Fields Institute Communications
Volume **00**, 0000

Normal forms for germs of analytic families of planar vector fields unfolding a generic saddle-node or resonant saddle

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Abstract. Normal form theory provides an algorithmic way to decide if two germs of planar vector fields with a saddle-node or a resonant saddle are equivalent under a C^N-change of coordinates, in which case, the normal forms are polynomial. However, in the analytic case, the formal change of coordinates to normal form generically diverges. An explanation of this is found by considering unfoldings of the vector fields and explaining the divergence in the limit process. We consider the orbital equivalence problem for germs of families of vector fields unfolding a generic saddle-node or resonant saddle and give a complete modulus of analytic classification for such families.

1 Introduction

The paper is a contribution to the general question of the equivalence problem for analytic vector fields on $\mathbb{R}^n$, namely

1. When are two germs of vector fields v and w locally orbitally equivalent?
2. When are two germs of families of vector fields v_ϵ and w_ϵ locally orbitally equivalent? (ϵ can be a multi-parameter.)

We can of course always suppose that the germs of vector fields and parameters are all localized at the origin.

Question 1 (resp. 2) has the answer "always" when $v(0), w(0) \neq 0$ (resp. $v_0(0), w_0(0) \neq 0$). This is just a consequence of the blow-box theorem which is valid for analytic families and implies that there is a unique equivalence class under the hypothesis that the vector field does not vanish.

A strategy to try to solve the equivalence problem and identify the different equivalence classes is to use normal form theory. Indeed the normal form can be seen as a canonical element of the equivalence class.

2000 *Mathematics Subject Classification.* 34C20, 34M25, 37G05.
This work was supported by NSERC in Canada.

If we look to the case of a nonzero family of vector fields the flow-box theorem states that the family is locally orbitally equivalent to

$$\begin{array}{rcl} \dot{x}_1 &=& 1 \\ \dot{x}_2 &=& 0 \\ \vdots && \\ \dot{x}_n &=& = 0 \end{array} \tag{1.1}$$

which can be seen as a normal form for the family.

So the first nontrivial case is when the vector field has a singular point at the origin $v(0) = 0$. In dimension 1 the problem has been completely solved by Kostov [7] (except for Question 2 in the case where $v_0 \equiv 0$). Indeed if $v'(0) \neq 0$, then for any family v_ϵ unfolding v there exists an analytic linearizing change of coordinates $h_\epsilon(x) = h(\epsilon, x)$ defined in a neighborhood of the origin in (x, ϵ) space. When $v'(0) = 0$ and $v^{(k+1)}(0) \neq 0$ (this is the codimension k case), then for any family v_ϵ unfolding v there exists an analytic change of coordinates $h_\epsilon(x) = h(\epsilon, x)$ defined in a neighborhood of the origin in (x, ϵ) and an analytic scaling of time bringing the family to the normal form

$$\dot{x} = (\epsilon_0 + \epsilon_1 x + \cdots + \epsilon_{k-1}x^{k-1} + x^{k+1})(1 + a(\epsilon)x^k). \tag{1.2}$$

For the rest of the paper we will limit ourselves to the two-dimensional case $n = 2$.

The node case. We consider a vector field v_0 which has a node at the origin with eigenvalues λ_1 and λ_2 such that $\lambda_1/\lambda_2 = \mu_0 \in \mathbb{R}^+$. Let v_ϵ be an unfolding of v_0 and μ_ϵ be the quotient of the eigenvalues at the singular point. It goes back to Poincaré that the family v_ϵ is locally orbitally equivalent to:

1. If $\mu_0 \notin \mathbb{N} \cup 1/\mathbb{N}$
$$\begin{array}{rcl} \dot{x} &=& x \\ \dot{y} &=& \mu_\epsilon y. \end{array} \tag{1.3}$$
2. If $\mu_0 \in \mathbb{N}$
$$\begin{array}{rcl} \dot{x} &=& x \\ \dot{y} &=& \mu_\epsilon y + a(\epsilon)x^{\mu_0}. \end{array} \tag{1.4}$$
3. If $\mu_0 \in 1/\mathbb{N}$
$$\begin{array}{rcl} \dot{x} &=& x + b(\epsilon)y^{1/\mu_0} \\ \dot{y} &=& \mu_\epsilon y. \end{array} \tag{1.5}$$

The saddle case. We consider a vector field v_0 which has a saddle at the origin with eigenvalues λ_1 and λ_2 such that $\lambda_1/\lambda_2 = -\mu_0 \in \mathbb{R}^-$.

1. If λ_1/λ_2 is irrational then there is a formal change of coordinates and a formal time scaling to the linear system
$$\begin{array}{rcl} \dot{x} &=& x \\ \dot{y} &=& -\frac{\mu_0}{y}. \end{array} \tag{1.6}$$
If μ_0 is diophantian (badly approximated by the rational numbers) then there exists an analytic change of coordinates and time scaling to the linear system. However if μ_0 is Liouvillian then *divergence is the rule and convergence is the exception* [6]. The question we ask is **"Why?"**

2. If $\lambda_2/\lambda_1 = -p/q$, then, in the generic (codimension 1) case, there exists a formal change of coordinates and a formal time scaling transforming the system to the polynomial normal form
$$\begin{array}{rcl} \dot{x} & = & x \\ \dot{y} & = & y(-\frac{p}{q} + u + au^2), \end{array} \tag{1.7}$$
where $u = x^p y^q$. Again *divergence is the rule.* It is very exceptional that the change of coordinate to normal form converges. Here also we ask **"Why?"**

The Hopf bifurcation case. We consider a vector field v_0 which has a weak focus of order 1 at the origin with eigenvalues $\lambda_{1,2} = \pm i\omega$. Then there is a formal change of coordinates and a formal time scaling to the system
$$\begin{array}{rcl} \dot{x} & = & -\omega y + x(x^2+y^2) + ax(x^2+y^2)^2 \\ \dot{y} & = & \omega x + y(x^2+y^2) + ay(x^2+y^2)^2. \end{array} \tag{1.8}$$
Again *divergence is the rule* and it is very exceptional that the change of coordinate to normal form converges. Again we ask **"Why?"**

The saddle-node case. We consider a vector field v_0 which has a saddle-node of codimension 1 at the origin. Then there exists a formal change of coordinates and a formal time scaling to the system
$$\begin{array}{rcl} \dot{x} & = & x^2 \\ \dot{y} & = & y(1+ax). \end{array} \tag{1.9}$$
Here also *divergence is the rule* and only exceptionally the change of coordinate to normal form converges. Again we ask **"Why?"**

These questions are all related as non resonant (resp. resonant) saddles appear in the perturbation of a resonant (resp. non resonant) saddle. And a saddle-node is the coallescence of a saddle and a node. Hence, to answer these questions, it is natural to study the unfoldings of these situations. Moreover as we are considering convergence of power series it is necessary to enlarge the variables (x, y) to the complex domain, namely to a neighborhood of $(0, 0)$ in $\mathbb{C}^2$. This point of view allows to unify the weak focus case with the saddle case as the ratio of eigenvalues of a weak focus is -1.

The spirit of the general answer is the following. The dynamics of the original system is very rich. It is much too rich to be encoded in the simple dynamics of the normal form which depends of at most one parameter. Hence the divergence of the normalizing series.

Strategy. We must learn to read the rich dynamics of the original system in order to solve the equivalence problem.

In Section 2 we discuss the example of the saddle-node. In Section 3 we discuss the equivalence problem for saddles and saddle-nodes via the holonomy map. In Section 4 we discuss analytic changes of coordinates to normal form. Finally in Section 5 we discuss applications to problems of finite cyclicity of graphics. We end up with perspectives.

2 The example of the saddle-node

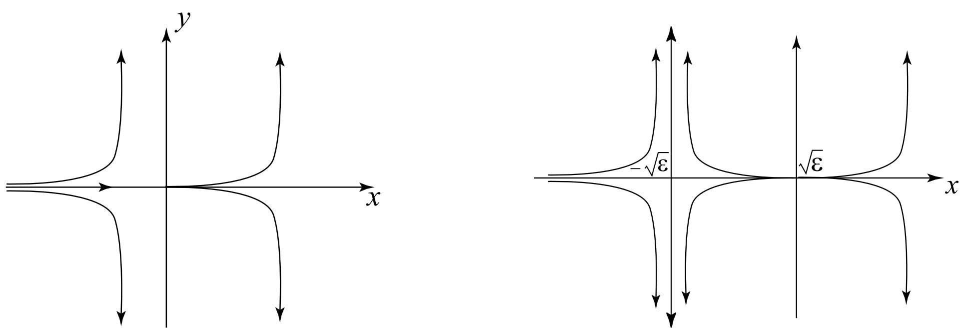
As mentioned above, if v_0 has a saddle-node of codimension 1 at the origin, then there exists a formal change of coordinates and a formal time scaling to the system

$$\begin{array}{rcl}\dot{x} & = & x^2\\ \dot{y} & = & y(1+ax).\end{array} \tag{2.1}$$

If v_η is a generic family unfolding the saddle node then, for any $k \in \mathbb{N}$, there exists a C^k-change of coordinates and parameters and a C^k time scaling bringing the family to the normal form

$$\begin{array}{rcl}\dot{x} & = & x^2-\epsilon\\ \dot{y} & = & y(1+a(\epsilon)x).\end{array} \tag{2.2}$$

We call (2.1) the *model* and (2.2) the *model family*. Their phase portrait appear in Figure 1.



(a) $\epsilon = 0$ (b) $\epsilon \neq 0$

Figure 1 The "model"

Starting with a single analytic vector field v_0 with a saddle-node at the origin it is possible to find an analytic change of coordinates and analytic time scaling to bring the system to the form

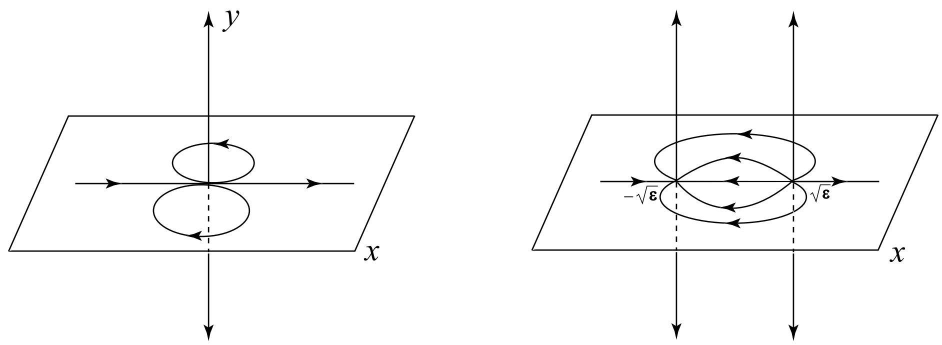
$$\begin{array}{rcl}\dot{x} & = & x^2\\ \dot{y} & = & f_0(x)+y(1+ax)+\sum_{j=2}^{\infty} f_j(x)y^j.\end{array} \tag{2.3}$$

The model has the analytic center manifold $y=0$. One obstruction to bring (2.3) to normal form is the non-existence of an analytic center manifold. Indeed there exists a formal center manifold

$$y=\sum_{n=2}^{\infty} b_n x^n, \tag{2.4}$$

but generically the series is divergent. The generic divergence is this case is formulated in this way. Suppose that the system (2.3) depends analytically on a finite number of parameters $\eta \in \mathbb{R}^n$. As soon as there exists a single value η_0 for which the series (2.4) is divergent, then the set of values of η for which the series is convergent is an analytic set. In particular divergence occurs on a dense open set.

To understand why divergence is the rule in this case we complexify x and y so that (x, y) is now defined on a neighborhood of the origin in $\mathbb{C}^2$ and we unfold. The model and model family appear in Figure 2.



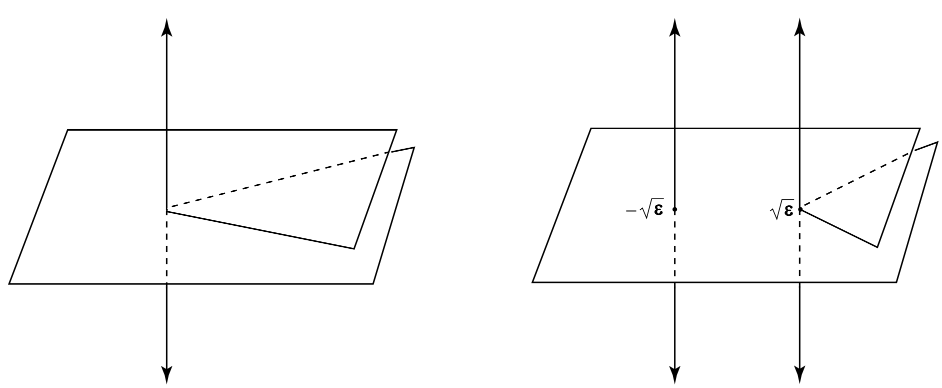
(a) $\epsilon = 0$ (b) $\epsilon \neq 0$

Figure 2 The "model"

Given a generic unfolding of v_0 it is possible to find an analytic change of coordinates and parameter and an analytic time scaling to bring the family to the form

$$\begin{aligned} \dot{x} &= x^2 - \epsilon \\ \dot{y} &= f_{0,\epsilon}(x) + y(1 + a(\epsilon)x) + \sum_{j=2}^{\infty} f_{j,\epsilon}(x)y^j. \end{aligned} \tag{2.5}$$

Let us now discuss the case where there exists no analytic center manifold, i.e. the series given in (2.4) is divergent. However this series is 1-sumable (or Borel-summable) [10] and yields a solution in a sectorial domain V of the universal covering of the x-space punctured at 0 of the form $V = \{\hat{x}; |\hat{x}| < r, arg(\hat{x}) \in (-\pi/2 + \delta, 5\pi/2 - \delta)\}$, where $r, \delta > 0$ are small. As we will not use the theory of summability we do not define it and refer the interested reader to [1]. This solution is ramified (see Figure 3a).



(a) $\epsilon = 0$ (b) $\epsilon \neq 0$

Figure 3 The invariant manifold

The Figure 3b) represents the case $\epsilon > 0$, i.e. the system has a saddle and a node. It is known that the saddle always has an analytic stable manifold with equation $y = g_\epsilon(x)$. Let us now discuss the local model at the node. We have two cases depending if the node is non resonant or resonant.

1. If the node is non resonant then the local model at the node is given by the linear system
$$\begin{aligned} \dot{x} &= \lambda_1 x \\ \dot{y} &= y \end{aligned} \tag{2.6}$$
with $\lambda_1 \notin 1/\mathbb{N}$. All solution curves (except $x = 0$) are of the form $y = Cx^{1/\lambda_1}$. They are all ramified but one! We get the following:

   Conclusion 1: *When we unfold a system with no analytic center manifold, then the analytic separatrices of the saddle and of the node do not match.*

2. If the node is resonant then the local model at the node is the normal form
$$\begin{aligned} \dot{x} &= \tfrac{x}{n} \\ \dot{y} &= y + Ax^n. \end{aligned} \tag{2.7}$$
If $A = 0$ then all solution curves at the node (except $x = 0$) are analytic of the form $y = Cx^n$. This case is obviously impossible when unfolding a system as in Figure 3a) and we are forced to have $A \neq 0$, yielding that all solutions (except $x = 0$) are of the form $y = nAx^n \ln x + Cx^n$. Hence we get:

   Conclusion 2: *When we unfold a system with no analytic center manifold then the node is non linearizable as soon as resonant. This is the "parametric resurgence phenomenon".*

Putting together Conclusions 1 and 2 we get:

Conclusion: *The divergence of the series giving the center manifold reflects an incompatibility between the two equilibrium points as their analytic separatrices do not match. For sequences of parameter values the incompatibility is carried by the singular point itself: this is the parametric resurgence phenomenon.*

The non analyticity of the center manifold is expressed by the divergence of the series (2.4) which is equivalent to the divergence of the change of coordinate $Y = y - \sum_{n=2}^{\infty} b_n x^n$ removing the term $f_0(x)$ in (2.3). This is the first step in transforming the system to the normal form which is our model.

The phenomenon we have described above is very general. If we were considering a system (2.3) in which $f_0 \equiv 0$ (the system has an analytic center manifold) we can look for a change of coordinates
$$Y = y + \sum_{k=0}^{\infty} \sum_{j=2}^{\infty} b_{k,n} x^k y^n \tag{2.8}$$
removing the terms $f_j(x)y^j$ in (2.3). This power series is generically divergent. Its sum is defined as a ramified function on a domain of the form $V \times W$, where W is a neighborhood of the origin in y-space and V is a sectorial domain of the universal covering of the x-space punctured at 0 of the form $V = \{\hat{x}; |\hat{x}| < r, arg(\hat{x}) \in (-3\pi/2 + \delta, 3\pi/2 - \delta)\}$, where $r, \delta > 0$ are small. If the unfolding is of the form (2.5) with $f_{0,\epsilon} \equiv 0$, then we can find an unfolded change of coordinates which

is regular at the node and ramified at the saddle, the ramification reflecting an incompatibility between normalizing changes of coordinates at the saddle and at the node. Here again we observe parametric resurgence phenomena for sequences of parameter values for which the saddle is resonant: for these parameter values the saddle is not integrable.

This last step remains true when the system (2.3) has no analytic center manifold, but the calculations cannot be done in a simple way on the series and the use of geometric methods to handle the proofs is necessary.

Remark 2.1 Many papers in the literature describe the fact that, whenever the normalizing series may diverge, then divergence is the rule and convergence the exception (see for instance [6]). The above geometric explanation for the divergence of the series for the center manifold explains why this is the case. It is indeed the general situation that the analytic separatrices of the saddle and the node do not match. It is also the generic situation that a resonant node be nonlinearizable. When these generic behaviours persist till the limit case of the saddle-node there exists no analytic center manifold.

3 The equivalence problem for resonant saddles and saddle-nodes

The equivalence problem is a problem in two-variables for 2-dimensional vector fields. We will introduce the holonomy of each separatrix (resp. of the strong separatrix) in the case of a saddle (resp. saddle-node). The holonomy is a 1-dimensional map. We can make a parallel between its use and the use of the Poincaré return map which allows to reduce the search for periodic trajectories of a 2-dimensional vector field to the search of fixed points of a 1-dimensional map. As the separatrices of an analytic saddle are analytic and similarly for the strong separatrix of a saddle-node we can always use an analytic change of coordinates transforming these separatrices to the coordinate axes.

To define the holonomy we need to extend the system to a neighborhood of the origin in $\mathbb{C}$. We also allow the time to be complex. So the trajectories are parametrized by open sets in $\mathbb{C}$ and hence are complex curves in $\mathbb{C}^2$, which we can think of as real 2-dimensional surfaces in $\mathbb{R}^4$. The trajectories are usually called the *leaves of the foliation* given by the differential equation.

Definition 3.1 We consider a saddle point

$$\begin{array}{rcl} \dot{x} & = & \lambda_1 x(1+h_1(x,y)) \\ \dot{y} & = & \lambda_2(y+h_2(x,y)) \end{array} \tag{3.1}$$

or a saddle-node

$$\begin{array}{rcl} \dot{x} & = & x^2(1+h_1(x,y)) \\ \dot{y} & = & \lambda_3 y+h_3(x,y)) \end{array} \tag{3.2}$$

of a 2-dimensional vector field, where $\lambda_j \neq 0$, $\lambda_1\lambda_2 < 0$ and $h_{1,2}(x,y) = O(|x,y|)$, $h_3(x,y) = o(|x,y|)$. We consider a section $\Sigma = \{|x| < \delta, y = y_0\}$ where $y_0, \delta > 0$. The holonomy of the y-separatrix is a map $f : U \subset \Sigma \to \Sigma$, where U is a neighborhood of 0 in Σ. Let $(x, y_0) \in U$. We lift the curve $y = y_0 e^{i\theta}$, $\theta \in [0, 2\pi]$ into the leaf of the foliation passing through $(x, y_0) \in U$. The end point is the point $(f(x), y_0)$, yielding the definition of f (Figure 4).

The following proposition is classical.

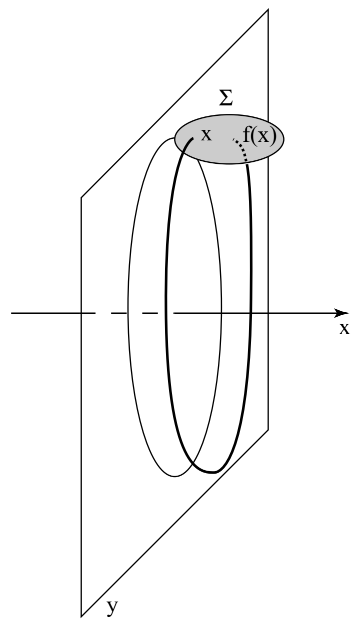


Figure 4 The holonomy map

Proposition 3.2 The holonomy f of the y-axis has the form

$$f(x) = \exp\left(2\pi i \frac{\lambda_2}{\lambda_1}\right) x + o(x) \tag{3.3}$$

for (3.1) and

$$f(x) = x + ax^2 + o(x^2), \tag{3.4}$$

with $a \neq 0$ for a saddle-node.

Remark 3.3 If we choose a different section $\Sigma_1 = \{y = y_1\}$ and if f_1 is the holonomy for the new section, then f and f_1 are conjugate, i.e. there exists an analytic diffeomorphism defined in a neighborhood of 0 such that $f_1 = h^{-1} \circ f \circ h$.

Theorem 3.4

1. *Two germs of vector fields* (3.1) *with a saddle at the origin and same formal normal form, are locally analytically orbitally equivalent if and only if the holonomies of their y-separatrices are conjugate.*
2. *Two germs of vector fields* (3.2) *with a saddle-node at the origin and same formal normal form, are locally analytically orbitally equivalent if and only if the holonomies of their strong separatrices are conjugate.*

Part 1 of the Theorem was proved by Mattei-Moussu [9] for resonant saddles and Pérez-Marco-Yoccoz [14] for non resonant saddles. Part 2 was proved by Martinet-Ramis [10].

The Theorem 3.4 shows that the equivalence problem for germs of vector fields with a saddle or saddle-node is reduced to the conjugacy problem for germs of diffeomorphisms with a fixed point at the origin and multiplier on the unit circle. The kind of results we will describe below applies to generic (codimension 1) resonant saddles ($\lambda_2/\lambda_1 \in \mathbb{Q}$) and saddle-nodes. But for the sake of simplicity we will limit ourselves to saddles with $\lambda_2/\lambda_1 = 1$ and saddle-nodes, which both have a holonomy map tangent to the identity. We limit ourselves to saddle points which are non integrable of order 1, i.e. orbitally analytically equivalent to the form

$$\begin{array}{rcl} \dot{x} & = & x \\ \dot{y} & = & -y(1 + Axy + o(|xy|). \end{array} \tag{3.5}$$

Then, under an adequate scaling for x, the holonomy of the y-axis has the form

$$f(x) = x + x^2 + o(x^2). \tag{3.6}$$

The conjugacy problem for germs of diffeomorphisms of the form (3.6) has been solved by Ecalle-Voronin ([4] and [16]). (Ecalle has also solved the conjugacy problem in the more general case of a resonant multiplier $\exp(2\pi i \frac{p}{q})$.)

Theorem 3.4 can be generalized to generic families unfolding a resonant saddle or a saddle-node.

Theorem 3.5 *(*[2] *and* [15]*) Two germs of generic families of analytic vector fields unfolding a vector field with a resonant saddle (resp. saddle-node) at the origin are analytically orbitally equivalent if and only if the families of their unfolded holonomies are conjugate.*

Hence the equivalence problem for germs of families of vector fields is reduced to the conjugacy problem for germs of families of diffeomorphisms

$$f_\epsilon(x) = x + x(x - \epsilon) + o(x^2), \tag{3.7}$$

in the saddle-case and

$$f_\epsilon(x) = x + x^2 - \epsilon + o(x^2), \tag{3.8}$$

in the saddle-node case. To solve the conjugacy problem we identify a complete modulus of analytic classification, so that two families are analytically conjugate if and only if they have the same modulus.

Before describing the modulus for the families we describe the Ecalle-Voronin modulus in the case $\epsilon = 0$. The principle is the following: Two germs of diffeomorphisms $f_1, f_2 : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ of the form

$$f(x) = x + x^2 + Ax^3 + o(x^3) \tag{3.9}$$

with same constant A (determined by the formal normal form) are conjugate if and only if they have the same orbit space. The Ecalle-Voronin modulus is one way to describe the orbit space. The orbit space is described in Figure 5: to explain its construction we first remark that the diffeomorphism is topologically like the time-one map of the vector field whose flow lines appear in Figure 5. We give ourselves a first fundamental domain limited by the curve ℓ_1 and its image $f(\ell_1)$. If we identify $x \in \ell_1$ with its image $f(x)$ the fundamental domain is conformally equivalent to a sphere S_1. The ends of the crescent limited by ℓ_1 and $f(\ell_1)$ correspond to the points 0 and ∞ on the sphere. All orbits of f (except that of 0) are represented by a most one point of the sphere. However there exists points in the neighborhood of 0 whose orbits have no representative on the sphere. To cover the orbit space we therefore need to take a second fundamental neighborhood limited by a second curve ℓ_2 and its image $f(\ell_2)$. As before we identify $x \in \ell_2$ with its image $f(x)$ and this fundamental domain is also conformally equivalent to a sphere S_2. But there exists points in the neighborhood of 0 (resp. ∞) in S_1 and S_2 which belong to the same orbit. So we need to identify a neighborhood of 0 (resp. ∞) in S_1 with a

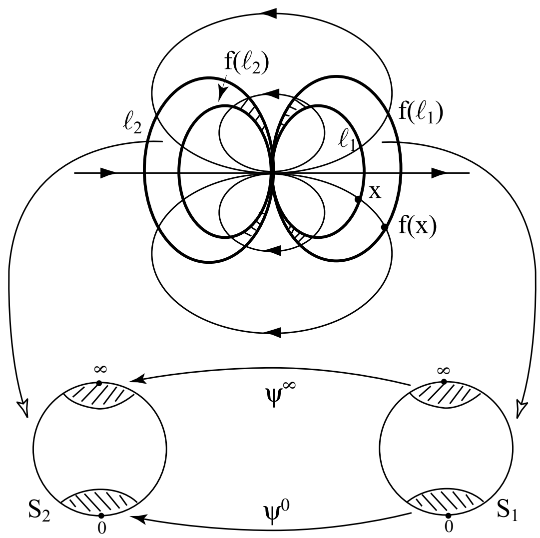


Figure 5 Orbit space of a generic diffeomorphim

neighborhood of 0 (resp. ∞) in S_2. This is done via an analytic diffeomorphism ψ^0 (resp. ψ^∞) sending 0 to 0 (resp. ∞ to ∞). The size of the neighborhoods of 0 and ∞ depend on the curves ℓ_i but what is intrinsic is the germs of analytic diffeomorphims:

$$\begin{cases} \psi^0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0) \\ \psi^\infty : (\mathbb{C}, \infty) \to (\mathbb{C}, \infty). \end{cases} \tag{3.10}$$

The only analytic changes of coordinates on S_j which preserve 0 and ∞ are the linear maps. If we choose different coordinates on S_j we get different germs $\overline{\psi}^0$ and $\overline{\psi}^\infty$. The equivalence relation corresponding to changes of coordinates on S_j preserving 0 and ∞ is

$$(\psi^0, \psi^\infty) \sim (\overline{\psi}^0, \overline{\psi}^\infty) \Longleftrightarrow \exists C_1, C_2 \begin{cases} \overline{\psi}^0(w) = C_2\psi^0(C_1 w) \\ \overline{\psi}^\infty(w) = C_2\psi^\infty(C_1 w). \end{cases} \tag{3.11}$$

Definition 3.6 The Ecalle-modulus of the diffeomorphism f is given by the tuple $(\psi^0, \psi^\infty)/\sim$.

All tuples (ψ^0, ψ^∞) are realizable as the Ecalle-Voronin modulus of a germ of diffeomorphism of the form (3.9).

Definition 3.7 The normal form of a germ of diffeomorphism of the form (3.9) is the time-one map of a vector field

$$\dot{x} = \frac{x^2}{1 + ax} \tag{3.12}$$

where $a = A - 1$. a is called the *formal invariant.*

Theorem 3.8 *([4] and [16]) For a germ of diffeomorphism of the form (3.9) there always exist a formal change of coordinate to normal form. This change of coordinate converges if and only if ψ^0 and ψ^∞ are linear.*

This theorem remains true for generic families unfolding (3.9). Considering such a family it is always possible to "prepare" the family so that the parameter becomes canonical.

Theorem 3.9 *We consider a generic germ of analytic family $f_\eta(x)$ depending on the parameter η unfolding the germ $f(x)$ given in (3.9) (the family is generic if $\frac{\partial f_\eta}{\partial \eta} \neq 0$). There exists a germ of analytic diffeomorphism $(x,\eta) \mapsto (y,\epsilon)$ such that in the coordinate y the diffeomorphism (3.9) becomes*

$$\overline{f}_\epsilon(y) = y + (y^2-\epsilon)(1 + B(\epsilon) + (y^2-\epsilon)h_\epsilon(y)), \tag{3.13}$$

where

- *The fixed points of $\overline{f}$ are given by $y_\pm = \pm\sqrt{\epsilon}$.*
- *The multipliers $\lambda_\pm$ of $y_\pm$ satisfy*

$$\frac{1}{\ln\lambda_+} - \frac{1}{\ln\lambda_-} = \frac{1}{\sqrt{\epsilon}}. \tag{3.14}$$

Moreover

$$a(\epsilon) = \frac{1}{\ln\lambda_+} + \frac{1}{\ln\lambda_-}, \tag{3.15}$$

so $a(\epsilon)$ is a "shift" between the two singular points.

Definition 3.10 The family (3.13) of Theorem 3.9 is called prepared. Its parameter ϵ is called the canonical parameter.

We now consider the conjugacy problem for two prepared families. Then necessarily the canonical parameter must be preserved as it is an analytic invariant of the family.

Theorem 3.11 [11] *Two germs of generic analytic famlies of diffeomorphisms*

$$f_\epsilon(x) = x + (x^2-\epsilon) + o(x^2) \tag{3.16}$$

are analytically conjugate if and only if they have the same unfolded Ecalle-Voronin modulus $((\psi^0_{\hat\epsilon}, \psi^\infty_{\hat\epsilon})/\sim)\hat\epsilon \in V$, where

- *V is a sectorial neighborhood of the origin in the universal covering of ϵ-space punctured at the origin. The radius $r(\delta)$ of V depends on its opening defined with the help of an arbitrarily small $\delta > 0$:*

$$V = \{\hat\epsilon; |\hat\epsilon| < r(\delta), arg(\hat\epsilon) \in (-\pi+\delta, 3\pi-\delta)\}. \tag{3.17}$$

- *The fundamental neighborhoods unfold as in Figure 6. As before we glue together the two curves ℓ_j and $f_\epsilon(\ell_j)$ limiting the fundamental neighborhoods, thus yielding domains which have the conformal structure of spheres $S_{j,\epsilon}$, $j = 1, 2$, the distinguished point 0 (resp. ∞) corresponding to the fixed point $-\sqrt{\epsilon}$ (resp. $\sqrt{\epsilon}$).*
- $$\begin{cases} \psi^0_{\hat\epsilon} : (\mathbb{C},0) \to (\mathbb{C},0) \\ \psi^\infty_{\hat\epsilon} : (\mathbb{C},\infty) \to (\mathbb{C},\infty). \end{cases} \tag{3.18}$$

  are germs of analytic diffeomorphisms depending analytically on $\hat\epsilon \neq 0$ and continuously on $\hat\epsilon$ near $\hat\epsilon = 0$.

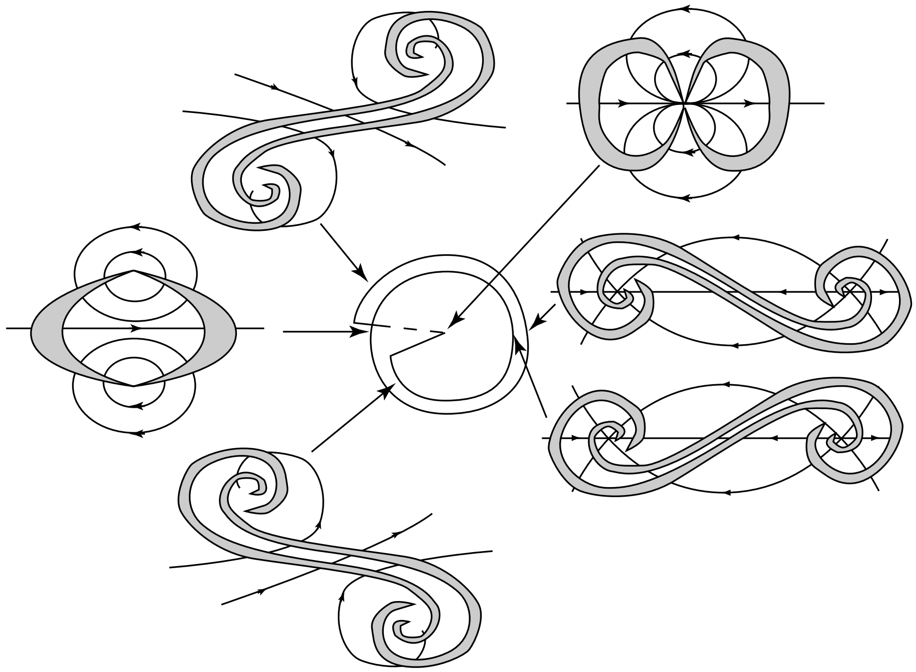


Figure 6 Orbit space of a generic family of diffeomorphims

Theorem 3.12 [15] *A germ of generic analytic family unfolding a saddle-node is analytically orbitally equivalent to a prepared family*

$$\begin{array}{rcl} \dot{x} & = & x^2 - \epsilon \\ \dot{y} & = & g_0(x)(x^2-\epsilon) + y(1+a(\epsilon)x) + O(y^2). \end{array} \tag{3.19}$$

The modulus $((\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}})/\sim)_{\hat{\epsilon}\in V}$ *for the unfolded holonomy of the* y*-separatrix with prepared form* (3.13) *is such that* $\psi^\infty_{\hat{\epsilon}}$ *is an affine transformation.*

Open questions:

1. Identify precisely the modulus space of $((\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}})/\sim)_{\hat{\epsilon}\in V}$ which are realizable as moduli of families of diffeomorphims. It is known that all (ψ^0, ψ^∞) are realizable for a single diffeomorphim ($\epsilon = 0$), see[4] and [16]. The difficulty is to identify precisely the dependence on $\hat{\epsilon}$ near $\hat{\epsilon} = 0$. The $(\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}})$ surely depend more than continuously on $\hat{\epsilon}$.
2. Derive similar theorems for the higher codimension case. The existence of fundamental domains for all values of the parameters has been done by Oudkerk ([12] and later work). It remains to organize them nicely in the parameter space. As appearing in Figure 6 there can be different non equivalent choices of fundamental domains to describe the orbit space for a single value of ϵ.

4 Changes of coordinates to normal form

We discuss the case of the generic saddle-node family (3.19).

Theorem 4.1 [15] *There exists a change of coordinate*

$$Y = y + \sum_{j=0}^{\infty} y^j h_j(\hat{\epsilon}, x) \tag{4.1}$$

defined on a domain $U \times D \times V$ where

- *U is a domain in the universal covering of x-space ramified at $\pm\sqrt{\epsilon}$ as in Figure 7,*
- *D is a neighborhood of 0 in y-space,*
- *V is a sectorial neighborhood of $\hat{\epsilon} = 0$ as in (3.17),*

bringing the family (3.19) to the model family

$$\begin{array}{rcl} \dot{x} & = & x^2 - \epsilon \\ \dot{Y} & = & Y(1 + a(\epsilon)x). \end{array} \tag{4.2}$$

The model family has a first integral

$$H(x, Y) = Y k_a(x, \epsilon), \tag{4.3}$$

with

$$k_a(x, \epsilon) = (x - \sqrt{\epsilon})^{-\frac{1+a\sqrt{\epsilon}}{2\sqrt{\epsilon}}} (x + \sqrt{\epsilon})^{\frac{1-a\sqrt{\epsilon}}{2\sqrt{\epsilon}}}. \tag{4.4}$$

This yields a first integral

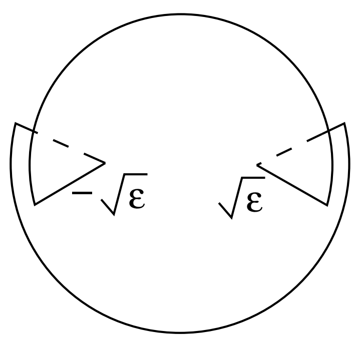


Figure 7 Domain U in x-space

$$\overline{H}_{\hat{\epsilon}}(x, y) \tag{4.5}$$

for (3.19) which is ramified at $x = \pm\sqrt{\epsilon}$. If we turn around $\sqrt{\epsilon}$ we get two branches $\overline{H}_{1,\hat{\epsilon}}$ and $\overline{H}_{2,\hat{\epsilon}}$ related in the following way

$$\overline{H}_{2,\hat{\epsilon}} = L(a, \hat{\epsilon}) \circ \psi^{\infty}_{\hat{\epsilon}}(\overline{H}_{1,\hat{\epsilon}}) \tag{4.6}$$

where $L(a, \hat{\epsilon})$ is a linear map. Similarly if we turn around $-\sqrt{\epsilon}$ we get two branches $\tilde{H}_{1,\hat{\epsilon}}$ and $\tilde{H}_{2,\hat{\epsilon}}$ related in the following way

$$\tilde{H}_{2,\hat{\epsilon}} = L(a, \hat{\epsilon}) \circ \psi^{0}_{\hat{\epsilon}}(\tilde{H}_{1,\hat{\epsilon}}). \tag{4.7}$$

5 Applications to problems of finite cyclicity of graphics

Definition 5.1 A graphic of a vector field is a union of singular points and characteristic trajectories joining them which is likely to produce limit cycles or periodic trajectories under perturbation.

Definition 5.2 A graphic Γ of a vector field v_0 has *finite cyclicity inside a family* v_λ unfolding v_0 (where λ is a multi-parameter) if there exists $N \in \mathbb{N}$, there exists $\epsilon > 0, \delta > 0$ such that, for any λ with $|\lambda| < \delta$, the vector field v_λ has at most N limit cycles $\gamma_1, \dots, \gamma_n$, $n \leq N$ such that $dist_H(\gamma_i, \Gamma) < \epsilon$, where $dist_H$ is the Hausdorff distance between compact sets. The graphic Γ has *finite absolute cyclicity* (or simply *finite cyclicity*) if N can be chosen independent of the C^∞ unfolding v_λ.

5.1 The lips. The lips is a continuum of graphics as in Figure 8 with two saddle-nodes and central transition through them. Each graphic is likely to create periodic solutions or limit cycles in a perturbation where the singular points disappear. In the finite cyclicity questions we are interested to give a bound on the number of limit cycles which can be created from a single limit periodic set in the bifurcation process.

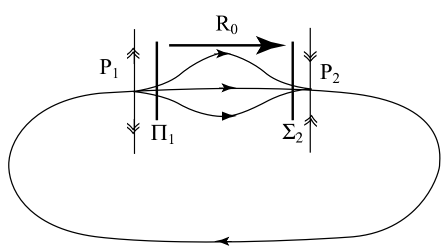


Figure 8 The lips

Theorem 5.3 [8] *We consider a C^∞ family of vector fields v_η having for v_0 lips as in Figure* (8)*. We consider C^k-normalizing changes of coordinates bringing the vector field in the neighborhood of P_1 and P_2 to the respective normal forms*

$$\begin{cases} \dot{x}_1 = x_1^2 - \epsilon_1 \\ \dot{y}_1 = y_1(1 + a_1 x_1) \end{cases} \qquad \begin{cases} \dot{x}_2 = x_2^2 - \epsilon_2 \\ \dot{y}_2 = -y_2(1 + a_2 x_2), \end{cases} \tag{5.1}$$

and sections parallel to the y_j-axes in the normalizing charts. We consider a graphic intersecting Π_1 in $y_{1,0}$. If the regular transition $R_0 : \Pi_1 \to \Sigma_2$ satisfies $R_0^{(n)}(y_{1,0}) \neq 0$, where $1 < n < k$, then the corresponding graphic has cyclicity less than or equal to n. Moreover there exists a perturbation of v_0 with exactly n limit cycles.

Proof The proof is easy. We consider sections Σ_i and Π_i, $i = 1, 2$, as in Figure 9. Limit cycles are given by zeros of the displacement map

$$V_\eta(y_1) = R_\eta(y_1) - D_2^{-1} \circ S_\eta^{-1} \circ D_1^{-1}(y_1). \tag{5.2}$$

As the system is integrable in the neighborhood of P_1 and P_2 this allows to calculate the Dulac maps $D_j : \Sigma_j \to \Pi_j$. These are linear maps of the form $D_1(y_1) = M(\epsilon_1) y_1$ with $\lim_{\epsilon_1 \to 0} M(\epsilon_1) = +\infty$ and $D_2(y_2) = m(\epsilon_2)$ with $\lim_{\epsilon_2 \to 0} m(\epsilon_2) = 0$.

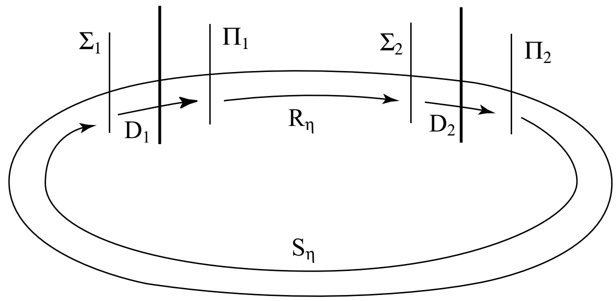


Figure 9 Transition maps for the lips

We exploit the freedom on the choice of normalizing coordinates to simplify the map S_η and bring it to a mere affine map

$$S_\eta(y_2) = A(\eta)y_2 + B(\eta), \tag{5.3}$$

with $A(\eta) > 0$ and $B(0) = 0$. Then $D_2^{-1} \circ S_\eta^{-1} \circ D_1^{-1}$ is affine. Hence

$$V_\eta^{(n)}(y_1) = R_\eta^{(n)}(y_1) \neq 0 \tag{5.4}$$

for (y_1, η) in a small neighborhood of $(0,0)$, which determines a neighborhood U_1 of the origin on the section Π_1. By Rolle's theorem there are at most n periodic solutions intersecting Π_1 on U_1. □

Remark 5.4 1. The proof illustrates the power of a good "preparation" to deal with problems of finite cyclicity. Indeed composing a Dulac map with $M(\epsilon_1)$ very large with one with $m(\epsilon_2)$ very small yields indeterminacy. The trick of transforming S_η to an affine map allows to see that the indeterminacy is limited to constant and linear terms.

2. The theorem may seem completely useless in practice for C^∞ vector fields because it is impossible in practice to check the hypothesis. Fortunately this is not so for analytic vector fields and we will see that there are a number of cases where the hypothesis can be checked without nearly any calculation.

Theorem 5.5 [3] *We consider an analytic family of vector fields with a saddle-node at the origin. It is possible to choose C^k normalizing changes of coordinates bringing the family to the normal form*

$$\begin{array}{rcl} \dot{X} & = & X^2 - \epsilon \\ \dot{Y} & = & Y(1 + a(\epsilon)X) \end{array} \tag{5.5}$$

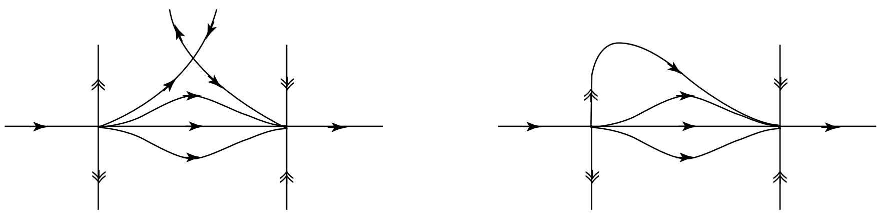
in such a way that there exists $X_0 > 0$ such that, for all values of the parameters, the sections $X = \pm X_0$ are analytic and parameterized by analytic coordinates.

Proof The proof uses the case $\epsilon = 0$ of Theorem 4.1. This particular case of Theorem 4.1 was proved in [3], with an additional step to prove that the change of coordinates can be taken real. □

Theorem 5.6 [3] *We consider an analytic family of vector fields v_η having for v_0 lips as in Figure 8.*

1. *If R_0 is nonlinear at one point $y_{1,0}$ it is nonlinear everywhere.*

2. *Each graphic of the lips has finite cyclicity if for instance one of the following conditions is satisfied*
   - *One of the graphics has an additional saddle point with ratio of eigenvalues different of -1 (Figure 10(a)).*



(a) (b)

Figure 10 Two types of bordering graphics

   - *The family of graphics ends in a graphic entering one saddle-node through the strong manifold and the other saddle-node through a center manifold (Figure 10(b)).*

Proof 1. is just analytic extension principle.

2. • Let $\lambda_1 < 0$ and $\lambda_2 > 0$ be the eigenvalues of the saddle point and $r = -\lambda_1/\lambda_2$. Let $y = y_1 - y_{1,0}$, where $y_{1,0}$ corresponds to the graphic through the saddle point. The transition map has the form $R_0(y) = y^r(C + O(y))$ with $C > 0$ in the neighborhood of the graphic through the saddle. This map is obviously nonlinear as soon as $r \neq 1$.
   • No affine map can send a semi-infinite domain on a finite one. Hence the map R_0 is non affine.

□

Remark 5.7 1. These results only use the modulus of the vector field ($\epsilon = 0$) and not the modulus of the family. This is because of the genericity condition $R_0^{(n)}(0) \neq 0$ for $n > 1$.

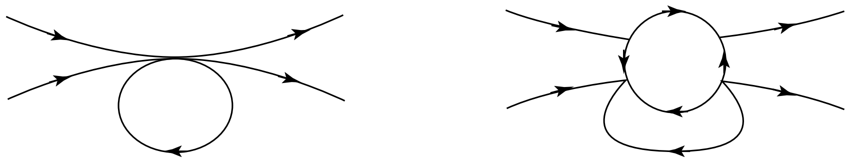
2. We have seen that the method of Theorem 5.6 is extremely powerful, since nearly no calculations are needed. The price to pay however is that we just get results of finite cyclicity but cannot determine the exact cyclicity.

In the non generic case where R_0 is linear, then we can hope for a full result of finite cyclicity for analytic families depending on a finite number of parameters and unfolding a vector field with lips. The tool would be an improved Theorem 4.1 where we would have a better control on the dependence on the parameter in the neighborhood of $\hat{\epsilon} = 0$.

5.2 The hp-graphic through a nilpotent elliptic point. We consider a nilpotent elliptic point of multiplicity 3 inside a C^∞ family of vector fields. Such a point has a 3-jet C^∞ orbitally equivalent to

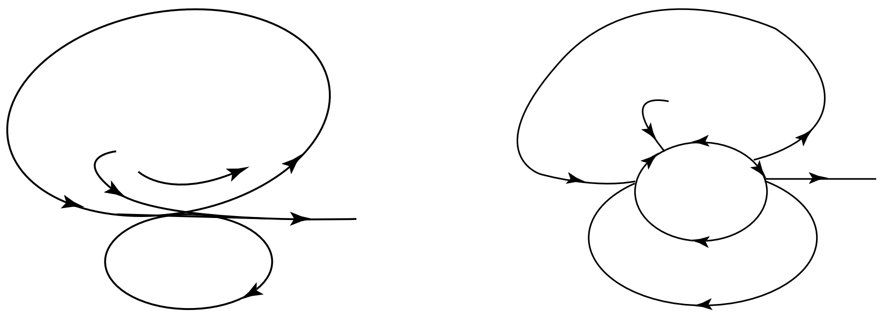
$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x^3 + y(bx + o(x)) + y^2 h(x,y) \end{aligned} \tag{5.6}$$

with $b > 2\sqrt{2}$. Such a point has a phase portrait as in Figure 11.



(a) The nilpotent point (b) Its (weighted) blow-up

Figure 11 A nilpotent elliptic point of multiplicity 3



(a) (b) with the blow-up of the nilpotent point

Figure 12 An hp-graphic

We consider an hp-graphic through such a point, i.e. a connection between a separatrix of a hyperbolic sector and a characteristic curve of a parabolic sector (Figure 12). The following theorem proved in [18] made an essential use of Theorem 5.5.

Theorem 5.8 *We consider an hp-graphic through a nilpotent elliptic point of multiplicity 3 in a C^∞ vector field v_0. There exists $N \in \mathbb{N}$ such that the graphic has finite cyclicty inside any C^∞ family of vector fields v_λ unfolding v_0.*

6 Perspectives

Sections 2 and 3 illustrate that the divergence of the normalizing series reflects that the dynamics of the system is much more complicated than that of the normal form. We have also seen that embedding a vector field in a family so as to unfold the situation and extending the phase variables to the complex domain allows to give a geometric explanation of the divergence and why the divergence is so often the rule.

Here we have only discussed the divergence of normalizing series in the resonant cases: the divergence can essentially be explained as the incompatibility between a finite number of objects: for instance two fixed points in the case of a generic diffeomorphism tangent to the identity, one fixed point and one periodic point of period q in the case of a generic diffeomorphism with multiplier $\exp(2\pi i \frac{p}{q})$, one fixed point and one periodic orbit in the case of a generic Hopf bifurcation, one saddle and one node in the case of a saddle-node. There is another source of divergence coming from the "small denominators". This occurs for instance in the case of a diffeomorphism

$$f(z) = \exp(2\pi i\alpha)z + o(z) \tag{6.1}$$

where α is irrational. It is also the case of a saddle point for which the quotient of eigenvalues is irrational. In both cases the system is formally linearizable but the linearizing change of coordinates is generically divergent when α (in the case of a diffeomorphism) or the quotient of eigenvalues (in the case of a saddle) is Liouvillian. Here again an explanation is suggested by unfolding. For instance if we embed the diffeomorphism (6.1) inside a family

$$f_\epsilon(z) = \exp(2\pi i(\alpha + \epsilon))z + o(z) \tag{6.2}$$

then we generically have the birth of a periodic orbit as soon as $\alpha+\epsilon = \frac{p}{q} \in \mathbb{Q}$. If α is Liouvillian (i.e. very well approximated by the rationals) then an infinite number of periodic orbits cannot escape from a neighborhood of the origin sufficiently rapidly and form an obstruction to linerizability. This was conjectured by Arnold and Yoccoz showed that it indeed occurs [17]. Pérez-Marco showed that there are also other kinds of obstructions to linearizability [13].

One interesting direction of research is to put together the two approaches: indeed small divisors phenomena occur in the unfolding of a resonant situation. The quadratic family

$$f(z) = \exp(2\pi i\alpha)z + z^2 \tag{6.3}$$

is a family for which the change of coordinate to normal form in the neighborhood of the origin diverges for all $\alpha \in \mathbb{R}$.

A second direction of research is to make a systematic study of the meaning of the divergence of the normalizing series for more complicated singular points.

In Section 5 we have shown the power of good normal forms in some applications to finite cyclicity problems. The results on normal forms presented here are still partial as the dependence on the parameters is not enough precise. For instance in the conjugacy problem for a germ of analytic family unfolding a germ of diffeomorphism (3.6) we are still missing a realization theorem describing exactly the modulus space, i.e. identifying precisely which pairs of germs $(\psi^0_{\hat\epsilon}, \psi^\infty_{\hat\epsilon})_{\hat\epsilon\in V}$ can be realized as the modulus of a germ of family. This problem is probably very difficult. However its solution would open new perspectives in finite cyclicity problems.

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