1$, this procedure gives another proof of the results of [8].

This paper is dedicated to Yulij Ilyashenko who introduced the third author to the Stokes phenomena.

2. Preliminaries

2.1. **The irregular nonresonant singularity of Poincaré rank k.** We consider the linear differential system (1.1). Performing an appropriate rotation in x and a linear change of coordinates in y, we can always suppose that the matrix $A(0)$ is diagonal, and that its (distinct) eigenvalues λ_j satisfy

$$\Re(\lambda_{q_1} - \lambda_{q_2}) > 0 \quad \text{if } q_1 < q_2. \tag{2.1}$$

Remark 2.1. *Once* (2.1) *is realized, there exists $\phi > 0$ such that* (2.1) *remains valid when we rotate the complex variable by any angle $\theta \in (-\phi, \phi)$.*

It is known that a (diagonal) formal normal form of (1.1) is given by

$$x^{k+1}y' = (\Lambda_0 + \Lambda_1 x + \cdots + \Lambda_k x^k)y, \tag{2.2}$$

where the Λ_j are diagonal matrices containing the $n(k+1)$ formal invariants of the system (1.1). Using a linear change of coordinate in y, which is polynomial in x, we can always suppose that the k-jet of $A(x)$ is given by (2.2).

The sectorial normalization theorem of Sibuya asserts that there exist a covering of a disk $\mathbb{D}_r$ in x-space with $2k$ sectors $\Omega_j^\pm$, $j = 1, 2, ..., k$, (see Figure 1), and over each sector $\Omega_j^\pm$, a linear change of coordinate $H_j^\pm(x)$ ($y \mapsto H_j^\pm(x)y$) transforming the normal form (2.2) to the system (1.1). If we take the $H_j^\pm(x)$ to be 0-tangent to the identity (i.e. $H_j^\pm(0) = I$), then all transformations $H_j^\pm$ have the same unique asymptotic expansion. Each sector $\Omega_j^\pm$ contains exactly one ray $\Re x^k = 0$. Its intersection with each neighbouring sector contains exactly one ray $\Im x^k = 0$ (see Figure 1). The sectors we choose are smaller than the ones used by Sibuya, but they are sufficiently large to ensure uniqueness as in Sibuya's theorem. The justification for this choice will become clear in Section 4.

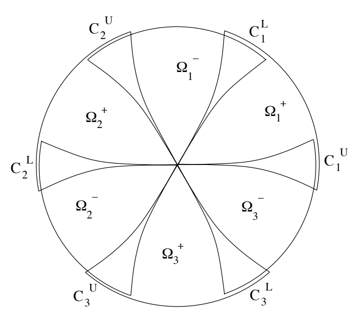


Figure 1. The sectors in x-space and the Stokes matrices, k=3.

Let us fix a diagonal fundamental matrix solution of (2.2),

$$F(x) = x^{\Lambda_k} \exp\left(-\sum_{j=0}^{k-1} \frac{\Lambda_j}{(k-j)x^{k-j}}\right),$$

chosen with the natural determination of the logarithm for $x \in \Omega_1^+$, and analytically continued over the other sectors $\Omega_j^\pm$ $(j \leq k)$ when turning in the positive direction. The monodromy (around $x = 0$, in the positive direction) of this diagonal solution is given by $e^{2\pi i\Lambda_k}$. The matrix

$$W_j^\pm(x) = H_j^\pm(x)F(x)$$

is a fundamental matrix solution of (1.1) over $\Omega_j^\pm$.

Notation 2.2. All indices for sectors $\Omega_j^\pm$, matrices $W_j^\pm(x)$, transformations $H_j^\pm(x)$, etc., are given (mod k), taking values in a range $1 \leq j \leq k$, so that $0 = k$.

We call

$$\begin{cases} \Omega_j^U = \Omega_{j-1}^- \cap \Omega_j^+, \\ \Omega_j^L = \Omega_j^+ \cap \Omega_j^-. \end{cases} \tag{2.3}$$

Over Ω_j^U (resp. Ω_j^L), the sectorial automorphism $(H_{j-1}^-(x))^{-1}H_j^+(x)$ (resp. $(H_j^+(x))^{-1}H_j^-(x)$) is determined by an upper (resp. lower) triangular constant Stokes matrix C_j^U (resp C_j^L) satisfying

$$(H_j^+(x))^{-1}H_j^-(x)F(x) = F(x)C_j^L, \quad \text{on } \Omega_j^L, \quad 1 \leq j \leq k, \tag{2.4}$$

$$(H_{j-1}^-(x))^{-1}H_j^+(x)F(x) = F(x)C_j^U, \quad \text{on } \Omega_j^U, \quad 2 \leq j \leq k, \tag{2.5}$$

and

$$(H_k^-(x))^{-1}H_1^+(x)F(x) = F(x)e^{-2\pi i\Lambda_k}C_1^U, \quad \text{on } \Omega_1^U. \tag{2.6}$$

It is common in the literature to omit the term $e^{-2\pi i\Lambda_k}$ in (2.6) so that all the matrices C_j^U and C_j^L be unipotent. Here, we rather choose the determination of $F(x)$ so that the monodromy (around $x = 0$, in the positive direction) of the fundamental matrix solution $W_0^-(x)$ be given by the product of the Stokes matrices in the right order. Hence the diagonal part of C_1^U is $e^{2\pi i\Lambda_k}$ (the matrices C_j^U for $j \neq 1$ and all matrices C_j^L are unipotent).

Let us now discuss one point of view to understand the particular fundamental matrix solution which we will be able to unfold for the family. For that purpose, let us call $w_j^{\pm,1}(x), \dots, w_j^{\pm,n}(x)$, the particular solutions given by the columns of $W_j^\pm(x)$. The solution $w_j^{\pm,i}(x)$ is asymptotic to $e^{-\frac{\lambda_i}{kx^k}}g_i(x)$ where $g_i(x)$ is a power series such that $g_i(0)$ is a multiple of e_i (and $\lambda_i = (\Lambda_0)_{ii}$). If we note $f \prec g$ for f flatter than g, i.e. $\frac{f}{g} = O(x)$, then we have

$$\begin{cases} e^{-\frac{\lambda_n}{kx^k}} \prec e^{-\frac{\lambda_{n-1}}{kx^k}} \prec \cdots \prec e^{-\frac{\lambda_1}{kx^k}}, & \text{on } \Omega_j^L, \\ e^{-\frac{\lambda_1}{kx^k}} \prec e^{-\frac{\lambda_2}{kx^k}} \prec \cdots \prec e^{-\frac{\lambda_n}{kx^k}}, & \text{on } \Omega_j^U. \end{cases} \tag{2.7}$$

So we have two natural flags on the vector space of solutions on $\Omega_j^\pm$, namely

$$\begin{cases} V_{j,n}^{\pm,L} \subset V_{j,n-1}^{\pm,L} \subset \cdots \subset V_{j,1}^{\pm,L}, & \text{on } \Omega_j^L, \\ V_{j,1}^{\pm,U} \subset V_{j,2}^{\pm,U} \subset \cdots \subset V_{j,n}^{\pm,U}, & \text{on } \Omega_j^U, \end{cases} \tag{2.8}$$

where $V_{j,i}^{\pm,L}$ (resp. $V_{j,i}^{\pm,U}$) is generated by $\{w_j^{\pm,n}, \dots, w_j^{\pm,i}\}$ (resp. $\{w_j^{\pm,1}, \dots, w_j^{\pm,i}\}$). These flags respect the natural order of flatness of solutions on the sectors Ω_j^U and Ω_j^L.

All vector subspaces $V_{j,i}^{\pm,L}$ and $V_{j,n-i+1}^{\pm,U}$ intersect transversally. Hence, $V_{j,i}^{\pm,L} \cap V_{j,n-i+1}^{\pm,U}$ has dimension 1. It has the solution $w_j^{\pm,i}(x)$ as a basis, $i = 1, \ldots, n$.

2.2. **Prenormal form.** Modulo a translation in x and a change of parameter, a generic unfolding of (1.1) has the form

$$p_\epsilon(x)y' = A(\epsilon, x)y, \tag{2.9}$$

where

$$p_\epsilon(x) = x^{k+1} + \sum_{j=0}^{k-1} \epsilon_j x^j, \tag{2.10}$$

$\epsilon = (\epsilon_0, ..., \epsilon_{k-1}) \in \mathbb{C}^k$ and A is a matrix of germs of analytic functions in (ϵ, x) at the origin satisfying $A(\vec{0}, x) = A(x)$ as in (1.1). For the rest of the paper, x and the parameter ϵ will be fixed.

We are interested in equivalence classes of systems (2.9) under the equivalence relations:

Definition 2.3. Two systems $y' = A(\epsilon, x)y$ and $z' = B(\epsilon, x)z$ are locally *analytically equivalent* (respectively *formally equivalent*) if there exists an invertible matrix T of germs of analytic functions of (ϵ, x) at the origin (respectively an invertible matrix of formal series in (ϵ, x)) such that the substitution $y = T(\epsilon, x)z$ transforms the first system into the second system.

In [8], it is shown that it is sufficient to consider the classification of the systems in their prenormal form. Generically, the linear change of coordinates to the formal normal form diverges. A family can be transformed by an analytic equivalence into a prenormal form, which is a perturbation of the formal normal form, and on which the formal invariants can easily be read.

Theorem 2.4. [8] *The family of systems (2.9) is analytically equivalent to a family in the prenormal form*

$$p_\epsilon(x)y' = \left(\Lambda(\epsilon, x) + p_\epsilon(x)R(\epsilon, x)\right)y, \tag{2.11}$$

where

$$p_\epsilon(x)y' = \Lambda(\epsilon, x)y = \left(\sum_{j=0}^{k} \Lambda_j(\epsilon)x^j\right)y \tag{2.12}$$

is the formal normal form, $\Lambda_j(\epsilon)$ are diagonal matrices of germs of analytic functions in ϵ at the origin, $p_\epsilon(x)$ is given by (2.10) and R is a matrix of germs of analytic functions in (ϵ, x) at the origin.

Hence, for the rest of the paper, it will always be sufficient to consider the systems (2.9) in prenormal form (2.11).

The fact that (2.12) is indeed the formal normal form of the system (2.11) will be proved in Theorem 3.2. When $\epsilon = \vec{0}$, the system (2.12) is the usual formal normal form.

2.3. **Size of the neighbourhood in ϵ and the set Σ_0 of generic values.** We will suppose that ϵ belongs to a polydisk $\mathbb{D}_\rho$ given by

$$\mathbb{D}_\rho = \{\epsilon; |\epsilon_j|^{\frac{1}{k+1-j}} < \rho, j = 0, \dots k-1\}, \tag{2.13}$$

supposing that the roots of the polynomial p_ϵ belong to the disk $\mathbb{D}_{r'}$ in x-space. The radius r' is taken smaller than r, and sufficiently small so that the ordering (2.1) on the eigenvalues of $\Lambda(\epsilon, x)$ at $(\epsilon, x) = (\vec{0}, 0)$ is kept for $(\epsilon, x) \in \mathbb{D}_\rho \times \mathbb{D}_r$.

The reason for this choice of $\mathbb{D}_\rho$ will become clear in Section 4.2.

Notation 2.5.
(1) We let $\Sigma_0 = \{\epsilon \mid \Delta(\epsilon) \neq 0\}$, where $\Delta(\epsilon)$ is the discriminant of $p_\epsilon(x)$, be the set of generic values of ϵ for which the zeros of $p_\epsilon(x)$ are distinct.
(2) For $\epsilon \in \Sigma_0$, we denote the zeros of $p_\epsilon(x)$ by x_l, with $l = 0, 1, ..., k$.

By the Riemann Removable Singularities Theorem, we will be able to extend to $\mathbb{D}_\rho$ analytic and bounded functions that we will define on $\Sigma_0 \cap \mathbb{D}_\rho$.

3. Formal invariants

3.1. **Complete system of formal invariants.** In this section, we will prove that the $\Lambda(\epsilon, x)$ in the prenormal form (2.11) contains all the information on the formal invariants. In order to prove that a prepared family (2.11) is formally equivalent to the formal normal form (2.12), we introduce the following decomposition of matrices:

Definition 3.1. Let $B \in \mathrm{Mat}(n \times n, \mathbb{C})$. The matrix B can decomposed as $B = \mathrm{Diag}(B) + \mathrm{Off}(B)$, where

- $\mathrm{Diag}(B)$ is the diagonal part of B,
- $\mathrm{Off}(B)$ is the off-diagonal part of B.

Theorem 3.2. *Two systems (2.11) are formally equivalent if and only if they have the same $\Lambda(\epsilon, x)$. Hence, the complete system of formal invariants of the systems (2.11) depends analytically on ϵ at $\epsilon = 0$, and is given by the $k+1$ diagonal matrices $\Lambda_0(\epsilon)$, ..., $\Lambda_k(\epsilon)$ in the polynomial part of degree k of the prenormal form.*

Proof. First, we will prove that a germ of family of systems (2.11) is formally equivalent to its formal normal form (2.12). We would like to apply the Poincaré-Dulac Theorem (see [6] p. 45) to the nonlinear system

$$\begin{cases} \dot{y} = (\Lambda(x, \epsilon) + p_\epsilon(x) R(\epsilon, x))\, y, \\ \dot{x} = p_\epsilon(x), \\ \dot{\epsilon} = \vec{0}, \end{cases} \tag{3.1}$$

corresponding to (2.11), in order to bring it to a similar form with $R(\epsilon, x)$ diagonal. But we need to show that $\Lambda(\epsilon, x)$ is not changed in the process.

We look for a linear change of coordinates $y = (I_n + B(\epsilon, x))Y$, with B off-diagonal, which would bring the $\dot{y}$ subsystem to $\dot{Y} = M(\epsilon, x)Y$, with M diagonal. We drop the dependence on ϵ and x. We have on one side $\dot{y} = \dot{B}Y + (I + B)MY$, and on the other side $\dot{y} = (\Lambda + pR)(I + B)Y$. Comparing the diagonal parts, we get

$$M = \Lambda + p\left(\mathrm{Diag}(R) + p\mathrm{Diag}(RB)\right), \tag{3.2}$$

which yields that Λ will not have been changed in the process. Comparing the off-diagonal parts and using (3.2), yields

$$\dot{B} = [\Lambda, B] + p\left(\mathrm{Off}(R) + \mathrm{Off}(RB) - B\mathrm{Diag}(R) - B\mathrm{Diag}(RB)\right).$$

We include this equation in a larger system

$$\begin{cases} \dot{B} = [\Lambda, B] + p\left(\mathrm{Off}(R) + \mathrm{Off}(RB) - B\mathrm{Diag}(R) - B\mathrm{Diag}(RB)\right), \\ \dot{x} = p, \\ \dot{\epsilon} = \vec{0}. \end{cases} \tag{3.3}$$

Since the eigenvalues corresponding to the off-diagonal entries of B are nonzero, then the formal series of $B(\epsilon, x)$ satisfying $B(0,0) = 0$ is just the center manifold of the system (3.3), which can be written as a power series in (ϵ, x) by the center manifold theorem.

We now have $p_\epsilon(x)Y' = (\Lambda(\epsilon, x) + p_\epsilon(x)S(\epsilon, x))Y$ with $S(\epsilon, x)$ diagonal. The transformation $Z = e^{-\int_0^x S(\epsilon,x)dx}Y$ leads to the formal normal form.

To prove the converse, let us first define $A^\alpha(\epsilon, x) = \Lambda^\alpha(\epsilon, x) + p_\epsilon(x)R^\alpha(\epsilon, x)$ with $\alpha = 1, 2$. Let us suppose that the two systems $p_\epsilon(x)y_1' = A^1(\epsilon, x)y_1$ and $p_\epsilon(x)y_2' = A^2(\epsilon, x)y_2$ are formally equivalent. Since each of them is formally conjugated to its formal normal form, there exists an invertible formal transformation $T(\epsilon, x)$ from the first system $p_\epsilon(x)z_1' = \Lambda^1(\epsilon, x)z_1$ to the second system $p_\epsilon(x)z_2' = \Lambda^2(\epsilon, x)z_2$, leading to

$$p_\epsilon(x)T'(\epsilon, x) + T(\epsilon, x)\Lambda^1(\epsilon, x) = \Lambda^2(\epsilon, x)T(\epsilon, x). \tag{3.4}$$

We expand $T(\epsilon, x)$, $\Lambda^j(\epsilon, x)$ as power series in ϵ with the multi-index notation: $T(\epsilon, x) = \sum_s T_s(x)\epsilon^s$, and $\Lambda^j(\epsilon, x) = \sum_s \Lambda_s^j(x)\epsilon^s$. We show by multi-induction on s that $\Lambda_s^1(x) = \Lambda_s^2(x) = \Lambda_s$, and that T_s is a constant diagonal matrix. For $s = 0$, we have

$$x^{k+1}T_0' = \Lambda_0^1 T_0' - T_0'\Lambda_0^2. \tag{3.5}$$

The result is known, but we sketch briefly the argument: (3.5) is a product of n^2 independent differential equations, all of the form

$$x^{k+1}t' = b(x)t, \tag{3.6}$$

with $b(x)$ a polynomial of degree at most k. Each equation of type (3.6) has no nonzero power series solution if $b(x)$ is non identically zero, and has constant solutions if $b(x) \equiv 0$. Also, since $T_0(0)$ conjugates $\Lambda_0^1(0)$ and $\Lambda_0^2(0)$, it must be a diagonal invertible matrix. Hence, we need to have $\Lambda_0^1 = \Lambda_0^2$, and T_0 must be a constant diagonal matrix.

Now, suppose that the result is known for all $s' = (s_{k-1}', \dots, s_0')$ less than $s = (s_{k-1}, \dots, s_0)$ (i.e. $s_j' \leq s_j$ for all j and there exists ℓ such that $s_\ell' < s_\ell$). Then T_s is solution of a system

$$x^{k+1}T_s' = \Lambda_0 T_s - T_s\Lambda_0 + D(x), \tag{3.7}$$

where $D(x)$ is a diagonal matrix that depends on $T_{s'}$ and $\Lambda_{s'}$ for smaller s'. The matrix D is a sum of expressions of the form $\Lambda_{s''}^2 T_{s'} - T_{s'}\Lambda_{s''}^1$ for $s'' + s' = s$. By induction all these expressions vanish except $(\Lambda_s^2 - \Lambda_s^1)T_0$, since diagonal matrices commute and $\Lambda_{s''}^2 = \Lambda_{s''}^1$ for s'' smaller than s. Hence, $D(x) = (\Lambda_s^2 - \Lambda_s^1)T_0$. We must show that a system of the form (3.7), with $D(x)$ a polynomial diagonal matrix of degree at most k has a formal solution in x only if $D \equiv 0$, and that the solution is a constant. The system (3.7) is also a product of n^2 independent differential equations. Outside the diagonal, the differential equation for the entries of T_s are of the type (3.6) with $b(x)$ nonzero, and hence the solutions vanish identically. The differential equations for each entry on the diagonal are of the form

$$x^{k+1}t' = d(x),$$

where $d(x)$ is a polynomial in x of degree at most k. A solution $t(x)$ of this equation can be a power series in x only if $d(x) \equiv 0$, in which case t will be constant. □

3.2. **The set of residue matrices at the Fuchsian singular points.**

Definition 3.3. A Fuchsian singular point x_l is resonant when at least two eigenvalues differ by an integer, which is equivalent to saying that the eigenvalues of the monodromy matrix around x_l are not all distinct.

When $\epsilon \in \Sigma_0$ (Notation 2.5) and a singular point x_l is not resonant, it is well-known that the local formal (and analytic) invariants (corresponding to a transformation centered at x_l and for fixed ϵ) are given by the (diagonal) residue matrices of the system (2.11). The set of all these invariants at the different Fuchsian singular points may be used to compute the complete system of formal invariants (i.e. $\Lambda(\epsilon, x)$):

Lemma 3.4. *The formal normal form* (2.12)*, hence the polynomial matrix* $\Lambda(\epsilon, x)$*, is uniquely determined from the set of matrices*

$$\mathcal{U}_l = \operatorname{diag}(\mu_\ell^1, \mu_\ell^2, ..., \mu_\ell^n) = \frac{\Lambda(\epsilon, x_l)}{p'_\epsilon(x_l)}, \quad l = 0, 1, ..., k, \tag{3.8}$$

which correspond to the residue matrices of the system (2.11) *at the Fuchsian singular points* x_l *.*

Proof. For $\epsilon \in \Sigma_0$, let $\Lambda_j(\epsilon, x)$ be the j-th entry of $\Lambda(\epsilon, x)$ on the diagonal. It is a polynomial of degree k. Its values at the $k+1$ singular points x_l are given by $p'_\epsilon(x_l)\mu_\ell^j$, where μ_ℓ^j is given by (3.8). The polynomial $\Lambda_j(\epsilon, x)$ is completely determined by Lagrange interpolation formula. Since it is determined for $\epsilon \in \Sigma_0$ and is analytic in ϵ, it is determined for $\epsilon \in \mathbb{D}_\rho$. □

Remark 3.5. *For a fixed non resonant* $\epsilon \in \Sigma_0$*, there are* $n(k+1)$ *eigenvalues of the residue matrices at the (non resonant) singular points, and* $n(k+1)$ *formal invariants. As in similar works (for instance* [8]*), it is observed that the number of formal invariants is exactly the same. This reflects the fact that there is no relation between the eigenvalues of the residue matrices at the different non resonant singular points up to the limit* $\epsilon = 0$.

3.3. **Asymptotic behavior of the solutions of the formal normal form.**

Lemma 3.6. *For* $\epsilon \in \Sigma_0 \cup \{0\}$*, the formal normal form* (2.12) *has the following fundamental matrix solution:*

$$\begin{aligned} F(\epsilon, x) &= diag(f_1(\epsilon, x), f_2(\epsilon, x), ..., f_n(\epsilon, x)) \\ &= \begin{cases} \prod_{l=0}^{k} (x - x_l)^{\mathcal{U}_l}, & \epsilon \in \Sigma_0, \\ x^{\Lambda_k(\vec{0})} \exp\left(-\sum_{j=0}^{k-1} \frac{\Lambda_j(\vec{0})}{(k-j)x^{k-j}}\right), & \epsilon = \vec{0}, \end{cases} \end{aligned} \tag{3.9}$$

where $\mathcal{U}_l$ *is given by* (3.8)*. The matrix representing the monodromy of* $F(\epsilon, x)$ *in the positive direction around a singular point* x_l *(*$l \in \{0, ..., k\}$*) is given by* $D_l = e^{2\pi i \mathcal{U}_l}$*.*

(The monodromy of $F(\epsilon, x)$ in the positive direction around the set of all singular points is represented by $\prod_{l=0}^{k} D_l = e^{2\pi i \Lambda_k(\epsilon)}$.)

In Section 4, we will construct (generic) unfoldings $\Omega^{\pm}_{j,\epsilon}$ of the sectors $\Omega^{\pm}_{j}$. The unfolded sectors $\Omega^{\pm}_{j,\epsilon}$ will intersect along two types of domains. One type will be denoted by $\Omega^{U}_{j,\epsilon}$ (resp. $\Omega^{L}_{j,\epsilon}$); it will converge to Ω^{U}_{j} (resp. Ω^{L}_{j}) when $\epsilon \to 0$ and will be attached to a Fuchsian singular point x_l. The other type will be attached to two Fuchsian singular points and will disappear when $\epsilon \to 0$. The construction of the sectors $\Omega^{\pm}_{j,\epsilon}$ will ensure (see Section 4.4) that we have the following asymptotic behavior of the solutions (3.9) over the sectors near the Fuchsian singular points x_l:

$$\lim_{x\to x_l} \frac{f_p(\epsilon,x)}{f_q(\epsilon,x)} = 0, \qquad \text{for } \begin{cases} q > p, & \text{if } x_l \text{ is attached to an } \Omega^{U}_{j,\epsilon}, \\ q < p, & \text{if } x_l \text{ is attached to an } \Omega^{L}_{j,\epsilon}. \end{cases} \tag{3.10}$$

Note that we have a similar behavior when $\epsilon = 0$ (see also (2.7)):

$$\lim_{\substack{x\to 0\\ x\in\Omega^{\dagger}_{j}}} \frac{f_p(\vec{0},x)}{f_q(\vec{0},x)} = 0, \quad \text{for } \begin{cases} q > p, & \text{if } \dagger = U, \\ q < p, & \text{if } \dagger = L. \end{cases} \tag{3.11}$$

The limit (3.10) may be used to prove that the unfolded Stokes matrices are triangular matrices.

4. Sectors in x-space and sectoral domains in parameter space

In order to obtain the complete system of analytic invariants, we first need to define the domains on which we take the variable and the parameter. We partition the generic set Σ_0 of the parameter space into a finite number of sectoral domains $S_s \subset \Sigma_0$; for ϵ in each sectoral domain, we construct $2k$ sectors in x-space covering the complement of the singular set of the equation; this construction varies continuously with ϵ in the sectoral domain. The construction is similar to the construction of the sectors in [10]. In [10], the number of sectoral domains S_s was not minimal, allowing a simpler construction of the sectors $\Omega^{\pm}_{j,\epsilon}$ in x. Here we will use a slightly more elaborate construction in order to use the minimal number C_k of sectoral domains in ϵ. We will only do the construction for values of the parameters $\epsilon \in \Sigma_0$ for which the singular points are Fuchsian, but it can be extended to all values of the parameters.

The sectors $\Omega^{\pm}_{j,\epsilon}$ in x-space will be unfoldings of the sectors $\Omega^{\pm}_{j}$ defined in Section 2.1 for $\epsilon = 0$. In particular, any compact set K included in some $\Omega^{\pm}_{j}$ will be included in $\Omega^{\pm}_{j,\epsilon}$ for ϵ sufficiently small. A sector $\Omega^{\pm}_{j,\epsilon}$ will be adherent to two singular points which coalesce to 0 when $\epsilon \to 0$.

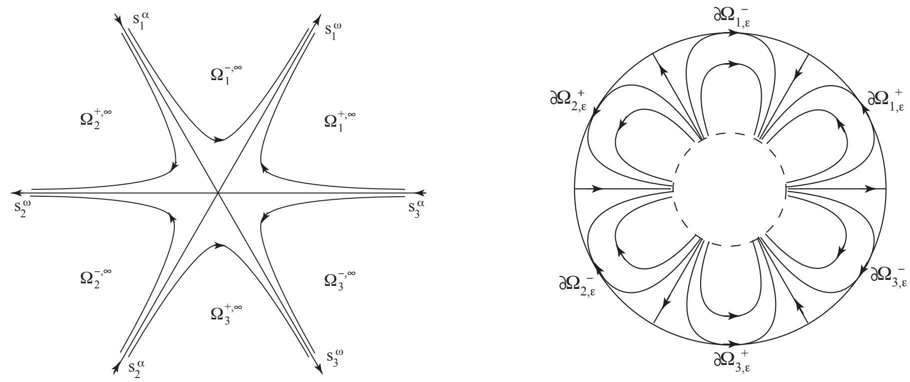
Each sector $\Omega^{\pm}_{j,\epsilon}$ is a union of "trajectories" of the polynomial vector field

$$v_\epsilon(x) = p_\epsilon(x)\frac{\partial}{\partial x} = (x^{k+1} + \epsilon_{k-1}x^{k-1} + \dots \epsilon_1 x + \epsilon_0)\frac{\partial}{\partial x}. \tag{4.1}$$

Solving the differential equation

$$\frac{dx}{dt} = p_\epsilon(x) \tag{4.2}$$

with complex time t, these trajectories will be, in general, the images in x-space of the lines $Im(t) =$ constant. Occasionally, the trajectories $Im(t) =$ constant will be



(a) Neighbourhood of infinity (b) Near $\partial\mathbb{D}_r$

FIGURE 2. The phase portrait of $v_\epsilon(x)$ at infinity and near the boundary of $\mathbb{D}_r$ when $k = 3$.

deformed to other images of real curves in t-space. The Fuchsian (resp. irregular) singular points of the system correspond to simple (resp. multiple) singular points of the vector field. Hence, the construction relies heavily on the phase portrait of the vector field $v_\epsilon(x)$.

The phase portrait of $v_\epsilon(x)$ has been extensively studied in the literature. It is organized by its pole at infinity (see Figure 2(a)), which yields for small ϵ the phase portrait of Figure 2(b) along the boundary of $\mathbb{D}_r$.

The pole at infinity has order $k-1$ (hence for $k = 1$, infinity is a regular point), and so has $2k$ separatrices, alternating between outwards and inwards, labelled cyclically anticlockwise around the point at infinity as $s_1^\omega, s_1^\alpha, s_2^\omega, s_2^\alpha, \ldots$. Between these sectors, on a disk around infinity, we have wedge shaped open sets $\Omega_{j,\epsilon}^{+,\infty}$ between s_{j-1}^α and s_j^ω, and $\Omega_{j,\epsilon}^{-,\infty}$ between s_j^ω and s_j^α. We want to extend these inwards away from infinity, expanding $\Omega_{j,\epsilon}^{\pm,\infty}$ to $\Omega_{j,\epsilon}^{\pm}$, to cover the plane.

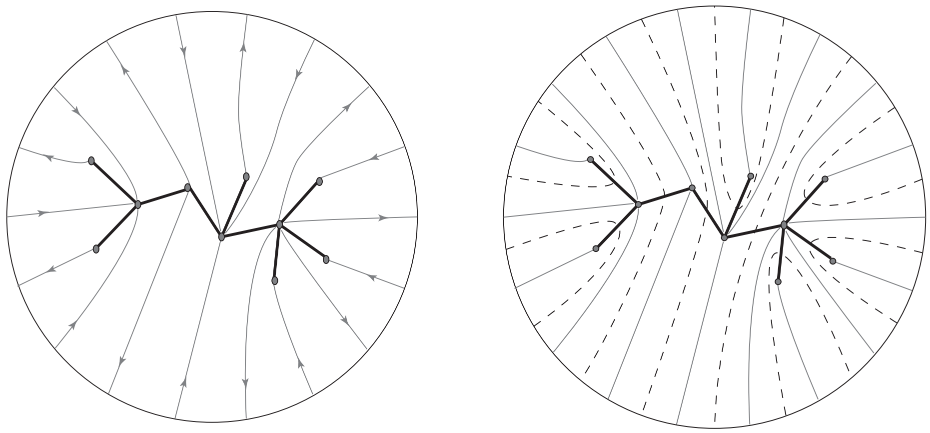
The separatrices move inwards from infinity, and can either come back to infinity (a homoclinic connection), or land at a singular point x_l ($l \in \{0, \ldots, k\}$). The latter is the generic case:

Definition 4.1. Let us define Σ_1 as the (open) subset of Σ_0 composed of values of ϵ for which the phase portait of v_ϵ has no homoclinic connection from separatrix to separatrix. By a straightforward contour integral along the homoclinic connection if there is one, one deduces that the absence of such a connection is guaranteed by the following generic condition: for any subset of $I \subset \{0, \ldots, k\}$ of the set of singular points, then

$$\sum_{j\in I} \frac{1}{p'_\epsilon(x_j)} \notin i\mathbb{R}. \tag{4.3}$$

In the generic condition detailed in Definition 4.1, extending the separatrices inwards, each sector $\Omega_{j,\epsilon}^{\pm}$ will be adherent to two singular points:

Definition 4.2. One of these points is an ω-limit point for the trajectories of v_ϵ (i.e. attracting along these trajectories), and the other an α-limit point (i.e. repelling



(a)

(b)

FIGURE 3. Division of a finite disk by the separatrices and chosen trajectories joining the singular points when $k = 9$. In (b) we highlight the lines determining the combinatorial invariant.

along these trajectories). For these reasons, the first point will be called an ω-point, and the second an α-point (see Figure 3).

Remark 4.3. *Note that since it is defined by a contour integral, the limit of the quantity* (4.3) *exists when several singular points, all having indices inside I or outside I, coalesce, and the contribution of the multiple point is equal to the formal invariant at the corresponding multiple point, which is defined as follows: if x_0 is the multiple point, then the vector field has the normal form $\frac{(x-x_0)^m}{1+a(x-x_0)^{m-1}}\frac{\partial}{\partial x}$ near x_0, and a is the formal invariant.*

4.1. **Defining the sectors over the complex plane.** While we are primarily concerned with the equations for x in a disk of radius r around the origin, the equation for x, $\dot{x} = p_\epsilon(x)$, has real trajectories defined over all of the complex plane. We therefore define the sectors $\Omega^{\pm}_{j,\epsilon}$ on the whole complex plane and then restrict them to $\mathbb{D}_r$. We begin by noting that the differential equation (4.2) is solved by

$$t = \int \frac{dx}{p_\epsilon(x)}.$$

This change of parameters has the singular points (zeros of p_ϵ) moved out to infinity in the t-plane, with the repulsive α-points moved to the $Re(t) \to -\infty$ limit, and the attractive ω-points moved to $Re(t) \to \infty$; the point at infinity in the x-plane (or rather several copies of it) lives in the finite portion of the t-plane since a pole is reached in finite time; the function $x(t)$ is multivalued there. Near such a point t_0 that is mapped to infinity, consecutive separatrices become horizontal half-lines moving out from t_0 in the positive and negative directions. This line in the t-plane is one half of the boundary of a strip of horizontal lines in t (with constant imaginary part), which get mapped to flow lines joining a fixed α-point x_α to a fixed ω-point x_ω (see Figure 4). One can continue in this way until one hits a horizontal line containing another preimage t_1 of infinity, with again two separatrices emanating

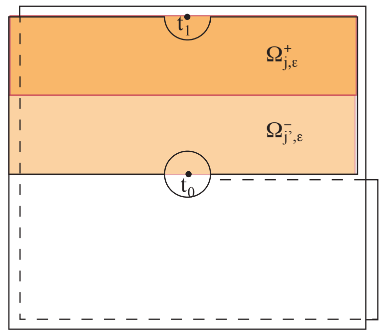


FIGURE 4. Two regions in the t-plane corresponding to sectors $\Omega^+_{j,\epsilon}$ and $\Omega^-_{j',\epsilon}$ in x-space, joining along $x(t)$. The disks correspond to the preimages of the complement of $\mathbb{D}_r$ in t-space. Each half line from ∞ to either t_0 or t_1 will be mapped to a separatrix.

from it. The result is a horizontal strip bounded by t_0 on one side and t_1 on the other side. Cut this strip in half, by the horizontal line through the midpoint of the segment joining t_0 and t_1. The bottom half of the strip will map to a region $\Omega^-_{j',\epsilon}$ in x-space, which is a triangle bounded on two sides by separatrices joining infinity to x_α, x_ω, and on the third by a flow line $x(t)$ joining x_α to x_ω; the top half will be a region $\Omega^+_{j,\epsilon}$, again a triangle bounded on two sides by separatrices, and on the third by $x(t)$. Repeating this construction gives $2k$ sectors, with half labelled by $+$, and half labelled by $-$. One such possibility, for $k = 9$, is shown in Figure 3. The Figure 4 shows two regions in t-plane corresponding to two sectors in x-space.

One sees that the sectors $\Omega^+_{j,\epsilon}$ and $\Omega^-_{j',\epsilon}$, are naturally paired. The possible such pairings are shown in [5] to be exactly the ones corresponding to the different ways of linking the $2k$ sectors of $\partial\mathbb{D}_r$ by k non-intersecting lines as in Figure 3(b). This partitions Σ_1 into C_k open connected regions in parameter space, denoted by $\widetilde{S}_s$. The number C_k is the k-th Catalan number

$$C_k = \frac{\binom{2k}{k}}{k+1}. \tag{4.4}$$

The sectoral regions $\widetilde{S}_s$ are simply connected, indeed contractible (see [5] and also Theorem 4.4 in [9]; we summarise this below, in Remark 4.10).

The sectoral domains in parameter space covering Σ_1 are strongly linked to the structure of the vector field $v_\epsilon(x)$ given by (4.1). They both have an invariance under rescaling, and we use this to introduce an equivalence relation on the parameter space.

4.2. **Equivalence relation on the parameter space.** Let us remark that v_ϵ is invariant under

$$(z, t, \epsilon_{k-1}, \dots, \epsilon_0) \mapsto (cz, c^{-k}t, c^2\epsilon_{k-1}, \dots, c^{k+1}\epsilon_0). \tag{4.5}$$

This is simply the equivalence relation induced by scaling the roots of p_ϵ, and suggests the following norm on ϵ.

Definition 4.4. Let us define

$$\| \epsilon \| := \max\{|\epsilon_0|^{\frac{1}{k+1}}, |\epsilon_1|^{\frac{1}{k}}, ..., |\epsilon_{k-1}|^{\frac{1}{2}}\}. \tag{4.6}$$

Remark 4.5. *When considering the norm defined by (4.6), all the roots of $p_\epsilon(x)$ are contained in a closed disk of radius at most $\sqrt{k}||\epsilon||$.*

Definition 4.6. In the parameter space, let us consider the following equivalence relation (motivated by (4.5)):

$$\begin{aligned} &\epsilon = (\epsilon_0, \epsilon_1, ..., \epsilon_{k-1}) \sim \epsilon' = (\epsilon_0', \epsilon_1', ..., \epsilon_{k-1}') \\ &\iff \exists \eta \in \mathbb{R}_{>0} : \epsilon_j' = \eta^{k+1-j}\epsilon_j, \quad j = 0, 1, ..., k-1. \end{aligned} \tag{4.7}$$

The open sets Σ_0 and Σ_1 in parameter space are unions of equivalence classes defined by the relation (4.7). In Section 4.5, we will construct an open covering of Σ_0 by open sectoral domains S_s such that each S_s has a vertex at $\epsilon = 0$ and is a union of equivalence classes (under the relation (4.7)).

4.3. **Sectors in the disk, $\epsilon \in \Sigma_1$, no homoclinic connection in the disk.** We will be interested in equations defined over the disk $\mathbb{D}_r$, even if the flow lines in x are defined over the whole complex plane. It is therefore important to ensure that the geometry described above restricts to the disk. As the singularities lie in a $\mathbb{D}_{r'} \subset \mathbb{D}_r$, the separatrices end up in $\mathbb{D}_{r'}$. However, in the generic situation described by Definition 4.1, even if the separatrices all enter $\mathbb{D}_r$, a separatrix may exit $\mathbb{D}_r$ before landing at a singular point. This splits $\mathbb{D}_r$ into two parts, each containing singular points. Although this phenomenon occurs on an open set in parameter space, we say that there is a homoclinic connection in the disk, in the sense of a trajectory from boundary to boundary. (Note that for ϵ in the polydisk $\mathbb{D}_\rho$, the relative measure of this open set goes to 0 as $\rho \to 0$.) We will study separately the cases when there is a homoclinic connection (in the disk) in 4.4.

If there is no homoclinic connection, the picture we have on the complex line extends well to the disk, and there is not much need to modify the construction. We will, however, also want to enlarge the sectors somewhat, as we will want to compare the solutions on the different sectors, and so would like them to overlap. We do this here simply by enlarging the horizontal strip in the t-plane.

4.4. **Sectors in the disk, $\epsilon \in \Sigma_1$, homoclinic connection in the disk.** Let us now consider what happens when we have a homoclinic connection in the disk. For this, it is perhaps best to consider what is happening in t-space (see Figure 5). Let t_0, t_1, be mapped to infinity in x-space. If $Im(t_0) = Im(t_1)$, one has a (true) homoclinic connection linking infinity to itself in x-space. Now suppose that $|Im(t_0) - Im(t_1)|$ is small. The outside of the disk $\mathbb{D}_\rho$ in x space corresponds to regions $D(t_i)$, which are approximate disks of radius $\frac{1}{r}$ in the imaginary direction around t_1, t_0. If $\frac{2}{r}$ is greater than $|Im(t_1) - Im(t_0)|$, one can have the horizontal line through t_0 entering and exiting the regions $D(t_1)$, and the horizontal line segments between the two regions $D(t_i)$ correspond to homoclinic connections in the disk (this in some sense justifies the terminology, as they are in some sense approximations of true homoclinic connections).

It is shown by Douady and Sentenac ([5] Corollary I.2.2.1) that if ϵ is small, the points t_1, t_0 are far apart; indeed in such a way that if the imaginary parts of the regions $D(t_i)$ overlap, their real parts are separated by a constant K, with K going to infinity uniformly if ϵ goes to zero. This then ensures that if ϵ is bounded

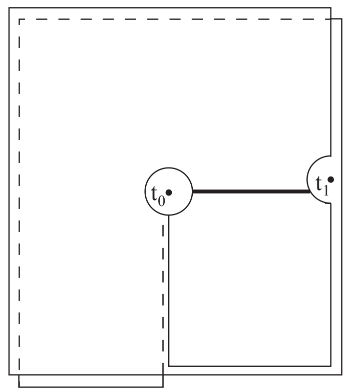


FIGURE 5. A homoclinic connection in the disk.

suitably, one can leave $D(t_0)$ horizontally for a distance of the order of $\frac{c}{\|\epsilon\|^k}$, and then move out to infinity along a curve $Im(e^{i\theta}t) = \text{constant}$, while avoiding $D(t_1)$, all the time keeping θ in an interval $[-\phi/2, \phi/2]$.

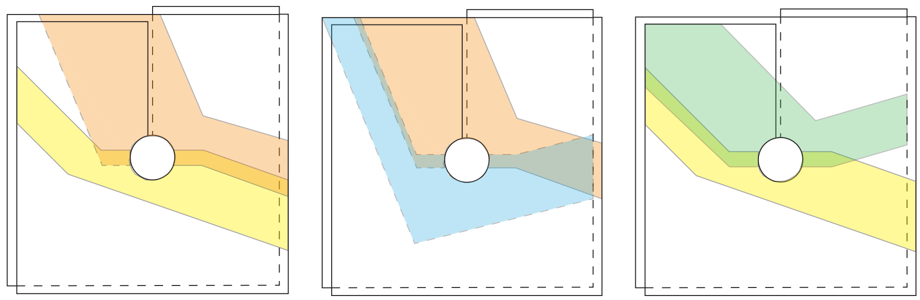
Lemma 4.7. *There exists $K < 0$ such that, for any ρ such $4\rho < r$, then, for $\| \epsilon \| < \rho$, the time along the part of a homoclinic loop separating singular points inside $\mathbb{D}_r$ is larger than $\frac{K}{\|\epsilon\|^k}$.*

Proof. Let $|z| = a \| \epsilon \|$. Then

$$|P_\epsilon(z)| \geq \| \epsilon \|^{k+1} \left(a^{k+1} - \sum_{j=0}^{k-1} a^j \right) > 0,$$

as soon as $a > 2$. Hence, all roots of $P_\epsilon(z)$ lie inside $2 \| \epsilon \| \mathbb{D}$ (where $\mathbb{D}$ is the unit disk). We can of course take ρ sufficiently small so that $4\rho < r$ (remember that $\| \epsilon \| < \rho$). Since all singular points are located inside the disk $2 \| \epsilon \| \mathbb{D}$, then any homoclinic path passing between the singular points has some points inside the disk $2 \| \epsilon \| \mathbb{D}$. The length of the path included in the disk $4 \| \epsilon \| \mathbb{D}$ is therefore larger than $2 \| \epsilon \|$. Over this disk, the speed of the vector field is $|P_\epsilon(z)| < C \| \epsilon \|^{k+1}$. Hence, the time spent inside $r\mathbb{D}$ is larger than $\frac{2\|\epsilon\|}{C\|\epsilon\|^{k+1}} = \frac{K}{\|\epsilon\|^k}$ for some positive K. □

Changing our paths in this way will allow, in essence, to modify the sectors and the system of paths that goes with them so that the homoclinic connection disappears. There is one thing that must be preserved in these modifications: as we shall see, the main tool for understanding the geometry of the unfolding is the flag given by the decay rate of solutions to the equation for y as one approaches the singular points. This decay rate can depend on the way one approaches the singular points, more precisely, it depends on the angle in t-space one uses to approach the singular point. Since we want our constructing to pass to the limit when $\epsilon \to 0$, we choose $\theta \in (-\frac{\phi}{2}, \frac{\phi}{2})$, where ϕ is the angle defined in Remark 2.1. The angle we use when there is no homoclinic connection in the disk is zero, so that one fixes the imaginary part of t and moves out to infinity in that direction. The flag for the solutions in y is then determined by the ordering of the real parts of the eigenvalues



(a) (b) (c)

FIGURE 6. Four regions in the t-plane corresponding to sectors in x-space when $k = 2$. Three of the four intersections are shown.

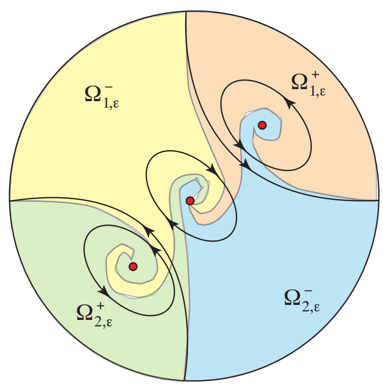


FIGURE 7. Some set of sectors approaching the singular points along spirals when $k = 2$.

of $\Re(\Lambda(\epsilon, x_l))$. If, instead, one chooses an angle θ, the controlling ordering is that of the eigenvalues of $\Re(e^{i\theta}(\Lambda(\epsilon, x_l))$. One can thus perturb the direction somewhat, keeping $|\theta| < \frac{\phi}{2}$. One can do this continuously over the sectoral domain (see Figures 6 and 7). (The condition $|\theta| < |\frac{\phi}{2}|$ allows to have, for ϵ sufficiently small, the asymptotic behavior described by (3.10) over the sectors near the Fuchsian singular points x_l.)

On the other hand, away from the singular point, we would like the shape of the sectors to pass to the limit. Hence, in the t-space, it is natural to choose paths that are horizontal for small and moderate t, and then deviate from the horizontal by an admissible angle θ for large t.

We also, as in the preceding case, fatten our sectors by extending the width of our strips.

This will be our modified system of sectors and curves; in short we have modified things so that there are no homoclinic connections on our disk, and the portrait on our disk is essentially the one we have on $\mathbb{C}$ for $\epsilon \in S_s$.

4.5. **From Σ_1 to Σ_0.** The approach used in Section 4.4 allows us to extend the sectoral domains from a covering of Σ_1 to a covering of Σ_0 in a straightforward way. We have decomposed Σ_1 into C_k sectoral domains $\widetilde{S}_s$, where the open sectors $\widetilde{S}_s$ are disjoint and invariant under the equivalence relation (4.7). Over each of these domains, we have decomposed the disk $\mathbb{D}_r$ into a union of $2k$ sectors $\Omega^{\pm}_{j,\epsilon}$, for all values of $\epsilon \in \Sigma_1 = \bigcup_{s=1}^{C_k} \widetilde{S}_s$. The subset $\Sigma_0 \backslash \Sigma_1$ is of real codimension 1, i.e. a real hypersurface in the parameter space. The boundary of $\widetilde{S}_s$ consists of hypersurfaces where homoclinic connections occur. These homoclinic connections are created by paths $Im(t) =$ constant.

Pushing the construction of Section 4.4 continuously allows to enlarge the $\widetilde{S}_s$ to larger sectoral domains S_s, the union of which covers Σ_0. For the extended sectoral domain S_s, we simply take an open neighbourhood of the closure of $\tilde{S}_s$ inside Σ_0, for which we can eliminate the homoclinic connection with a angle $\theta \in (-\frac{\phi}{2}, \frac{\phi}{2})$ while keeping all singular points of the same type as in $\tilde{S}_s$.

This extension of the sectoral domain can be made to branch around the discriminant locus Δ; the extension may have self-intersections, as one moves around Δ. This is no problem and we will indeed use these auto-intersections in Section 6.

4.6. **The structure of sectoral domains.** It is worth recalling the description of Douady and Sentenac of the sectoral domain $\widetilde{S}_s$. As we saw, the sectors $\Omega^{\pm}_{j,\epsilon}$ corresponding to an element of ϵ of Σ_1 were constructed as the images of strips in the t-space, which mapped bijectively to the union of pairs $\Omega^{+}_{j,\epsilon}$, $\Omega^{-}_{j',\epsilon}$; it is this pairing that defines the combinatorial invariant. The strip is bounded by two lines which are union of separatrices; they each contain points t_j (above), $t_{j'}$ (below) mapping to infinity in x-space. Douady and Sentenac show that the differences $\tau_j = t_j - t_{j'}$, for each pair in the combinatorial invariant, form a complete invariant for the polynomial vector field. In short, the sectoral domain is the product of k copies of the upper half plane. The differences $t_j - t_{j'}$ can be thought of as the complex times in getting from infinity to infinity along the dotted curves of Figure 3(b).

The points for which τ_j are pure imaginary correspond to the points for which all $p'_\epsilon(x_\ell)$ are real, and for which the trajectories are simplest. This defines a core, isomorphic to $(\mathbb{R}^+)^k$, to each stratum.

Definition 4.8. The real codimension k subset of $\widetilde{S}_s$ for which all singular points have real eigenvalues is called the *organizing center* of S_s.

On the enlarged sectoral domains S_s, we have constructed $2k$ sectors $\Omega^{\pm}_{j,\epsilon}$ in x-space depending continuously on $\epsilon \in S_s$. The process associates two non equivalent (i.e. some corresponding boundaries are not attached to the same singular point) sets of sectors $\{\Omega^{\pm}_{j,\epsilon}\}$ to a parameter value ϵ belonging to the intersection of two sectoral domains S_{s_1} and S_{s_2}.

Notation 4.9. Whenever necessary, we will add a subscript S_s to $\Omega^{\pm}_{j,\epsilon}$ and other quantities to highlight the dependence of the construction on the sectoral domain S_s.

The construction passes to the limit in a sectoral domain S_s when several singular points coalesce in a multiple point. In that case, the width of the strip becomes infinite with the full curve disappearing at infinity. When the α-point and ω-point

(Definition 4.2) coalesce, such a (generalized) half-plane parameterizes a sector of the form described in Section 2.1.

Remark 4.10. *$\tilde{S}$ is parametrized by $(\tau_1, \ldots, \tau_k) \in \mathbb{H}^k$ and is hence simply connected.*

5. The modulus of analytic classification

We now consider ϵ in a sectoral domain S_s in parameter space, and corresponding sets of sectors $\{\Omega^{\pm}_{j,\epsilon}\}$ in x-space. We have seen that for each sector $\Omega^{+}_{j,\epsilon}$, there is a unique sector $\Omega^{-}_{j',\epsilon}$ such that $\Omega^{+}_{j,\epsilon}$ and $\Omega^{-}_{j',\epsilon}$ are adherent to the same two singular points. This allows us to define a bijection

$$\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\},$$

by $j \mapsto j'$.

5.1. The geometry of a covering $\{\Omega^{\pm}_{j,\epsilon}\}$ of $\mathbb{D}_r$.

Definition 5.1. The intersection of the two sectors $\Omega^{+}_{j,\epsilon}$ and $\Omega^{-}_{\sigma(j),\epsilon}$ has one connected component adherent to the two singular points. We call it a *gate sector* and denote it by $\Omega^{G}_{j,\sigma(j),\epsilon}$. Let us recall that we have enlarged our sectors $\{\Omega^{\pm}_{j,\epsilon}\}$ so that they overlap on more than a single curve $x(t)$ from the ω-point to the α-point. The gate sector is an open set, in essence fattening $x(t)$.

Definition 5.2. If a connected component of the intersection of two sectors is adherent to a single singular point, and its boundary has a nonempty intersection with $\partial\mathbb{D}_r$, it is called an *intersection sector*. We denote by $\Omega^{L}_{j,\epsilon}$ (resp. $\Omega^{U}_{j,\epsilon}$) the intersection sector included into $\Omega^{+}_{j,\epsilon} \cap \Omega^{-}_{j,\epsilon}$ (resp. $\Omega^{+}_{j,\epsilon} \cap \Omega^{-}_{j-1,\epsilon}$). It is attached to an ω-point (resp. an α-point), and can be thought of as the fattening of the separatrix emerging from infinity going to the singular point.

The Fuchsian singularities are all tied together by the closure of the gate sectors, whose skeleton is a graph with vertices, the singular points, and edges, the curves $x(t)$ from ω-point to α-point; this graph is in fact a tree. For $k > 1$, the intersection of the closure of two arbitrary sectors (of type $\Omega^{\pm}_{j,\epsilon}$) consists of one of the following (see Figure 8):

(1) one singular point (for instance, $\Omega^{-}_{1,\epsilon} \cap \Omega^{-}_{3,\epsilon}$),
(2) two singular points and a gate sector (for instance, $\Omega^{+}_{2,\epsilon} \cap \Omega^{-}_{3,\epsilon}$),
(3) one singular point and an intersection sector (for instance, $\Omega^{+}_{1,\epsilon} \cap \Omega^{-}_{4,\epsilon}$),
(4) two singular points, a gate sector and an intersection sector (for instance, $\Omega^{-}_{2,\epsilon} \cap \Omega^{+}_{3,\epsilon}$).

(In the case $k = 1$, the intersection of the closure of the two sectors consists of two singular points, a gate sector and two intersection sectors.)

5.2. Bases of solutions of (2.11).

In this section, we will define particular bases of solutions to the equation (2.11) over each sector $\Omega^{\pm}_{j,\epsilon}$. The comparison of these bases of solutions over the intersection sectors (Section 5.3) will lead to the analytic invariants. Over each sector $\Omega^{+}_{j,\epsilon}$ (resp. $\Omega^{-}_{j,\epsilon}$) the n-dimensional space of solutions is denoted by $V^{+}_{j,\epsilon}$ (resp. $V^{-}_{j,\epsilon}$). We will show that the space of solutions has over each $\Omega^{\pm}_{j,\epsilon}$ two natural flags (sequences of nested subspaces) coming from the respective order of flatness of solutions as one approaches the two singular points of the sector,

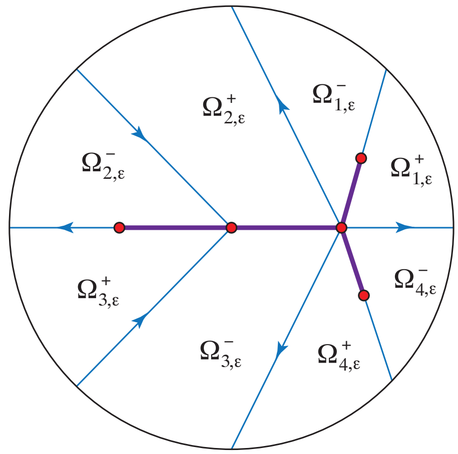


FIGURE 8. The different types of intersection of the closure of two arbitrary sectors (of type $\Omega^{\pm}_{j,\epsilon}$).

unfolding the flags of (2.8). Let us recall (see Section 2.3) that the eigenvalues $\lambda_i(\epsilon, x_l)$ of the diagonal matrix $\Lambda(\epsilon, x_l)$ have distinct real parts ordered as:

$$\Re(\lambda_1(\epsilon, x_l)) > \Re(\lambda_2(\epsilon, x_l)) > \cdots > \Re(\lambda_n(\epsilon, x_l)). \tag{5.1}$$

For a fixed ϵ in the organizing centre of the sectoral domain, let us consider a singular point x_ℓ attached to an $\Omega^U_{j,\epsilon}$ (resp. $\Omega^L_{j,\epsilon}$). Then $p'_\epsilon(x_\ell) \in \mathbb{R}^+$ (resp $p'_\epsilon(x_\ell) \in \mathbb{R}^-$), i.e. the point is an α-limit (resp. ω-limit) point. Hence, if the eigenvalues $\lambda_i(\epsilon, x_l)$ are distinct and real, there are simple growth rates of a sequence of n solutions to the equation at the singularity, whose leading order terms are:

$$\begin{cases} (x-x_\ell)^{\mu_\ell^n} \prec (x-x_\ell)^{\mu_\ell^{n-1}} \prec \cdots \prec (x-x_\ell)^{\mu_\ell^1}, & \text{on } \Omega^L_{j,\epsilon}, \\ (x-x_\ell)^{\mu_\ell^1} \prec (x-x_\ell)^{\mu_\ell^2} \prec \cdots \prec (x-x_\ell)^{\mu_\ell^n}, & \text{on } \Omega^U_{j,\epsilon} \text{ (resp. } \Omega^U_{j+1,\epsilon}), \end{cases}$$

where μ_ℓ^i is given by (3.8).

When the eigenvalues are not all real, i.e. ϵ not in the organizing center (Definition 4.8), then the direction of approach is important, and indeed, one can vary the flag by choosing different paths into the singularity (typically, spirals of different type). This is where the uniformisation of the system in t-space (which also intervenes in the definition the sectors), is essential. With x, y as functions of t, the equation is (3.1).

We have built our sectors from the image in x-space of the horizontal strips in t-space. Our direction of approach to the singularities will be along $Im(t) =$ constant, staying within the strip, as one goes to plus or minus infinity. This gives spirals in x-space approaching the singularity, if ϵ is not in the organizing centre, but they are of fixed type. The existence of the flag, and its continuity as one varies ϵ (even moving out of Σ_0) is given by the following theorem of Levinson [4] (see Coddington and Levinson [3], Theorem 8.1, p. 92).

Theorem 5.3. *Let a system of linear differential equations of the form*

$$\dot{y} = \left(\widetilde{\Lambda}_0(\epsilon) + \widetilde{\Lambda}(\epsilon, t) + P(\epsilon, t)\right) \cdot y, \tag{5.2}$$

be given on the real line, for which $\widetilde{\Lambda}_0$ *is diagonal, with distinct real parts of the eigenvalues,* $\widetilde{\Lambda}(\epsilon,t)$ *is also diagonal, with limit zero at* $t=\infty$ *and*

$$\int_0^\infty \left|\frac{d}{dt}(\widetilde{\Lambda}(\epsilon,t))\right| dt<\infty,\qquad \int_0^\infty |P(\epsilon,t)|\,dt<\infty. \tag{5.3}$$

Then, setting $\lambda_\ell(\epsilon,t)$, $\ell=1,\dots,k$, *to be the successive eigenvalues of* $\widetilde{\Lambda}_0(\epsilon)+\widetilde{\Lambda}(\epsilon,t)$, *there exist* $t_0\in(0,\infty)$ *and solutions* $\phi_{\ell,\epsilon}(t)$ *of the system for* $t\in(t_0,\infty)$ *with*

$$\lim_{t\to\infty}\phi_{\ell,\epsilon}(t)\cdot\exp\left(-\int_{t_0}^t\lambda_\ell(\tau)d\tau\right)=v_\ell(\epsilon),$$

for $v_\ell(\epsilon)$ *a non-zero eigenvector of* $\widetilde{\Lambda}_0(\epsilon)$ *corresponding to* $\lambda_\ell(\infty)$*. If* $\widetilde{\Lambda}_0(\epsilon)$*,* $\widetilde{\Lambda}(\epsilon,t)$ *and* $P(\epsilon,t)$ *depend continuously (resp. analytically) on* ϵ *over compact sets in* t*-space, with the integrals in* (5.3) *uniformly bounded, then the solutions can be chosen depending continuously (resp. analytically) on* ϵ*.*

This theorem applies to our situation: indeed, when we consider a slanted strip we perform the generalized change of variable $t=e^{-i\theta}\int\frac{dx}{p_\epsilon(x)}$ (the case of a horizontal strip corresponding to $\theta=0$), with $\theta\in(0,\frac{\phi}{2})$ and ϕ defined in Remark 2.1. Then, $\widetilde{\Lambda}_0(\epsilon)+\widetilde{\Lambda}(t,\epsilon)=e^{i\theta}\Lambda(\epsilon,x(t))$, and $P(\epsilon,t)=e^{i\theta}p_\epsilon(x(t))R(\epsilon,x(t))$, for Λ,p,R analytic in their arguments, and $x(t)$ converging towards a singular point (we multiply time by -1 when the singular point is of α-type). We have that $\frac{d}{dt}\widetilde{\Lambda}(\epsilon,t)=\frac{d}{dx}\widetilde{\Lambda}(\epsilon,t)e^{-i\theta}p_\epsilon(x)$; hence,

$$\int_0^\infty\left|\frac{d}{dt}(\widetilde{\Lambda}(\epsilon,t))\right|dt=\int\left|\frac{d}{dx}(\Lambda(\epsilon,x))\right||dx|<K,$$

where the bound K can be chosen uniform in ϵ, since we approach the singularity along logarithmic spirals $|x|=ae^{ib\arg(x)}$, the length of which is uniformly bounded if we force b to remain bounded away from 0 (this is achieved by taking the sectoral domains S_s not too large). The second part of (5.3) is similar. Indeed,

$$\int_0^\infty|P(\epsilon,t)|\,dt=\int|R(\epsilon,x)|\;|dx|$$

is also uniformly bounded.

For finding the flags, it suffices to consider one line in t-space, the image of which in x-space is contained in a sector $\Omega^U_{j,\epsilon}$ or $\Omega^L_{j,\epsilon}$, but the flag is of course intrinsic and independent on the choice of the line and we could have chosen to consider the parametric families of parallel lines in t-space corresponding to the sector $\Omega^U_{j,\epsilon}$ or $\Omega^L_{j,\epsilon}$ (parameterized by an additional parameter).

Note that the original theorem of Levinson does not include the dependence on the parameter ϵ stated in Theorem 5.3. For that reason, we will say a word on the proof. The solution $\phi_{\ell,\epsilon}$ is found by a successive approximation as a fixed point of an integral operator, starting from the ℓ-th eigensolution of the diagonal system $\dot{y}=\left(\widetilde{\Lambda}_0(\epsilon)+\widetilde{\Lambda}(\epsilon,t)\right)\cdot y$. The minimal distance between the real parts of the eigenvalues of $\widetilde{\Lambda}_0(\epsilon)+\widetilde{\Lambda}(\epsilon,t)$ remains bounded from below for (ϵ,x) sufficiently small, and θ well chosen depending on ϵ. Under this condition, if we let $P(\epsilon,t)=(r_{j,\ell}(\epsilon,t))_{j,\ell=1}^n$, the crucial ingredient is to choose t_0 so that $\int_{t_0}^\infty\sum_{j,\ell}|r_{j,\ell}(\epsilon,t)|\,dt$ be sufficiently small in order that the integral operator becomes a contraction. This can be achieved uniformly in ϵ, again because we approach the singularity along logarithmic spirals $|x| = ae^{ib\arg(x)}$, the length of which is uniformly bounded.

Getting the analyticity in ϵ is no problem, as long as the input is varying analytically, e.g., within a sectoral domain: we know that the solutions of the system depend analytically on ϵ. Hence it suffices to choose a normalization depending analytically on ϵ: for instance we normalize the value of the ℓ-th coordinate of the ℓ-th vector of the basis at a fixed point inside $\Omega^{\pm}_{j,\epsilon}$. The beauty of the use of Theorem 5.3 is that it gives us the flag and its analyticity in ϵ, regardless whether the Fuchsian singular point is resonant or not.

One can also see this independence of the flag from resonance from the other way of solving the equations over Σ_0, which would be to look for Frobenius series solutions. Indeed, while the solution of the full ODE $y' = p_\epsilon(x)A(x)y$ gives us an infinite sequence of values of ϵ converging to zero for which there are resonances, the quotiented problem $y'B = p_\epsilon(x)A(x)yB$, where B is the subgroup of upper triangular matrices, does not; the matrix entries for which resonances occur all lie in B, and are quotiented out.

Theorem 5.3 also gives a clear picture of what happens at the boundary of the $\Omega^{\pm}_{j,\epsilon}$: basically our solutions behave well as long as the curves $x(t)$ do not bifurcate.

Summarising, we have proved the following theorem:

Theorem 5.4. *The space of solutions $V^{+}_{j,\epsilon}$ over $\Omega^{+}_{j,\epsilon}$ has two natural flags defined from the asymptotics of solutions over its boundary sectors $\Omega^{L}_{j,\epsilon}$ and $\Omega^{U}_{j,\epsilon}$:*

$$\begin{cases} V^{+,L}_{j,n,\epsilon} \subset V^{+,L}_{j,n-1,\epsilon} \subset \cdots \subset V^{+,L}_{j,1,\epsilon} = V^{+}_{j,\epsilon}, \\ V^{+,U}_{j,1,\epsilon} \subset V^{+,U}_{j,2,\epsilon} \subset \cdots \subset V^{+,U}_{j,n,\epsilon} = V^{+}_{j,\epsilon}. \end{cases}$$

Similarly, the space of solutions $V^{-}_{j,\epsilon}$ over $\Omega^{-}_{j,\epsilon}$ has two natural flags defined from the asymptotics of solutions over its boundary sectors $\Omega^{L}_{j,\epsilon}$ and $\Omega^{U}_{j+1,\epsilon}$:

$$\begin{cases} V^{-,L}_{j,n,\epsilon} \subset V^{-,L}_{j,n-1,\epsilon} \subset \cdots \subset V^{-,L}_{j,1,\epsilon} = V^{-}_{j,\epsilon}, \\ V^{-,U}_{j,1,\epsilon} \subset V^{-,U}_{j,2,\epsilon} \subset \cdots \subset V^{-,U}_{j,n,\epsilon} = V^{-}_{j,\epsilon}. \end{cases}$$

Here the dimension of $V^{\pm,L}_{j,i,\epsilon}$ is $n-i+1$; that of $V^{\pm,U}_{j,i,\epsilon}$ is i. The flags depend analytically on ϵ in a fixed sectoral domain S_s. They have a continuous limit along the boundary of Σ_0, as long as the paths $x_\epsilon(t)$ and their limit points $x_\epsilon(\pm\infty)$ vary continuously.

As we have seen, the flags vary continuously. There are two for each sector, one labeled by L, one by U; at $\epsilon = 0$, these are transverse, in the sense that the intersections of the subspaces of the L-flag and the subspaces of the U-flag have minimal dimension. This property then extends to ϵ small.

Proposition 5.5. *If ρ is chosen sufficiently small, the vector subspaces $V^{\pm,L}_{j,i,\epsilon} \cap V^{\pm,U}_{j,i,\epsilon}$, $i = 1,\dots,n$, all have dimension 1. One can form a basis of solutions $\mathcal{B}^{\pm}_{j,\epsilon}$ of $V^{\pm}_{j,\epsilon}$ by choosing one vector in each of these intersections; the choices are determined up to the action of $(\mathbb{C}^*)^n$.*

5.3. **Normalized bases of** (2.11) **and transition matrices.** We now choose some normalization for the bases of Proposition 5.5 (i.e. adequate multiples of the vectors of each basis) so that all bases $\mathcal{B}^{\pm}_{j,\epsilon}$, $j = 1, \dots, k$ are completely determined by $\mathcal{B}^{+}_{1,\epsilon}$.

Definition 5.6. Let $W^{\pm}_{j,\epsilon}$ be the fundamental matrix solution whose columns are given by the vectors of the basis $\mathcal{B}^{\pm}_{j,\epsilon}$ of Proposition 5.5. We define the *unfolded Stokes matrices*, for $j = 1, 2, ..., k$, by

$$\begin{cases} W^{+}_{j,\epsilon} = W^{-}_{j-1,\epsilon} C^{U}_{j,\epsilon}, & \text{on } \Omega^{U}_{j,\epsilon}, \\ W^{-}_{j,\epsilon} = W^{+}_{j,\epsilon} C^{L}_{j,\epsilon}, & \text{on } \Omega^{L}_{j,\epsilon}. \end{cases} \tag{5.4}$$

The flag structure guarantees that the comparison of the fundamental matrix solutions over an intersection sector is a triangular matrix. We normalize the bases $\mathcal{B}^{\pm}_{j,\epsilon}$ so that the unfolded Stokes matrices $C^{U}_{j,\epsilon}$ and $C^{L}_{j,\epsilon}$ have diagonal part I (i.e. the identity matrix), except for $C^{U}_{1,\epsilon}$ which has diagonal part $e^{2\pi i\Lambda_k(\epsilon)}$. In order to have uniform convergence of the elements of $\mathcal{B}^{\pm}_{j,\epsilon}$ to the elements of $\mathcal{B}^{\pm}_{j,0}$ over compact subspaces of $\Omega^{\pm}_{j}$, it suffices that it be the case for $\mathcal{B}^{+}_{1,\epsilon}$. One possibility for achieving this is to take some $x_0 \in \mathbb{R}^+$ sufficiently large (for instance $x_0 = \frac{3r}{4}$), and to choose the i-th element $\mathcal{B}^{+}_{1,\epsilon}$, so that its i-th coordinate at x_0 be equal to 1. (Note that the basis $\mathcal{B}^{\pm}_{j,0}$ is precisely a basis of a fundamental matrix solution $W^{\pm}_{j}$ of Section 2.1.)

This normalization provides the following theorem, which is an unfolding of Sibuya's sectorial normalization theorem

Theorem 5.7. *We consider a germ of family of systems* (2.9) *in prenormal form. For each sectoral domain S_s in Σ_0, and the corresponding associated sectors $\Omega^{\pm}_{j,\epsilon,S_s}$, there exist linear changes of coordinates $H^{\pm}_{j,\epsilon,S_s}$ ($y \mapsto H^{\pm}_{j,\epsilon,S_s} y$) transforming the formal normal form* (2.12) *to the system* (2.9)*, with continuous limits $H^{\pm}_{j}$ at $\epsilon = 0$ independent of S_s. The convergence to $H^{\pm}_{j}$ is uniform over compact sets of $\Omega^{\pm}_{j}$. Moreover the map $H^{\pm}_{j,\epsilon,S_s}$ is uniformly bounded in the neighbourhood of the two singular points x_ℓ adherent to $\Omega^{\pm}_{j,\epsilon,S_s}$.*

Proof. We consider the fundamental matrix solutions $W^{\pm}_{j,\epsilon,S_s}$ of Definition 5.6. Then the maps $H^{\pm}_{j,\epsilon,S_s}$ are simply given by

$$H^{\pm}_{j,\epsilon,S_s} = W^{\pm}_{j,\epsilon,S_s} F_{\epsilon}^{-1}, \tag{5.5}$$

where F_ϵ is the solution of the formal normal form (2.12) using the standard determination of the logarithm on $\Omega^{+}_{1,\epsilon,S_s}$ and analytically continued over the other sectors $\Omega^{\pm}_{j,\epsilon,S_s}$ when turning in the positive direction. From their construction the columns of $W^{\pm}_{j,\epsilon,S_s}$ have asymptotic expansion at the singular points which are the same as the asymptotic expansion of the columns of F_ϵ, and the coefficients of the expansion are uniformly bounded and bounded away from 0. This yields the uniform boundedness of $H^{\pm}_{j,\epsilon,S_s}$ in the neighbourhood of the singular points x_ℓ adherent to $\Omega^{\pm}_{j,\epsilon,S_s}$. □

Definition 5.8. We call *gate matrices* the transition matrices $C^G_{i,\sigma(i),\epsilon}$ representing the passage from $\Omega^+_{i,\epsilon}$ to $\Omega^-_{\sigma(i),\epsilon}$ through a gate sector $\Omega^G_{i,\sigma(i),\epsilon}$, satisfying

$$(H^-_{\sigma(j),\epsilon,S_s}(x))^{-1}H_{j,\epsilon,S_s}(x)^+F_\epsilon(x) = F_\epsilon(x)C^G_{j,\sigma(j),\epsilon,S_s}.$$

Remark 5.9. *The product in the right order of the gate matrices (or their inverses) attached to a singular point yields the monodromy of the fundamental matrix solution* (3.9) *of the formal normal form* (2.12) *around a Fuchsian singular point* x_l.

6. The theorem of analytic classification

Theorem 6.1. *Two germs of generic unfoldings of linear differential systems,* $p_\epsilon(x)y' = A(\epsilon,x)y$ *and* $p_\epsilon(x)z' = \overline{A}(\epsilon,x)z$ *with an irregular non resonant singularity of Poincaré rank k, are analytically equivalent if and only if*

- *they have the same formal invariants (i.e. the same formal normal form* (2.12)*);*
- *For each germ of sector* S_s *of the covering of* Σ_0*, they have equivalent collections of unfolded (normalized) Stokes matrices* $\{C^\dagger_{j,\epsilon,S_s}\}$*,* $\dagger \in \{L,U\}$*, where two collections* $\{C^\dagger_{j,\epsilon,S_s}\}$ *and* $\{\overline{C}^\dagger_{j,\epsilon,S_s}\}$ *are equivalent if there exists a family of diagonal matrices* D_{ϵ,S_s} *depending analytically on* $\epsilon \in S_s$ *with continuous invertible limit at* $\epsilon = 0$ *independent of* S_s*, such that, for all j and* $\dagger$*,*
$$D_{\epsilon,S_s}C^\dagger_{j,\epsilon,S_s} = \overline{C}^\dagger_{j,\epsilon,S_s}D_{\epsilon,S_s}.$$
In particular a germ of generic unfolding of linear differential system, $p_\epsilon(x)y' = A(\epsilon,x)y$ *is diagonalizable, and hence analytically equivalent to its formal normal form, if and only if all its unfolded Stokes matrices are diagonal.*

Proof. Let us first suppose that two systems are analytically equivalent. We can of course suppose that the two families of systems are in prenormal form. We have already seen (in Theorem 3.2) that they have the same formal invariants. Moreover, we can always restrict r and ρ so that we have the same sectors $\Omega^\pm_{j,\epsilon,S_s}$ covering $\mathbb{D}_r$ for $\epsilon \in S_s \cap \mathbb{D}_\rho$. Then, a normalization of the family $p_\epsilon(x)z' = \overline{A}(\epsilon,x)z$ over a sector $\Omega^\pm_{j,\epsilon,S_s}$ is obtained as the composition of an equivalence between that system and the system $p_\epsilon(x)y' = A(\epsilon,x)y$, with a normalization of the system $p_\epsilon(x)y' = A(\epsilon,x)y$ over $\Omega^\pm_{j,\epsilon,S_s}$. It follows that the collections of unfolded Stokes matrices over S_s are equivalent; indeed, as the data that defines them is geometric, this is not surprising.

Conversely, let consider two germs of families, $p_\epsilon(x)y' = A(\epsilon,x)y$ and $p_\epsilon(x)z' = \overline{A}(\epsilon,x)z$, with same formal invariants and equivalent collections of unfolded Stokes matrices. As before, we can restrict r and ρ so that they are the same for the two families. We first show that the two families are equivalent by means of a linear equivalence $z = T_{\epsilon,S_s}(x)y$ over each sectoral domain S_s, where all the T_{ϵ,S_s} have the same limit T_0 for $\epsilon \in S_s$ and then tending to 0. We then correct to a linear equivalence single-valued in $\epsilon \in \Sigma_0 \cup \{0\}$, i.e independent of the sectoral domain. The map T_{ϵ,S_s} is defined as follows

$$T_{\epsilon,S_s} = \overline{H}^\pm_{j,\epsilon,S_s}D_{\epsilon,S_s}(H^\pm_{j,\epsilon,S_s})^{-1}, \qquad \text{on} \quad \Omega^\pm_{j,\epsilon,S_s}.$$

We first need to show that this map is single-valued in x, that is, independent of the sectors $\Omega^{\pm}_{j,\epsilon,S_s}$. First, it glues uniformly over the intersection sectors $\Omega^U_{j,\epsilon}$ and $\Omega^L_{j,\epsilon}$ since D_{ϵ,S_s} commutes with $F^{\pm}_{j,\epsilon}$. Also, when we consider a gate sector $\Omega^G_{j,\sigma(j),\epsilon,S_s}$ (see Definition 5.1), the transition matrix $C^G_{j,\sigma(j),\epsilon,S_s}$ through the gate sector is diagonal since the same flags have been used for the two sectors $\Omega^+_{j,\epsilon,S_s}$ and $\Omega^-_{\sigma(j),\epsilon,S_s}$. The diagonal matrix $C^G_{j,\sigma(j),\epsilon,S_s}$ depends only on the eigenvalues at the singular points and on the normalization we have chosen for the unfolded Stokes matrices. Hence the two systems have the same transition matrices over the gate sectors. From this it follows that T_{ϵ,S_s} is uniform in x. Also, Theorem 5.7 ensures that $\lim_{\substack{x\to x_l\\ x\in\Omega^{\pm}_{j,\epsilon,S_s}}} H^{\pm}_{j,\epsilon,S_s}$ is bounded when x_ℓ is adherent to $\Omega^{\pm}_{j,\epsilon,S_s}$. Moreover, the limit maps T_0 for $\epsilon = 0$ are independent of S_s.

We now need to show that we can construct a uniform equivalence $T_\epsilon(x)$ between the two systems, that is, which is both independent of the sectoral domain S_s and extended to $\mathbb{D}_\rho \setminus \Sigma_0$. Here is the strategy. Let us first show how to do this on Σ_0. We consider pairs of intersecting sectoral domains S_s and $S_{s'}$ and the corresponding T_{ϵ,S_s} and $T_{\epsilon,S_{s'}}$. Then the map $T_{\epsilon,S_s}(T_{\epsilon,S_{s'}})^{-1}$ is an automorphism of the second system for $\epsilon \in S_s \cap S_{s'}$ and $x \in \mathbb{D}_r$. Hence we need to analyze the automorphisms of families of systems of the form (2.9). This is done in the following lemma.

Lemma 6.2. *The automorphism group of each system of the form* (2.9) *is a direct sum of m copies of $\mathbb{C}^*$, with $m \le n$.*

Proof. In some sense, as the model system has symmetries $(\mathbb{C}^*)^n$ it will follow that the symmetries of any deformation of it will be a subgroup of this group. Explicitly, for each system with $\epsilon \in S_s$ we consider the corresponding set of sectors $\Omega^{\pm}_{j,\epsilon}$ and transformations $H^{\pm}_{j,\epsilon}$ from the normal form to the system over $\Omega^{\pm}_{j,\epsilon}$. For the normal form, it is obvious that the symmetry group is the group of invertible diagonal matrices which is isomorphic to $(\mathbb{C}^*)^n$, and an element of the symmetry group is given by $\operatorname{diag}(c_{1,\epsilon}, \dots, c_{n,\epsilon})$. Now symmetries of (2.9) need to commute with the unfolded Stokes matrices. If some unfolded Stokes matrices are non diagonal, then this forces some of the $c_{j,\epsilon}$ to take equal values. Hence the number m of independent $c_{j,\epsilon}$ is equal to the maximum number of blocks in a common block diagonal form for all unfolded Stokes matrices after applying a given permutation of rows and columns to all unfolded Stokes matrices simultaneously. To emphasize the structure of direct sum we write the corresponding automorphism of (2.9) as $G_\epsilon \circ \operatorname{diag}(c_{1,\epsilon}, \dots, c_{n,\epsilon})$, with the understanding that some of the $c_{j,\epsilon}$ may be equal. □

End of proof of Theorem 6.1. Hence,

$$T_{\epsilon,S_s}(T_{\epsilon,S_{s'}})^{-1} = \Xi_{s,s'}(\epsilon) = G_\epsilon \circ \operatorname{diag}(c_{1,s,s'}(\epsilon), \dots, c_{n,s,s'}(\epsilon)).$$

For fixed j, each set $\{c_{j,s,s'}\}$ gives Cousin data for the covering S_s of Σ_0. Since Σ_0 is a Stein manifold, there exist functions $c_{j,s}(\epsilon)$ defined on each S_s, such that for each $s \neq s'$, $c_{j,s}(\epsilon)^{-1}c_{j,s'}(\epsilon) = c_{j,s,s'}(\epsilon)$. The corrected $\Theta_{\epsilon,S_s}(x) = G_\epsilon \circ \operatorname{diag}(c_{1,s}(\epsilon), \dots, c_{n,s}(\epsilon))T_{\epsilon,S_s}(x)$ coincide on the overlaps of sectoral domains, and can be glued together to obtain a global $\Theta_\epsilon(x)$, defined for $(\epsilon, x) \in \Sigma_0 \times \mathbb{D}_r$. From their construction, the maps $H^{\pm}_{j,\epsilon,S_s}$ have a continuous limit at points of $\mathbb{D}_\rho \setminus \Sigma_0$, allowing to extend the T_{ϵ,S_s} to these points. If we could prove that the functions $c_{j,s}$ were bounded near the points of $\hat{\mathbb{D}}_\rho \setminus \Sigma_0$, we could fill the holes analytically and

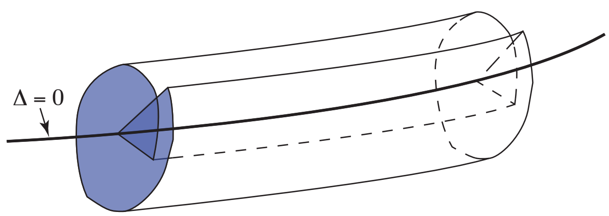


Figure 9. An auto-intersection of S_s providing a tubular neighbourhood of $\Delta = 0$.

the theorem would follow. But there is a simpler way: we will add Cousin data to cover a generic subset of $\mathbb{D}_\rho \setminus \Sigma_0$, such that the remaining points form a subvariety of codimension 2; indeed, for these remaining points we can apply Hartogs' theorem. We will enlarge Σ_0 to Σ_0', where Σ_0' is the union of Σ_0 plus the generic points of $\Delta = 0$ where exactly two singular points coalesce, and we will construct an open covering of Σ_0' by adding to the sectoral domain S_s some open neighbourhoods of the points of $\Sigma_0' \setminus \Sigma_0$.

The problem of considering what happens when exactly two zeros of $p_\epsilon(x)$ coalesce as one varies ϵ reproduces, in a parametrised form (the extra parameters being essentially the location of the other zeros) the $k = 1$ case studied in [8]. Indeed, the coalescence of exactly two singular points at a given value ϵ' of the parameter is a codimension 1 phenomenon with the generic unfolding parameter $\eta = \Delta$ in the neighbourhood of ϵ'. Let η' be $(k-1)$-parameters transverse to η, so that $\epsilon \mapsto (\eta, \eta')$ is a biholomorphic change of parameters. Hence, a neighbourhood of ϵ' is of the form $E = \mathbb{D} \times \mathbb{D}'$, where $\mathbb{D}$ (resp. $\mathbb{D}'$) is an open disk in η-space (resp. polydisk in η'-space).

Let S_s be a sectoral domain adherent to ϵ'. We can extend S_s in a ramified way around $\eta = 0$ so that it self-intersects (see Figures 9 and 10).

An essential ingredient is that the construction of the sectors $\Omega^{\pm}_{j,\epsilon,S_s}$ and transformations $H^{\pm}_{j,\epsilon,S_s}$ have continuous limits when $\epsilon \in S_s$ tends to ϵ' (and a finer property described below). For η in a small neighbourhood $\mathbb{D}$ of 0, there are two zeros of p_ϵ in a small disk, coalescing at $\eta = 0$ to a double zero. Taking the parameter η around a small circle around the origin interchanges the two zeros. The issue is that at some point, the separatrices converging to the zeros bifurcate, and change endpoints. Hence, we need to work with $\hat\eta$ in the double covering of η-space punctured at $\eta = 0$. We extend S_s in a ramified way so that for, each fixed η', $\hat\eta$ belongs to a ramified cover $\hat{\mathbb{D}}$ of $\mathbb{D}$. On the auto-intersection of $\hat{\mathbb{D}}$, the map

$$T_{(\hat\eta e^{2\pi i},\eta'),S_s}\big(T_{(\hat\eta,\eta'),S_s}\big)^{-1} = \Xi_{\epsilon',\cap}(\epsilon) = G_{\epsilon,S_s} \circ \operatorname{diag}(d_1'(\eta), \dots, d_n'(\eta))$$

is an automorphism of the second system. We find invertible functions $d_j(\hat\eta)$ defined on $\hat{\mathbb{D}} \cap \{|\eta| < \rho'\}$ for some $\rho' > 0$ such that $d_j(\hat\eta e^{2i\pi})(d_j(\hat\eta))^{-1} = d_j'(\eta)$ (details in Lemma 6.3 below). Replacing $T_{(\hat\eta,\eta'),S_s}$ by

$$\Theta_{(\eta,\eta'),S_s} = G_{(\eta,\eta'),S_s} \circ \operatorname{diag}(d_1(\hat\eta)), \dots, d_n(\hat\eta))T_{(\hat\eta,\eta'),S_s}$$

yields a uniform $\Theta_{(\eta,\eta'),S_s}$ over $\mathbb{D} \setminus \{\eta = 0\}$. Note that the $d_j'(\eta)$ have the same limit on all rays in $\hat E$ because of the chosen normalization in Section 5.3. Since it is a one-dimensional phenomenon depending analytically on η', there is an explicit

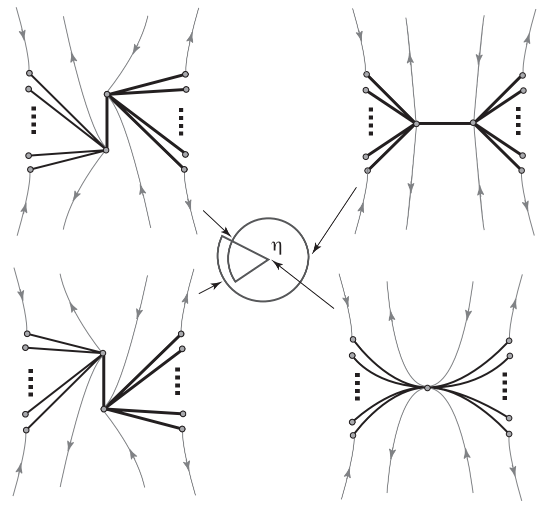


FIGURE 10. The change of skeleton depending on η, where $\eta = 0$ corresponds to a generic point of $\Delta = 0$. Of course on the left figures the gate sectors should be spiraling at the middle singular points.

formula for the $d_j(\hat{\eta})$, which has a finite limit at $\eta = 0$. Hence we can extend $\Theta_{(\eta,\eta'),S_s}$ to $\eta = 0$ in an analytic way. The details are exactly the same as in [8], and this is why we are brief here.

Adding the collection of $\Theta_{(\eta,\eta'),S_s}$ to the former Θ_ϵ, yields a collection of equivalences between the two systems for $\epsilon \in \Sigma_0'$. Their comparisons two by two are again symmetries of the second system, hence diagonal matrices, the coefficients of which provide Cousin data. Since Σ_0' is again a Stein manifold, we can solve the Cousin problem and correct the collection of equivalences to a uniform equivalence over Σ_0'. We can extend it to a uniform equivalence over $\mathbb{D}_\rho$ by Hartogs' theorem.

As for showing that a system $p_\epsilon(x)y' = A(\epsilon, x)y$ is diagonalizable, and hence analytically equivalent to its formal normal form, if and only if all its unfolded Stokes matrices are diagonal, it suffices to remark that the diagonal part of the normalized Stokes matrices are uniquely determined by the formal normal form, and to apply the first part of the theorem.

□

Lemma 6.3. *There exists nonzero functions $d_j(\hat{\eta})$ defined on $\hat{\mathbb{D}}$ such that*

$$d_j(\hat{\eta}e^{2i\pi})(d_j(\hat{\eta}))^{-1} = d_j'(\eta). \tag{6.1}$$

Proof. We change to the variable $\nu = \sqrt{\hat{\eta}}$. Then $\hat{\mathbb{D}}$ gives a sector $\overline{V}$ if we take $\arg\hat{\eta} \in (-\pi - 2\delta, \pi + 2\delta)$ and we take a symmetric sector $\widetilde{V} = e^{\pi i}\overline{V}$. These two sectors have two intersections parts which we call V^+ when $\arg\nu \in (\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta)$ and V^- when $\arg\nu \in (-\frac{\pi}{2} - \delta, -\frac{\pi}{2} + \delta)$. The map $d_j'(\eta)$ yields two maps $f^+(\nu) = \log(d'(\eta))$ and $f^-(\nu) = f^+(-\nu)$ over V^+ and V^- respectively. We look for maps $\overline{f}$

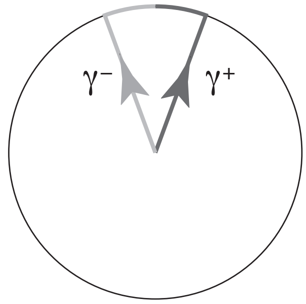


FIGURE 11. The paths $\gamma^{\pm}$ in the proof of Lemma 6.3.

(resp. $\tilde{f}$) over $\overline{V}$ (resp. $\widetilde{V}$) such that

$$\begin{cases} \overline{f} - \tilde{f} = f^+, & \text{on } V^+, \\ \tilde{f} - \overline{f} = f^-, & \text{on } V^-. \end{cases} \tag{6.2}$$

For that purpose, we look for a map $\overline{\overline{f}}$ such that $\overline{\overline{f}}(e^{2\pi i}\nu) - \overline{\overline{f}}(\nu) = f^+(\nu)$ on V^+. And we get the corresponding map $\tilde{\tilde{f}}(\nu) = \overline{\overline{f}}(-\nu)$. The maps $\overline{f}$ and $\tilde{f}$ are then obtained from $\overline{\overline{f}} + \tilde{\tilde{f}}$ on the two simply connected sectors of the intersections of the domains of $\overline{\overline{f}}$ and $\tilde{\tilde{f}}$.

For the construction of $\overline{\overline{f}}$, we split the boundary of V^+ in two half-pieces, γ_+ and γ_-, each containing one ray and half of the boundary arc (see Figure 11). We define

$$\overline{\overline{f}}(\nu) = \begin{cases} \int_{\gamma_+} \frac{f^+(\zeta)}{\zeta-\nu}\, d\zeta, & \arg(\nu) \in (-\frac{3\pi}{2} - \delta, -\frac{\pi}{2} + \delta), \\ \int_{\gamma_-} \frac{f^+(\zeta)}{\zeta-\nu}\, d\zeta, & \arg(\nu) \in (-\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta). \end{cases}$$

Let $\gamma = \gamma^+ \cup \{-\gamma^-\}$ the closed curve with interior V^+. Let us assume for the moment that the integrals defining $\overline{\overline{f}}(\nu)$ converge. Then, for $\nu \notin V^+$, the integral $\int_\gamma \frac{f^+(\zeta)}{\zeta-\nu}\, d\zeta$ vanishes, yielding that the two definitions of $\overline{\overline{f}}(\nu)$ coincide for $\arg(\nu) \in (-\frac{\pi}{2} - \delta, -\frac{\pi}{2} + \delta)$. From Cauchy integral theorem we also have that $\int_\gamma \frac{f^+(\zeta)}{\zeta-\nu}\, d\zeta = f^+(\nu)$ for $\nu \in V^+$, and hence the solution obviously satisfies $\overline{\overline{f}}(e^{2\pi i}\nu) - \overline{\overline{f}}(\nu) = f^+(\nu)$ on V^+.

We are only left with proving that the integrals defining $\overline{\overline{f}}(\nu)$ converge. For this purpose, it suffices to prove that $f^+(\nu)| \leq c|\nu|$ for some positive c. Many estimates of this kind are derived in detail in [2] and [10], and hence we are sketchy on the details. This will follow from an estimate on the fundamental matrix solutions $W^{\pm}_{j,(\eta,\eta'),S_s}$ of the form $|W^{\pm}_{j,(\eta,\eta'),S_s} - W^{\pm}_{j,(0,\eta'),S_s}| < |c'(x)|\,|\eta|$, which will in turn come from the application of Theorem 5.3 in Theorems 5.4 and 5.7. We need of course limit ourselves to the intersection sectors adherent to the two singular points which coalesce for $\eta = 0$. The vectors allowing to define the flags over $\Omega^{\pm}_{j,(\eta,\eta'),S_s}$ in applying Theorem 5.3 are fixed points of an integral operator. The integrand itself is C^1 in η, but the path on which we integrate for $\eta \neq 0$ corresponds to a horizontal segment of length of the order $\frac{c}{\nu}$ in t-space followed by a slanted half-line when $\eta \neq 0$. We then divide each integral (the one for $\eta \neq 0$ and the one for $\eta = 0$)

into two parts: a finite part on the horizontal segment and an infinite part on the slanted half-line. The difference of the integrands in the finite parts is less than $c_1|\eta|$ for some positive c_1 while we integrate along a segment of length of the order $\frac{c}{\nu}$. Hence the difference of the integrals on the finite parts is of the order of $c_2|\nu|$. Also each infinite part is less than $c_3|\nu|$. Indeed, the integrand is bounded, we transform this integral in t into an integral in x and we can take the slanted part so that the length of the corresponding curve in x be less than $c_4|\nu|$. This comes from the fact that the distance between the holes (the preimages of the complement of $\mathbb{D}_r$) for that special strip we need to pass is of the order of $C/\sqrt{\eta}$. Indeed, it is the two points that are coallescing for $\eta = 0$ which give the order of magnitude of the distance between the holes.

The function d_j in (6.1) is simply given by $d_j(\eta) = \exp(\overline{f}(\sqrt{\eta}))$. $\square$

7. Geometry of the unfolding and solutions with logarithmic terms

The unfolding we have obtained allows us to see that the geometry of solutions, at the values of the parameter for which the system has an irregular singular point, is tied to the geometry of the system at the Fuchsian singular points, and that this is mediated by the monodromy matrices of our flag data.

When a Fuchsian singular point x_l is resonant (Definition 3.3), the solutions at x_l generically have logarithmic terms and the local system is non diagonalizable. This comes from the fact that the monodromy matrix, generically, has Jordan block(s) corresponding to its equal eigenvalues.

The parametric resurgence phenomenon already described in [8] establishes that when the Stokes matrices at $\epsilon = 0$ are sufficiently generic, the resonant singularities are *forced* to have logarithmic terms for sequences of resonant values of the parameter ϵ converging to $\epsilon = 0$.

As an example, consider values of ϵ in a sector domain S_s and a Fuchsian singular point x_ℓ of ω-type to which are attached m intersection sectors $\Omega^L_{j_1,\epsilon}, \dots, \Omega^L_{j_m,\epsilon}$ and m gates sectors. Because the eigenvalues of the residue matrix at x_ℓ are of the form $\frac{\lambda_i(\epsilon)}{p_\epsilon(x_\ell)}$, it is clear that we can have accumulations of resonances at x_ℓ of a given type $\lambda_i(\epsilon) = \lambda_{i'}(\epsilon) + n$ for $n \in \mathbb{N}$ with $n \to \infty$. Let us suppose that the monodromy around x_ℓ starting in an appropriate sector has the form

$$M_\epsilon = C^L_{j_1,\epsilon} D_1 \dots C^L_{j_m,\epsilon} D_m,$$

where the $D_i = (C_{j_{i+1},\sigma(j_{i+1}),\epsilon})^{-1}$ are the inverses of the diagonal gate matrices introduced in Definition 5.8 and uniquely determined from the formal invariants of the system. The matrix M_ϵ is lower triangular and, because the $C^L_{j_i,\epsilon}$ are unipotent, its diagonal part is $D_1 \dots D_m$, which is precisely the diagonal part of the triangular matrix

$$N_\epsilon = C^L_{j_1,0} D_1 \dots C^L_{j_m,0} D_m.$$

Note that the dependence of this matrix on ϵ is only through the gate matrices D_i. We decide to approach $\epsilon = 0$ along sequences $\{\epsilon^m\}$ of values of ϵ for which the matrices D_i are constant. Then N_ϵ is constant along this sequence. We consider the particular sequences $\{\epsilon^m\}$ for which M_ϵ has multiple eigenvalues and some structure of Jordan blocks, with at least one non trivial Jordan block (such sequences generically exist when the Stokes matrices $C^L_{j_i,\epsilon}$ are not all diagonal). Since the $C^L_{j_i,\epsilon}$ depend continuously in ϵ, if N_{ϵ^m} has Jordan blocks, then M_{ϵ^m} is forced to have Jordan blocks at least as large for sufficiently large m. This phenomenon has been studied in detail for the case $k = 1$ in Corollary 4.33 of [8].

8. Conclusion

In this paper we have identified a complete modulus of analytic classification for germs of analytic families of linear differential systems at an irregular non resonant singularity of Poincaré rank k. The next natural question is to identify the moduli space: it has been solved in the case $k = 1$ in [8], and we are reasonably confident that the same strategy could be applied here, even if the problem is significantly more difficult. We hope to address this question in the near future.

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