MODULI SPACE OF UNFOLDED DIFFERENTIAL LINEAR SYSTEMS WITH AN IRREGULAR SINGULARITY OF POINCARÉ RANK 1

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Abstract. In the paper [8], we have identified the moduli space of generic unfoldings of linear differential systems with a nonresonant irregular singularity of Poincaré rank 1 for classification under analytic equivalence. The modulus of the unfolding of a linear differential system is the unfolding of the modulus of the system. It consists in formal invariants and an unfolding of the Stokes matrices. In the realization part, we have identified the realizable moduli. However, the necessary and sufficient condition for realizing unfoldings of Stokes matrices was quite obscure. In this paper we explore this condition and we determine the realizable moduli depending analytically on the parameter in dimensions 2 and 3. In dimension 2, all realizable unfoldings of Stokes matrices can be chosen depending analytically on the parameter. In dimension 3, not all pairs of Stokes matrices have realizable analytic unfoldings.

1. Introduction

The singularities of linear differential systems play a central role in mathematics and mathematical physics. The simplest irregular singularity is the nonresonant irregular singularity of Poincaré rank 1 which occurs at $x = 0$ in a system of the form

$$y' = \frac{A(x)}{x^2} y, \tag{1}$$

with $x \in (\mathbb{C}, 0)$, $y \in \mathbb{C}^n$, and A a matrix of germs of analytic functions in x at the origin such that $A(0)$ has distinct eigenvalues. Many mathematicians have contributed to the analytic classification of such systems and to the identification of the moduli space. A final statement can be found in [1], and a proof of the realization of an admissible modulus in [10].

The irregular singularities of Poincaré rank 1 are double singular points. It is hence natural to interpret the modulus of the system (1) as the limit, when $\epsilon \to 0$, of a modulus of analytic classification for

$$y' = \frac{A(x, \epsilon)}{x^2 - \epsilon} y, \tag{2}$$

on a neighborhood $\mathbb{D}_r$ containing the two regular singularities (which are indeed Fuchsian singularities). This was conjectured by Bolibruch, Arnold and Ramis (with slightly different statements). Particular cases and related questions were

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studied by Ramis [7], Duval ([4] and [3]), Zhang ([13] and [12]) and Schäfke [9]. Glutsyuk [5] studied the general case, but he limited himself to a sectorial domain S_G of the origin in the parameter space. For parameter values in S_G, the difficult case of resonant regular singularities is avoided. The method of Glutsyuk is then to compare two bases of solutions which are eigenvectors of the monodromy near each singular point, and which we will call *eigensolutions*. Together with the eigenvalues at each singular point, the comparison between the bases of eigensolutions at each singular point yields a modulus of analytic classification of (2) for a fixed value of $\epsilon \in S_G$. In order to be able to cover a full neighborhood of the origin in the parameter space, we have used a completely different approach. We have covered $\mathbb{D}_r$, punctured at the two singular points, with two sectors adherent to the singular points as in Figure 1. Over each such sector, we have constructed an

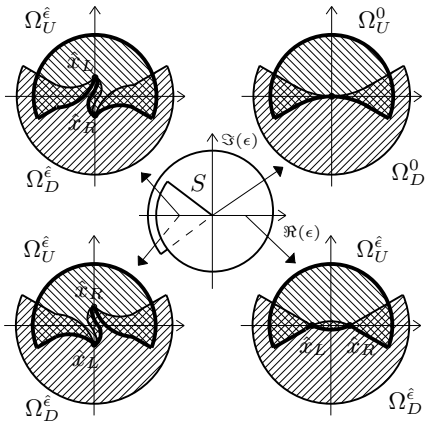

FIGURE 1. Sectorial domains in x, covering $\mathbb{D}_r$, depending on $\hat\epsilon \in S \cup \{0\}$ (we take, when $\hat\epsilon \in S$, $\arg(\hat\epsilon) \in (\pi - \gamma, 3\pi + \gamma)$ for a well-chosen $\gamma > 0$).

(almost) unique basis of solutions with given simultaneous asymptotic behavior at the singular points. The formal part of the modulus is obtained from the coefficients in a formal normal form (for a fixed value of the parameter such that the two Fuchsian singular points are nonresonant, it corresponds to the collection of the eigenvalues of the residue matrices at the two singular points). The analytic part of the modulus consists in unfolded Stokes matrices obtained from the comparison of the bases of solutions on the left and right connected components of the intersection of the two sectors. However, the price to pay to cover all parameter values in a neighborhood of the origin is that our description is ramified in the parameter ϵ and uses the parameter $\hat\epsilon$ lying in a sector S of opening greater than 2π of the universal covering of ϵ-space punctured at the origin. In particular, we get two different complete systems of invariants for the same differential system on $S_\cap$, the auto-intersection part of the projection of S in the ϵ-space. When considering the modulus space, a minimal necessary condition for realizing a family of unfolded Stokes matrices and formal invariants is that they describe families that are analytically equivalent (over $S_\cap \times \mathbb{D}_r$). We have shown in [8] that this condition is (essentially) also sufficient. But, this condition is difficult to state and quite obscure. Indeed, when $n = 2$, it forces the existence of representatives of the unfolded

Stokes matrices depending analytically on the parameter. In particular, constant Stokes matrices are realizable when $n = 2$. This paper explores the case $n \geq 2$ with great emphasis on the case $n = 3$. The necessary and sufficient condition gives the equality of two matrix invariants up to left and right multiplication by adequate invertible diagonal matrices. The vanishing of an element of these matrix invariants has an intrinsic geometric meaning. We first get rid of the diagonal matrices and give absolute invariants. In the generic case, there are $(n-1)^2$ such invariants. In the case $n = 3$, we then explore if it is possible to realize Stokes matrices depending analytically on ϵ and we see that this is possible only very exceptionally. But, to our surprise, moduli composed of Stokes matrices depending analytically on ϵ can occur for systems that are really 3-dimensional and cannot be decomposed in direct products of lower dimensional systems. Moreover, this can already be seen when the parameter vanishes: not all Stokes matrices have realizable analytic unfoldings.

2. Preliminaries

Let us consider the generic unfoldings of the linear differential systems (1). After linear change in y and affine transformation in x, we can assume that $A(0)$ is diagonal with eigenvalues $\lambda_{1,0}, ..., \lambda_{n,0}$ such that

$$\Re(\lambda_{q,0} - \lambda_{j,0}) > 0, \quad q < j, \tag{3}$$

and that the generic unfoldings of (1) are written as

$$(x^2 - \epsilon)y' = A(\epsilon, x)y, \tag{4}$$

with $\epsilon \in \mathbb{C}$ and A a matrix of germs of analytic functions in (ϵ, x) at the origin satisfying $A(0, x) = A(x)$. We have shown in [8] that (4) is locally analytically equivalent (see Definition 2.1) to a prenormal form

$$(x^2 - \epsilon)y' = \big(\Lambda_0(\epsilon) + \Lambda_1(\epsilon)x + (x^2 - \epsilon)R(\epsilon, x)\big)\, y, \tag{5}$$

where R is a matrix of germs of analytic functions in (ϵ, x) at the origin, and where Λ_0 and Λ_1 are diagonal matrices of germs of analytic functions in ϵ at the origin, which are the formal invariants of the system (note that $\Lambda_0(0) = A(0)$).

We classify the systems (5) (and hence (4)) under the following equivalence relations:

Definition 2.1. Two systems $(x^2 - \epsilon)y' = A(\epsilon, x)y$ and $(x^2 - \epsilon)z' = B(\epsilon, x)z$, with $A(\epsilon, x)$ and $B(\epsilon, x)$ matrices of germs of analytic functions at the origin, are locally *analytically equivalent* (respectively *formally equivalent*) if there exists an invertible matrix P of germs of analytic functions of (ϵ, x) at the origin (respectively an invertible matrix of formal series in (ϵ, x)) such that the substitution $y = P(\epsilon, x)z$ transforms one system into the other.

The complete system of formal invariants is given by the polynomial part of degree 1 in the prenormal form (5), hence it is totally described by a system

$$(x^2 - \epsilon)z' = (\Lambda_0(\epsilon) + \Lambda_1(\epsilon)x)\, z \tag{6}$$

that we call the *model system*. Its diagonal fundamental matrix of solutions is

$$F(\epsilon, x) = \begin{cases} (x - x_R)^{\frac{1}{2x_R}\Lambda_0(\epsilon) + \frac{1}{2}\Lambda_1(\epsilon)} (x - x_L)^{\frac{1}{2x_L}\Lambda_0(\epsilon) + \frac{1}{2}\Lambda_1(\epsilon)}, & \epsilon \neq 0, \\ x^{\Lambda_1(0)} \exp\left(-\frac{\Lambda_0(0)}{x}\right), & \epsilon = 0, \end{cases} \tag{7}$$

where

$$x_L = \sqrt{\epsilon}, \quad x_R = -\sqrt{\epsilon} \tag{8}$$

(in S illustrated in Figure 1, since we take $\arg(\hat{\epsilon}) \in (\pi - \gamma, 3\pi + \gamma)$ for a well-chosen $\gamma > 0$, the indices in (8), meaning left and right, make sense when $\sqrt{\hat{\epsilon}}$ is real, hence negative for $\hat{\epsilon} \in S$).

Notation 2.2. We denote by $S_\cap$ the auto-intersection of the domain S of the parameter (see Figure 2). If $\bar{\epsilon} \in S$ and $\bar{\epsilon}e^{2\pi i} \in S$, then we write $\bar{\epsilon} \in S_\cap$ and $\tilde{\epsilon} \in S_\cap$, where

$$\tilde{\epsilon} = \bar{\epsilon}e^{2\pi i}. \tag{9}$$

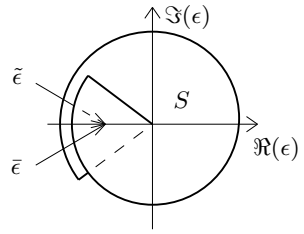

FIGURE 2. Example of values of $\bar{\epsilon}$ and $\tilde{\epsilon} = \bar{\epsilon}e^{2\pi i}$ in $S_\cap$.

Notation 2.3. We write the hat symbol over quantities that depend on $\hat{\epsilon} \in S$ (for example $\hat{x}_L$). We write the symbols $\tilde{}$ (respectively $\bar{}$) over quantities that depend on $\tilde{\epsilon} \in S_\cap$ (respectively $\bar{\epsilon} \in S_\cap$). When we use the hat symbol for values of the parameter in $S_\cap$, we mean that $\hat{\epsilon}$ could either be $\bar{\epsilon}$ or $\tilde{\epsilon}$.

To define the analytic part of the modulus of a system (5), we construct the open domains $\cup_{\hat{\epsilon}\in S}\{\hat{\epsilon}\} \times \Omega_D^{\hat{\epsilon}}$ (respectively $\cup_{\hat{\epsilon}\in S}\{\hat{\epsilon}\} \times \Omega_U^{\hat{\epsilon}}$) illustrated in Figure 1.

Remark 2.4. *The way we construct the sectorial domains $\Omega_D^{\hat{\epsilon}}$ and $\Omega_U^{\hat{\epsilon}}$ ensures that, near the singular points, the asymptotic behavior of the solutions of the model system is similar to the one at $\epsilon = 0$:*

$$\lim_{\substack{x \to \hat{x}_l \\ x \in \Omega_D^{\hat{\epsilon}} \cap \Omega_U^{\hat{\epsilon}}}} \frac{(F(\epsilon, x))_{jj}}{(F(\epsilon, x))_{qq}} = 0, \quad \textit{for} \begin{cases} q > j, & \textit{if } l = R, \\ q < j, & \textit{if } l = L. \end{cases} \tag{10}$$

The following theorem was proved in [8].

Theorem 2.5. [8] *On $\cup_{\hat{\epsilon}\in S}\{\hat{\epsilon}\} \times \Omega_D^{\hat{\epsilon}}$ (respectively $\cup_{\hat{\epsilon}\in S}\{\hat{\epsilon}\} \times \Omega_U^{\hat{\epsilon}}$), there exist invertible transformations $y = H_D(\hat{\epsilon}, x)z$ (respectively $y = H_U(\hat{\epsilon}, x)z$) from (5) to its model (6), where $H_D(\hat{\epsilon}, x)$ and $H_U(\hat{\epsilon}, x)$ are invertible matrices that are unique up to right multiplication with a diagonal matrix $\hat{K}$ satisfying the conditions in Definition 2.12.*

Sketch of proof. We give this sketch of proof for H_U to explain the important underlying idea of the construction of the fundamental matrix of solutions $H_U F$ over

Ω_U. The vector space V of solutions over Ω_U has a flag of invariant subspaces given by the asymptotic behavior of solutions at $\hat{x}_R$:

$$V_R^1 \subset V_R^2 \subset \cdots \subset V_R^n.$$

It has a second flag of invariant subspaces given by the asymptotic behavior of solutions at $\hat{x}_L$:

$$V_L^n \subset V_L^{n-1} \subset \cdots \subset V_L^1.$$

For small $\hat{\epsilon}$, these flags are transversal (because they are transversal at the limit $\epsilon = 0$), and each intersection $V_R^j \cap V_L^{n-j+1}$, $j = 1, \ldots, n$ is one-dimensional. The columns of $H_U F$ are bases of each of these subspaces. In particular, the first column is an eigensolution at $\hat{x}_R$ and the last column is an eigensolution at $\hat{x}_L$. □

Remark 2.6. *We construct the transformations $H_D(\hat{\epsilon}, x)$ and $H_U(\hat{\epsilon}, x)$ so they have the same bounded and invertible limit when $x \to \hat{x}_l$, for $l = L, R$.*

Over each connected component of the intersection of $\Omega_D^{\hat{\epsilon}}$ and $\Omega_U^{\hat{\epsilon}}$ (Figure 3), $H_D(\hat{\epsilon}, x)^{-1} H_U(\hat{\epsilon}, x)$ are automorphisms of the model system acting on a fundamental matrix of solutions. These automorphisms lead to the existence of matrices $\hat{C}_R$ and $\hat{C}_L$ satisfying

$$H_D(\hat{\epsilon}, x)^{-1} H_U(\hat{\epsilon}, x) = \begin{cases} F_D(\hat{\epsilon}, x) \hat{C}_R (F_D(\hat{\epsilon}, x))^{-1}, & \text{on } \Omega_R^{\hat{\epsilon}}, \\ F_D(\hat{\epsilon}, x) \hat{C}_L (F_D(\hat{\epsilon}, x))^{-1}, & \text{on } \Omega_L^{\hat{\epsilon}}, \\ I, & \text{on } \Omega_C^{\hat{\epsilon}}, \end{cases} \tag{11}$$

where $F_D(\hat{\epsilon}, x)$ is the restriction to $\Omega_D^{\hat{\epsilon}}$ of the fundamental matrix of solutions (7) of the model system. The matrices $\hat{C}_R$ and $\hat{C}_L$, that we call the *unfolded Stokes matrices*, are respectively an upper triangular and a lower triangular unipotent matrix (this follows from the proof of Theorem 2.5 and Remarks 2.4 and 2.6). They depend analytically on $\hat{\epsilon} \in S$ and converge when $\hat{\epsilon} \to 0$ ($\hat{\epsilon} \in S$) to the Stokes matrices (see [6], pp.351-372, for the case $\epsilon = 0$).

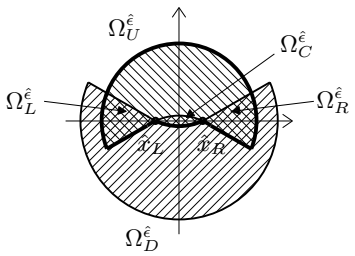

FIGURE 3. The connected components of the intersection of the sectorial domains $\Omega_D^{\hat{\epsilon}}$ and $\Omega_U^{\hat{\epsilon}}$, case $\hat{x}_L = \sqrt{\hat{\epsilon}} \in \mathbb{R}_-^*$.

The product $W = H_D(\hat{\epsilon}, x) F_D(\hat{\epsilon}, x)$ is a fundamental matrix of solutions of (5) on $\Omega_D^{\hat{\epsilon}}$ that can be analytically continuated (through $\Omega_C^{\hat{\epsilon}}$) to the ramified domain $V^{\hat{\epsilon}} = \Omega_D^{\hat{\epsilon}} \cup \Omega_U^{\hat{\epsilon}}$ illustrated in Figure 4 (which could have a spiraling part around $\hat{x}_R$ and $\hat{x}_L$, as in Figure 1).

To explain how the transition matrices between branches of W depend on the Stokes matrices, let us consider the monodromy operators $\hat{M}_L$ and $\hat{M}_R$ around

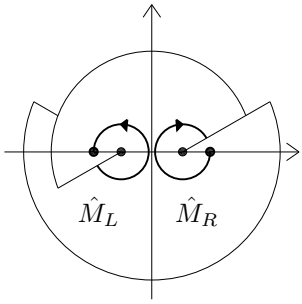

FIGURE 4. Illustration of the definition of the monodromy operators $\hat{M}_L$ and $\hat{M}_R$ on the ramified domain $V^{\hat{\epsilon}} = \Omega_D^{\hat{\epsilon}} \cup \Omega_U^{\hat{\epsilon}}$, case $\hat{x}_L = \sqrt{\hat{\epsilon}} \in \mathbb{R}_-^*$.

the singular points as illustrated in Figure 4 (i.e. in the positive direction around $x = \hat{x}_L$ and in the negative direction around $x = \hat{x}_R$).

Lemma 2.7. *The monodromy matrices representing the action of $\hat{M}_l$ on the fundamental matrix of solutions $F(\epsilon, x)$ of the model system are*

$$\hat{D}_R = e^{-2\pi i\left(\frac{-1}{2\sqrt{\hat{\epsilon}}}\Lambda_0(\epsilon)+\frac{1}{2}\Lambda_1(\epsilon)\right)} \qquad \text{and} \qquad \hat{D}_L = e^{2\pi i\left(\frac{1}{2\sqrt{\hat{\epsilon}}}\Lambda_0(\epsilon)+\frac{1}{2}\Lambda_1(\epsilon)\right)}. \tag{12}$$

Proposition 2.8. *For $l = L, R$, the monodromy matrix representing the action of $\hat{M}_l$ on the fundamental matrix of solutions W is $\hat{C}_l\hat{D}_l$.*

Proof. Starting on $\Omega_R^{\hat{\epsilon}}$, the operator $\hat{M}_R$ acting on W gives $H_U(\hat{\epsilon}, x)F_D(\hat{\epsilon}, x)\hat{D}_R$. Starting on $\Omega_L^{\hat{\epsilon}}$, the operator $\hat{M}_L$ acting on W gives $H_U(\hat{\epsilon}, x)F_D(\hat{\epsilon}, x)\hat{D}_L$. The proposition follows from (11). □

Definition 2.9. We call an *eigensolution at $x = \hat{x}_l$* a solution that is an eigenvector of the monodromy operator around the singular point $x = \hat{x}_l$.

When the singular points are nonresonant, the fundamental matrix of solutions W has the property that its j^{th} column is a linear combination of j eigensolutions at $x = \hat{x}_R$, or $n - j + 1$ eigensolutions at $x = \hat{x}_L$, with coefficients depending on the entries of the unfolded Stokes matrices and the formal invariants, as detailed by Definition 2.10 and Proposition 2.11.

Definition 2.10. For values of $\hat{\epsilon}$ such that the diagonal entries of $\hat{D}_l$ are distinct (in particular for $\hat{\epsilon} \in S_\cap$), we define $\hat{T}_l$ as the unipotent triangular matrix diagonalizing the matrix $\hat{C}_l\hat{D}_l$:

$$(\hat{T}_l)^{-1}\hat{C}_l\hat{D}_l\hat{T}_l = \hat{D}_l \quad l = L, R, \tag{13}$$

($\hat{T}_L$ is lower triangular while $\hat{T}_R$ is upper triangular).

Proposition 2.11. *For values of $\hat{\epsilon}$ such that the diagonal entries of $\hat{D}_l$ are distinct, the columns of the fundamental matrix of solutions $W\hat{T}_l$ are eigensolutions at $x = \hat{x}_l$: the j^{th} column of $W\hat{T}_l$ is a nonzero multiple of the Floquet solution (for example* [11] *p. 25) given by*

$$\hat{w}_{j,l}(x) = (x - \hat{x}_l)^{\frac{1}{2\hat{x}_l}(\Lambda_0(\epsilon))_{jj}+\frac{1}{2}(\Lambda_1(\epsilon))_{jj}}\hat{g}_{j,l}(x), \tag{14}$$

with $\hat{g}_{j,l}(x) = e_j + O(|x - \hat{x}_l|)$ an analytic function of x in a region containing $x = \hat{x}_l$ but no other singular point (e_j is the j^{th} column of the identity matrix).

In particular, when $n = 2$, W is composed of two eigensolutions at different singular points : $\hat{w}_{1,R}(x)$ and $\hat{w}_{2,L}(x)$.

The modulus $\mathcal{M}$ of the systems (5) is given by the complete system of formal invariants together with equivalence classes of unfolded Stokes matrices:

Definition 2.12. The unfolded Stokes matrices are *equivalent* if they are conjugated by the same invertible diagonal matrix $\hat{K}$ which depends analytically on $\hat{\epsilon} \in S$, has an invertible limit at $\epsilon = 0$ and satisfies

$$|\bar{K} - \tilde{K}| \leq c|\bar{\epsilon}| \quad \text{over } S_\cap, \quad \text{for some } c \in \mathbb{R}_+. \tag{15}$$

Theorem 2.13. [8] *Two families of systems of the form (5) with the same model system (6) are analytically equivalent if and only if their unfolded Stokes matrices are equivalent (see Definition 2.12).*

It is possible to choose representatives of the equivalence classes of unfolded Stokes matrices that are $\frac{1}{2}$-summable in ϵ.

Here we want to characterize the set of moduli: under which conditions a set $\mathcal{M}$, consisting of formal invariants (depending analytically on ϵ) and of a germ of pairs of unfolded Stokes matrices (defined up to conjugacy and depending analytically on $\hat{\epsilon}$ with continuous limit at $\epsilon = 0$), is realizable as the modulus of a system (4)? Any such $\mathcal{M}$ is realizable as the modulus of a system (4) depending analytically on $\hat{\epsilon}$. But an additional condition is needed so that the system (4) be uniform in ϵ. We call it the *auto-intersection relation*, since it links the two presentations on $S_\cap$, the auto-intersection of S. The auto-intersection relation implies that the two systems for $\bar{\epsilon}$ and $\tilde{\epsilon} = \bar{\epsilon}e^{2\pi i}$ are equivalent with analytic dependence on $(\bar{\epsilon}, x) \in S_\cap \times \mathbb{D}_r$ and uniform convergence on compact sets of $\mathbb{D}_r$ when $\bar{\epsilon} \to 0$. It is obtained from the uniqueness, up to different normalizations, of eigensolutions. It states the invariance (up to left and right multiplication by invertible diagonal matrices) of transition matrices between fundamental matrices of eigensolutions.

We now make the auto-intersection relation precise. Let the symbol $\hat{}$ denotes $\bar{}$ or $\tilde{}$. For $\hat{\epsilon} \in S_\cap$, the singular points (8) have nonzero imaginary part. We call x_U, the one with positive imaginary part and x_D the other one:

$$x_U = \bar{x}_L = \tilde{x}_R, \qquad x_D = \bar{x}_R = \tilde{x}_L. \tag{16}$$

We consider the monodromy around the singular points as illustrated in Figure 5 (in this way, the base points for the monodromy around the lower singular point ($M_{\tilde{x}_L}$ and $M_{\bar{x}_R}$) belongs to $\Omega_D^{\bar{\epsilon}} \cap \Omega_D^{\tilde{\epsilon}}$, whereas the base points for the monodromy around the upper singular point ($M_{\bar{x}_L}$ and $M_{\tilde{x}_R}$) are taken on $\Omega_U^{\bar{\epsilon}} \cap \Omega_U^{\tilde{\epsilon}}$).

Let us define $F_U(\hat{\epsilon}, x)$ as the analytic continuation of $F_D(\hat{\epsilon}, x)$ to $\Omega_U^{\hat{\epsilon}}$ when passing through $\Omega_R^{\hat{\epsilon}}$, hence satisfying

$$F_U(\hat{\epsilon}, x) = \begin{cases} F_D(\hat{\epsilon}, x), & \text{on } \Omega_R^{\hat{\epsilon}}, \\ F_D(\hat{\epsilon}, x)e^{2\pi i \Lambda_1(\epsilon)}, & \text{on } \Omega_L^{\hat{\epsilon}}, \\ F_D(\hat{\epsilon}, x)\hat{D}_R^{-1}, & \text{on } \Omega_C^{\hat{\epsilon}}. \end{cases} \tag{17}$$

We consider the following fundamental matrices of eigensolutions

$$\begin{array}{l} \bar{W}_{x_U} = H_U(\bar{\epsilon},x)F_U(\epsilon,x)\bar{D}_R\bar{T}_L\bar{D}_R^{-1}, \\ \tilde{W}_{x_U} = H_U(\tilde{\epsilon},x)F_U(\epsilon,x)\tilde{D}_R\tilde{T}_R\tilde{D}_R^{-1}, \\ \bar{W}_{x_D} = H_D(\bar{\epsilon},x)F_D(\epsilon,x)\bar{T}_R, \\ \tilde{W}_{x_D} = H_D(\tilde{\epsilon},x)F_D(\epsilon,x)\tilde{T}_L, \end{array} \tag{18}$$

and their analytic continuation along the related loop. We have that $\bar{W}_{x_U}$ and $\tilde{W}_{x_U}$ (respectively $\bar{W}_{x_D}$ and $\tilde{W}_{x_D}$) are fundamental matrices of solutions composed of eigenvectors of M_{x_U} (respectively M_{x_D}), and they converge uniformly on compact sets of Ω_U^0 (respectively Ω_D^0) when $\hat{\epsilon} \to 0$ on $S_\cap$.

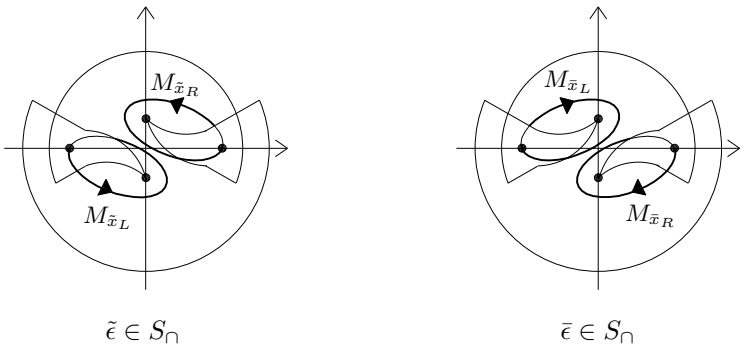

FIGURE 5. Illustration of the monodromy operators $M_{\hat{x}_l}$ around the singular points.

Let $\hat{E}_{L,x_D\to x_U}$ be the (left) transition matrix such that, over a fixed compact set of Ω_L^0 sufficiently far from the singular points,

$$\hat{E}_{L,x_D\to x_U} = \hat{W}_{x_D}^{-1}\hat{W}_{x_U}. \tag{19}$$

(The right transition matrix obtained from the comparaison of the fundamental matrices of solutions over Ω_R^0 can be considered too.)

Remark 2.14. *A transition matrix $\hat{E}_{L,x_D\to x_U}$ can be calculated using* (19)*,* (18)*,* (17)*,* (12)*,* (11) *and* (13)*. For example, we have*

$$\begin{array}{ll} \bar{E}_{L,x_D\to x_U} & = \bar{W}_{x_D}^{-1}\bar{W}_{x_U} \\ & = \bar{T}_R^{-1}F_D^{-1}(\epsilon,x)H_D^{-1}(\bar{\epsilon},x)H_U(\bar{\epsilon},x)F_U(\epsilon,x)\bar{D}_R\bar{T}_L\bar{D}_R^{-1} \\ & = \bar{T}_R^{-1}F_D^{-1}(\epsilon,x)H_D^{-1}(\bar{\epsilon},x)H_U(\bar{\epsilon},x)F_D(\epsilon,x)e^{2\pi i\Lambda_1(\epsilon)}\bar{D}_R\bar{T}_L\bar{D}_R^{-1} \\ & = \bar{T}_R^{-1}F_D^{-1}(\epsilon,x)H_D^{-1}(\bar{\epsilon},x)H_U(\bar{\epsilon},x)F_D(\epsilon,x)\bar{D}_L\bar{T}_L\bar{D}_R^{-1} \\ & = \bar{T}_R^{-1}\bar{C}_L\bar{D}_L\bar{T}_L\bar{D}_R^{-1} \\ & = \bar{T}_R^{-1}\bar{T}_L\bar{D}_L\bar{D}_R^{-1} \\ & = \bar{T}_R^{-1}\bar{T}_Le^{2\pi i\Lambda_1(\epsilon)}. \end{array} \tag{20}$$

As the fundamental matrices of eigensolutions around a regular singular point are unique up to different normalizations, the transition matrix $\hat{E}_{L,x_D\to x_U}$ must be invariant up to left and right multiplication by diagonal matrices, in both presentations $\bar{\epsilon}$ and $\tilde{\epsilon}$. In a more general situation, this comes implicitely from the paper [5] of A. Glutsyuk. When applying it to our situation, it results into:

Proposition 2.15. *The two systems for $\bar{\epsilon}$ and $\tilde{\epsilon} = \bar{\epsilon}e^{2\pi i}$ are equivalent, with analytic dependence on $(\bar{\epsilon}, x) \in S_\cap \times \mathbb{D}_r$ and uniform convergence on compact sets of $\mathbb{D}_r$ when $\bar{\epsilon} \to 0$, if and only if there exist $\bar{Q}_U$ and $\bar{Q}_D$ invertible diagonal matrices depending analytically on $\bar{\epsilon} \in S_\cap$, with an invertible limit at $\epsilon = 0$, and such that*

$$\bar{Q}_D \bar{E}_{L,x_D \to x_U} = \tilde{E}_{L,x_D \to x_U} \bar{Q}_U. \tag{21}$$

The exact expression of the transition matrices $\hat{E}_{L,x_D \to x_U}$ (see Remark 2.14) in relation (21) gives

$$\bar{Q}_D \bar{N}_L = \tilde{N}_L \bar{Q}_U, \tag{22}$$

where

$$\begin{aligned} \tilde{N}_L &= \tilde{D}_L \tilde{T}_L^{-1} \tilde{T}_R \tilde{D}_L^{-1} = \tilde{E}_{L,x_D \to x_U} e^{-2\pi i \Lambda_1(\epsilon)}, \\ \bar{N}_L &= \bar{T}_R^{-1} \bar{T}_L = \bar{E}_{L,x_D \to x_U} e^{-2\pi i \Lambda_1(\epsilon)}. \end{aligned} \tag{23}$$

Remark 2.16. *Although the same letter is used, we see, by relation (23), that $\tilde{N}_L$ and $\bar{N}_L$ are different expressions that are not obtained by the replacements $\hat{\epsilon} = \tilde{\epsilon}$ or $\hat{\epsilon} = \bar{\epsilon}$ into some $\hat{N}_L$.*

What we call the auto-intersection relation includes (22) and an extra condition on the diagonal matrices $\bar{Q}_s$ coming from properties of $H_s(\hat{\epsilon}, x)$ over $S_\cap$.

Definition 2.17. We say that the *auto-intersection relation* is satisfied if, in addition to the conditions of Proposition 2.15, we have

$$|\bar{Q}_s - I| < c_s |\bar{\epsilon}|, \quad \text{for some } c_s \in \mathbb{R}_+, \quad \bar{\epsilon} \in S_\cap, \quad s = U, D. \tag{24}$$

The auto-intersection relation for the family (5) is satisfied. Moreover, it is a necessary condition for the realization of the modulus of analytic classification:

Theorem 2.18. [8] *Let be given :*

- *a formal normal form (i.e. diagonal matrices $\Lambda_0(\epsilon)$ and $\Lambda_1(\epsilon)$ of formal invariants which depend analytically on ϵ at the origin),*
- *a sector $S' = \{\hat{\epsilon} : |\hat{\epsilon}| < \rho', \arg(\hat{\epsilon}) \in (\pi - \gamma, 3\pi + \gamma)\}$, for some well-chosen ρ' and $\gamma > 0$, with γ depending on Λ_0 (see Sections 4.3 to 4.5 of [8] for the specifications),*
- *an equivalence class (see Definition 2.12) of unfolded Stokes matrices $\hat{C}_R$ and $\hat{C}_L$, which are respectively an upper triangular and a lower triangular unipotent matrix depending analytically on $\hat{\epsilon} \in S'$ (ρ' can be chosen smaller if necessary) and having a bounded limit when $\hat{\epsilon} \to 0$ on S'.*

If the auto-intersection relation is satisfied on S', then there exist $r > 0$, a sector $S = \{\hat{\epsilon} : |\hat{\epsilon}| < \rho, \arg(\hat{\epsilon}) \in (\pi - \gamma, 3\pi + \gamma)\}$ for some $\rho < \min\left(\rho', \frac{r^2}{2}\right)$ and a system $(x^2 - \epsilon)y' = A(\epsilon, x)y$ ($y \in \mathbb{C}^n$) characterized by these analytic invariants, with $A(\epsilon, x)$ analytic over $\mathbb{D}_\rho \times \mathbb{D}_r$.

The matrix $\hat{N}_L$ appearing in the auto-intersection relation tends to the Stokes matrix C_L when $\epsilon \to 0$. In fact, we obtain a more precise statement given by Proposition 2.20.

Definition 2.19. We say that a quantity c is *exponentially close to* 0 *in* $\sqrt{\epsilon}$ if it satisfies $|c| < be^{-\frac{a}{\sqrt{|\epsilon|}}}$ for some $a, b \in \mathbb{R}_+^*$.

Proposition 2.20. *The entries of the matrix $\hat{N}_L - \hat{C}_L$ are exponentially close to* 0 *in $\sqrt{\epsilon}$ in the sense of Definition 2.19. Hence, the diagonal entries of $\hat{N}_L$ are always different from zero over $S_\cap$ (we choose the radius of $S_\cap$ sufficiently small).*

Before giving a proof of Proposition 2.20, let us introduce the quantities $\hat{\Delta}_{sj,l}$.

Definition 2.21. Let us define

$$\hat{\Delta}_{sj,l} = (\hat{D}_l)_{ss}(\hat{D}_l^{-1})_{jj}, \tag{25}$$

where $\hat{D}_l$ is given by (12).

Lemma 2.22. *From (9), we have*

$$\bar{\Delta}_{sj,R} = \tilde{\Delta}_{js,L}, \quad \bar{\Delta}_{sj,L} = \tilde{\Delta}_{js,R}. \tag{26}$$

When $s < j$ and $l = L, R$, $\tilde{\Delta}_{sj,l}$ is exponentially close to 0 *in $\sqrt{\epsilon}$ (in the sense of Definition 2.19).*

Proof of Proposition 2.20. Relation (13) can be used to express recursively (off-diagonal) entries of $\hat{T}_l$ and $\hat{T}_l^{-1}$ in terms of the entries of $\hat{C}_l$:

$$(\hat{T}_l)_{ij}(1 - \hat{\Delta}_{ij,l}) = (\hat{C}_l)_{ij} + \sum_{\substack{i<k<j,\, l=R \\ j<k<i,\, l=L}} (\hat{C}_l)_{ik}(\hat{T}_l)_{kj}\hat{\Delta}_{kj,l}, \tag{27}$$

and

$$(\hat{T}_l^{-1})_{ij}(\hat{\Delta}_{ij,l} - 1) = (\hat{C}_l)_{ij} + \sum_{\substack{i<k<j,\, l=R \\ j<k<i,\, l=L}} (\hat{T}_l^{-1})_{ik}(\hat{C}_l)_{kj}. \tag{28}$$

Together with Lemma 2.22, equalities (27) and (28) (or the ones obtained by multiplication on each side of the equality by $\hat{\Delta}_{ji,l}$) imply that, on $S_\cap$, entries from the following matrices are exponentially close to 0 in $\sqrt{\epsilon}$:

$$\bar{C}_L - \bar{T}_L, \qquad I - \bar{T}_R^{-1}, \tag{29}$$

$$I - \tilde{T}_L^{-1}, \qquad \tilde{C}_R - \tilde{T}_R, \tag{30}$$

$$\tilde{D}_L\tilde{T}_L^{-1}\tilde{D}_L^{-1} - \tilde{C}_L, \qquad \tilde{D}_L\tilde{T}_R\tilde{D}_L^{-1} - I \tag{31}$$

(relations (30) can be used when obtaining (31) from (27) and (28)). We then decompose $\hat{N}_l - \hat{C}_l$ as

$$\begin{array}{ll} \bar{N}_L - \bar{C}_L & = \bar{T}_R^{-1}\bar{T}_L - \bar{C}_L = (\bar{T}_R^{-1} - I)\bar{T}_L - (\bar{C}_L - \bar{T}_L), \\ \tilde{N}_L - \tilde{C}_L & = \tilde{D}_L\tilde{T}_L^{-1}\tilde{T}_R\tilde{D}_L^{-1} - \tilde{C}_L \\ & = \tilde{D}_L\tilde{T}_L^{-1}\tilde{D}_L^{-1}(\tilde{D}_L\tilde{T}_R\tilde{D}_L^{-1} - I) + (\tilde{D}_L\tilde{T}_L^{-1}\tilde{D}_L^{-1} - \tilde{C}_L) \end{array} \tag{32}$$

to prove the statement of the proposition. □

3. Compatibility relations on $S_\cap$

We will now have a closer look at the relation (22), with the goal of translating it directly in terms of the complete system of analytic invariants. This will then be done in full detail in dimensions $n = 2$ and $n = 3$.

3.1. **General case $n \geq 2$.** First, let us now reformulate relation (22) and find conditions for which it is satisfied.

Definition 3.1. Let us define the (left) *normalized transition invariant* as

$$\hat{G} = \hat{N}_L diag\left\{\frac{1}{(\hat{N}_L)_{11}}, \frac{1}{(\hat{N}_L)_{22}}, ..., \frac{1}{(\hat{N}_L)_{nn}}\right\}. \tag{33}$$

(Proposition 2.20 ensures that $diag\left\{(\hat{N}_L)_{11}, (\hat{N}_L)_{22}, ..., (\hat{N}_L)_{nn}\right\}$ is invertible).

Remark 3.2. *Elements $(\hat{G})_{ij}$ and $(\hat{N}_L)_{ij}$ vanish at the same order on $S_\cap \cup \{0\}$.*

Theorem 3.3. *The auto-intersection relation is satisfied if and only if there exists an invertible diagonal matrix $\bar{Q}$ depending analytically on $\bar{\epsilon} \in S_\cap$, with an invertible limit at $\epsilon = 0$, such that*

$$\bar{Q}\bar{G} = \tilde{G}\bar{Q}, \tag{34}$$

and

$$|\bar{Q} - I| < c|\bar{\epsilon}|, \quad \text{for some } c \in \mathbb{R}_+, \quad \bar{\epsilon} \in S_\cap. \tag{35}$$

Proof. In terms of the normalized transition invariant, the relation (22) can be written as

$$\bar{Q}_D\bar{G} = \tilde{G}\bar{Q}_U diag\left\{\frac{(\tilde{N}_L)_{11}}{(\bar{N}_L)_{11}}, \frac{(\tilde{N}_L)_{22}}{(\bar{N}_L)_{22}}, ..., \frac{(\tilde{N}_L)_{nn}}{(\bar{N}_L)_{nn}}\right\}. \tag{36}$$

Since the normalized transition invariant has 1's on the diagonal, we must have

$$\bar{Q}_D = \bar{Q}_U diag\left\{\frac{(\tilde{N}_L)_{11}}{(\bar{N}_L)_{11}}, \frac{(\tilde{N}_L)_{22}}{(\bar{N}_L)_{22}}, ..., \frac{(\tilde{N}_L)_{nn}}{(\bar{N}_L)_{nn}}\right\}. \tag{37}$$

□

Definition 3.4. The matrix G is said to be reducible it there exists a permutation matrix P such that PGP^{-1} is the direct sum of diagonal blocs.

Lemma 3.5. *If some element $(\bar{G})_{ij}$ (respectively $(\tilde{G})_{ij}$) of the normalized transition invariant $\bar{G}$ (respectively $\tilde{G}$) vanishes, it corresponds to the following geometric property: the j-th eigensolution at the upper singular point x_U has no component of the i-th eigensolution at the lower singular point x_D.*

Theorem 3.6. *The existence of an invertible diagonal matrix $\bar{Q}$ such that relation (34) is satisfied is equivalent to have*

- $(\bar{G})_{ij} \equiv 0 \iff (\tilde{G})_{ij} \equiv 0$,
- $(\bar{G})_{ij}$ *and* $(\tilde{G})_{ij}$ *vanishing at the same order on* $S_\cap \cup \{0\}$, $\forall i \neq j$,
- *a finite set of compatibility relations involving only elements of $\bar{G}$ and $\tilde{G}$.*

The number of compatibility relations is equal to

$$n^2 - 2n - (\textit{number of zeros in } \bar{G}) + (\textit{number of irreducible blocs of } \bar{G}). \tag{38}$$

In particular, this number is $(n-1)^2$ in the most generic case where $\bar{G}$ has only nonzero entries.

Proof. From Theorem 3.3, we see that the existence of an invertible diagonal matrix $\bar{Q}$ such that relation (34) is satisfied yields relations between elements of $\bar{G}$ and $\tilde{G}$. Indeed, denoting

$$\bar{Q} = diag\{q_1, q_2, ..., q_n\}, \tag{39}$$

relation (34) is equivalent to

$$q_i(\bar{G})_{ij} = (\tilde{G})_{ij} q_j, \tag{40}$$

which yields the first part of the theorem.

Let us count the number of compatibility relations. First, let us suppose that $\bar{G}$ is irreducible. Without loss of generality, we can assume that $q_n = 1$. From $i = 2$ to $i = n$, let us consider the $i \times i$ lower right submatrices of $\bar{G}$ and $\bar{Q}^{-1}\tilde{G}\bar{Q}$ that must be equal. Each time we increase i, each new entry which is not identically 1 or 0 allows either to

- calculate one q_i that has not been calculated before,
- obtain a new compatibility relation.

Since there are $(n-1)$ indeterminates (q_1, q_2,...,q_{n-1}), the number of nonequivalent compatibility relations is

$$n^2 - n - (\text{number of zeros in } \bar{G}) - (n-1). \tag{41}$$

If $\bar{G}$ is reducible, we repeat this process for each irreducible bloc of $\bar{G}$. Each time, we can take one indeterminate equal to 1 without loss of generality. The number of indeterminates is then

$$n - (\text{number of irreducible blocs of } \bar{G}), \tag{42}$$

yielding the general formula (38).

If some element $(\bar{G})_{ij}$ of the normalized transition invariant $\bar{G}$ vanishes, then $(\tilde{G})_{ij}$ must vanish simultaneously by Lemma 3.5 and conversely. If $(\bar{G})_{ij}$ vanishes at a value $\bar{\epsilon}_0$ to order k, then, in a perturbation of the family (5) we could have up to k zeros of the perturbed $(\bar{G})_{ij}$, since all $(\bar{G})_{ij}$ are realizable for $\bar{\epsilon}$ in a small neighborhood of $\bar{\epsilon}_0$. When $\epsilon_0 = 0$, it is possible to take a perturbation where the k zeros would belong to $S_\cap$, and hence, to apply the same argument. This is why the order of vanishing of $(\bar{G})_{ij}$ and $(\tilde{G})_{ij}$ must be the same. □

Corollary 3.7. *If $\bar{G}$ is irreducible and has $(n-1)^2$ elements that are identically zero, then the set of compatibility relations in Theorem 3.6 is empty.*

Proof. Theorem 3.6 gives the number of compatibility relations which is

$$n^2 - 2n - (\text{number of zeros in } \bar{G}) + 1 = (n-1)^2 - (\text{number of zeros in } \bar{G}) = 0. \tag{43}$$

□

Theorem 3.8. *In the generic case where all entries of $\bar{G}$ do not vanish identically, the compatibility relations of Theorem 3.6 can be chosen as*

$$(\tilde{G})_{ij}(\tilde{G})_{ji} = (\bar{G})_{ij}(\bar{G})_{ji}, \quad 1 \le i < j \le n, \tag{44}$$

and

$$(\bar{G})_{ij}(\bar{G})_{jn}(\bar{G})_{ni} = (\tilde{G})_{ij}(\tilde{G})_{jn}(\tilde{G})_{ni}, \quad 1 \le i < j < n. \tag{45}$$

Proof. Without loss of generality, we take $q_n = 1$. We choose the $(n-1)$ equations of (40) with $j = n$ to evaluate all the indeterminates:

$$q_i = \frac{(\tilde{G})_{in}}{(\bar{G})_{in}}, \quad i = 1, 2, ..., n-1. \tag{46}$$

Then, for each entry that is not on the diagonal or on the last column of the equal matrices $\bar{G}$ and $\bar{Q}^{-1}\tilde{G}\bar{Q}$, we must add a compatibility relation: each relation of the form (40) must agree with (46). The n^{th} row gives

$$\frac{1}{q_i} = \frac{(\tilde{G})_{ni}}{(\bar{G})_{ni}}, \quad i = 1, 2, ..., n-1, \tag{47}$$

and hence, to agree with (46), we must have (44) for $j = n$ and $1 \leq i < n$. The entries above the diagonal (except for the last column) give

$$\frac{q_i}{q_j} = \frac{(\tilde{G})_{ij}}{(\bar{G})_{ij}}, \quad 1 \leq i < j < n, \tag{48}$$

and hence, to agree with (46), we must have (45). Of couse, if some $(\bar{G})_{ij}$ vanishes, the expressions still make sense if $(\tilde{G})_{ij}$ vanishes to the same order. Finally, the entries under the diagonal (except for the last row) give

$$\frac{q_j}{q_i} = \frac{(\tilde{G})_{ji}}{(\bar{G})_{ji}}, \quad 1 \leq i < j < n. \tag{49}$$

In order that (49) agrees with (48), we must have (44) with $j \neq n$. □

When simplifying the compatibility relations in terms of $\hat{N}_L$, one may prefer to work with polynomial expressions instead of rational expressions. In our case, this is possible if we add one relation:

Theorem 3.9. *In the generic case where all entries of $\bar{G}$ do not vanish identically, the set of compatibility relations of Theorem 3.6 can be replaced by the set of relations*

$$\prod_{i=1}^{n} (\bar{N}_L)_{i\sigma(i)} = \prod_{i=1}^{n} (\tilde{N}_L)_{i\sigma(i)}, \quad \sigma \in \mathcal{U} \cup \{(1)\}, \tag{50}$$

where $\mathcal{U}$ is the subset of the symmetric group S_n composed of

- *all the 2-cycles $\sigma = (ij)$, for $1 \leq i < j \leq n$,*
- *all the 3-cycles $\sigma = (ijn)$, for $1 \leq i < j < n$.*

Proof. Let us suppose that the conditions of Theorem 3.6 are satisfied. Then, we have

$$(\bar{G})_{ij} = (\tilde{G})_{ij} \frac{q_j}{q_i}, \quad 1 \leq i, j \leq n. \tag{51}$$

For any σ in the symmetric group S_n, if we replace the n relations (51) into $\prod_{i=1}^{n} (\tilde{G})_{i\sigma(i)}$, we get

$$\prod_{i=1}^{n} (\bar{G})_{i\sigma(i)} = \prod_{i=1}^{n} (\tilde{G})_{i\sigma(i)} \frac{\prod_{k=1}^{n} q_k}{\prod_{j=1}^{n} q_j} = \prod_{i=1}^{n} (\tilde{G})_{i\sigma(i)}, \tag{52}$$

implying

$$\det \bar{G} = \det \tilde{G}. \tag{53}$$

Since $\det \bar{N}_L = \det \tilde{N}_L = 1$ (recall that the definition of $\hat{N}_L$ is given by (23)), relations (33) and (53) imply

$$\prod_{i=1}^{n} (\bar{N}_L)_{ii} = \prod_{i=1}^{n} (\tilde{N}_L)_{ii}. \tag{54}$$

Replacing (33) into the compatibility relations (44) and (45) and the multiplication by (54) yields (50). On the other hand, let us suppose that the set of compatibility relations of Theorem 3.6 is replaced by the set of relations (50). With $\sigma = (1)$, we have (54). With other σ's, the division by (54) yields the compatibility relations (44) and (45). □

The compatibility relations in the generic case depend on specific products of the elements of the unfolded Stokes matrices called *invariants products*, and become simpler when written in terms of these products.

Definition 3.10. Let us define

$$\hat{c}_{ij} = \begin{cases} (\hat{C}_L)_{ij}, & i > j, \\ (\hat{C}_R)_{ij}, & i < j, \\ 1, & i = j, \end{cases} \tag{55}$$

and, for any cycle σ in the symmetric group S_n, the invariant product corresponding to σ

$$\hat{a}_\sigma = \prod_{i=1}^{n} \hat{c}_{i\sigma(i)}. \tag{56}$$

The invariant products are, generically, analytic invariants:

Proposition 3.11. *Two families of systems of the form (5) with the same model system (6) and all quantities $\hat{c}_{ij}$ and $\hat{c}'_{ij}$ not identically zero are analytically equivalent if and only if*

- *$\hat{c}_{ij}$ and $\hat{c}'_{ij}$ vanish at the same order on $S \cup \{0\}$,*
- *for any cycle σ in $\mathcal{U}$, with $\mathcal{U}$ as in Theorem 3.9, their corresponding invariant products are equal on S ($\hat{a}_\sigma = \hat{a}'_\sigma$),*
- *we have*

$$\left| \frac{\bar{c}_{in}}{\bar{c}'_{in}} - \frac{\tilde{c}_{in}}{\tilde{c}'_{in}} \right| \leq k_i |\bar{\epsilon}| \quad \text{over } S_\cap, \quad \text{for some } k_i \in \mathbb{R}_+, \quad i = 1, 2, ..., n-1. \tag{57}$$

Proof. Starting from Theorem 2.13, the proof is similar to that of Theorem 3.8. □

3.2. **The case $n = 2$.** In this section, we apply the last results to the case $n = 2$, where we can simplify the conditions of Theorem 3.6 to express them directly in terms of the analytic invariants. In fact, when $n = 2$ (and no more generically when $n > 2$), the (i, j) entries of $\hat{G}$ are proportional to the (i, j) entries of the unfolded Stokes matrices and the compatibility condition can be simplified to an elegant expression given in Theorem 3.12 (Proposition 3.20 shows how this is no more the case generically when $n = 3$).

Theorem 3.12. *In the case $n = 2$, there exists an invertible diagonal matrix $\bar{Q}$ such that relation (34) is satisfied if and only if:*

- *the elements of each pair*

$$\begin{array}{c}\bar{c}_{12} \text{ and } \tilde{c}_{12},\\ \bar{c}_{21} \text{ and } \tilde{c}_{21}\end{array} \tag{58}$$

vanish simultaneously at the same order on $S_\cap \cup \{0\}$*, or both vanish identically,*
- *and the product* $\hat{a}_{(12)}$ *is analytic at* $\epsilon = 0$*.*

Proof. Note that the analyticity of $\hat{a}_{(12)}$ at $\epsilon = 0$ is equivalent to showing that

$$\bar{c}_{12}\bar{c}_{21} = \tilde{c}_{12}\tilde{c}_{21}. \tag{59}$$

Using (23), (13), (25) and (26), we obtain the relations between the entries of $\hat{N}_L$ and those of the unfolded Stokes matrices. When $n = 2$ (and no more generically when $n > 2$), we have

$$\begin{array}{l}(\bar{N}_L)_{22} = (\tilde{N}_L)_{11} = 1,\\ (\bar{N}_L)_{11} = 1 + \bar{c}_{12}\bar{c}_{21}\tilde{\Delta}_{12,L}(\tilde{\Delta}_{12,L} - 1)^{-1}(\tilde{\Delta}_{12,R} - 1)^{-1},\\ (\tilde{N}_L)_{22} = 1 + \tilde{c}_{12}\tilde{c}_{21}\tilde{\Delta}_{12,L}(\tilde{\Delta}_{12,L} - 1)^{-1}(\tilde{\Delta}_{12,R} - 1)^{-1},\\ (\tilde{N}_L)_{12} = \tilde{c}_{12}\tilde{\Delta}_{12,L}(1 - \tilde{\Delta}_{12,R})^{-1},\\ (\tilde{N}_L)_{21} = \tilde{c}_{21}(1 - \tilde{\Delta}_{12,L})^{-1},\\ (\bar{N}_L)_{12} = \bar{c}_{12}\tilde{\Delta}_{12,L}(1 - \tilde{\Delta}_{12,L})^{-1},\\ (\bar{N}_L)_{21} = \bar{c}_{21}(1 - \tilde{\Delta}_{12,R})^{-1}.\end{array} \tag{60}$$

The first part (58) comes from Theorem 3.6, Remark 3.2 and (60). The second part (59) is automatically satisfied in nongeneric cases. In generic cases, it is obtained from Theorem 3.8 since the only compatibility relation is

$$(\tilde{G})_{ij}(\tilde{G})_{ji} = (\bar{G})_{ij}(\bar{G})_{ji} \tag{61}$$

and reduces to (59) using (33) and (60). □

Corollary 3.13. *When* $n = 2$*, it is always possible to choose an analytic representative of the equivalence classes of unfolded Stokes matrices, and the following cases can occur for* $n_{12}, n_{21} \in \mathbb{N} = \{0, 1, \dots\}$*:*

(1) $c_{12}(\epsilon) = \epsilon^{n_{12}}$*,* $c_{21}(\epsilon) = \epsilon^{n_{21}} g(\epsilon)$*, with* g *analytic satisfying* $g(0) \neq 0$*;*
(2) $c_{12}(\epsilon) \equiv 0$*,* $c_{21}(\epsilon) = \epsilon^{n_{21}}$*;*
(3) $c_{12}(\epsilon) = \epsilon^{n_{12}}$*,* $c_{21}(\epsilon) \equiv 0$*;*
(4) $c_{12}(\epsilon) = c_{21}(\epsilon) \equiv 0$*.*

Remark 3.14. *Let us explain why Theorem 3.12 is natural. The coefficient* $(\hat{N}_L)_{12}$ *(respectively* $(\hat{N}_L)_{21}$*), which is proportional to* $\hat{c}_{12}$ *(respectively* $\hat{c}_{21}$*), tells us how much the second (respectively first) eigensolution at* x_U *contains of the first (respectively second) eigensolution at* x_D *(since from* (19) *and* (23)*, we have* $\hat{W}_{x_U} = \hat{W}_{x_D}\hat{E}_{L,x_D\to x_U} = \hat{W}_{x_D}\hat{N}_L e^{2\pi i \Lambda_1(\epsilon)}$*). These geometric informations have to be preserved in both presentations on* $S_\cap$*, taking into account that the eigensolutions are only defined modulo a constant. If we make the calculation in the case of nonvanishing* $\hat{c}_{12}$ *and* $\hat{c}_{21}$ *on* $S_\cap \cup \{0\}$*, we obtain that only the product* $\hat{a}_{(12)}$ *matters and that we must have the identity* $\bar{a}_{(12)} = \tilde{a}_{(12)}$*.*

3.3. **The generic case $n = 3$.** When $n \geq 3$, the situation is generically completely different from that in the case $n = 2$. In this section, we investigate the compatibility relations for $n = 3$ given by the system of conditions (44) and (45). It may happen that some of these conditions imply that $\hat{G}$ has some identically zero entry (and this would correspond to a nongeneric case).

Notation 3.15. We write

$$\Lambda_q(\epsilon) = diag\{\lambda_{1,q}(\epsilon), ..., \lambda_{n,q}(\epsilon)\}, \quad q = 0, 1, \tag{62}$$

where $\Lambda_q(\epsilon)$ are the diagonal matrices from (5).

Definition 3.16. Let us define

$$\hat{q} = \hat{a}_{(12)}\hat{a}_{(23)} - \hat{a}_{(132)} - \hat{a}_{(123)}, \tag{63}$$

$$\hat{\Delta}_{sj} = \sqrt{\hat{\Delta}_{sj,L}\hat{\Delta}_{sj,R}} = e^{\frac{\pi i}{\sqrt{\epsilon}}((\lambda_{s,0}(\epsilon))-(\lambda_{j,0}(\epsilon)))} \tag{64}$$

and

$$\delta_{sj} = \sqrt{\hat{\Delta}_{sj,L}\hat{\Delta}_{js,R}} = e^{\pi i(\lambda_{s,1}(\epsilon)-\lambda_{j,1}(\epsilon))}. \tag{65}$$

Lemma 3.17. *We have*

$$\hat{\Delta}_{sj,L} = \hat{\Delta}_{sj}\delta_{sj}, \quad \hat{\Delta}_{sj,R} = \hat{\Delta}_{sj}\delta_{sj}^{-1}, \tag{66}$$

$$\delta_{sj} = \delta_{sk}\delta_{kj}, \tag{67}$$

$$\hat{\Delta}_{sj} = \hat{\Delta}_{sk}\hat{\Delta}_{kj}. \tag{68}$$

For $n = 3$, contrary to the case $n = 2$, the analyticity at $\epsilon = 0$ of all the invariant products $\hat{a}_\sigma$ is neither necessary, nor sufficient for the compatibility relations to be satisfied:

Theorem 3.18. *Let us suppose that, for all $\sigma \in S_3$, the invariant product $\hat{a}_\sigma$ defined by (56) is analytic in ϵ at $\epsilon = 0$ (equivalently $\tilde{a}_\sigma = \bar{a}_\sigma$, which we write as a_σ). Let us define the following quantities which are then analytic in ϵ.*

$$\begin{array}{ll} \mathcal{A}\delta_{12} & = a_{(123)}(1-{\delta_{12}}^2) + a_{(12)}(a_{(13)}{\delta_{12}}^2 - a_{(23)}), \\ \mathcal{B}\delta_{23} & = a_{(123)}(1-{\delta_{23}}^2) + a_{(23)}{\delta_{23}}^2(a_{(12)} - a_{(13)}), \\ \mathcal{C}\delta_{12}\delta_{23} & = (1-{\delta_{12}}^2{\delta_{23}}^2)(a_{(12)}a_{(23)} - 2a_{(123)}) \\ & \quad + a_{(123)}(a_{(23)}{\delta_{23}}^2({\delta_{12}}^2-1) + a_{(12)}{\delta_{12}}^2({\delta_{23}}^2-1)) \\ & \quad + {\delta_{23}}^2({\delta_{12}}^2{a_{(12)}}^2a_{(23)} - a_{(12)}{a_{(23)}}^2) \\ & \quad + a_{(13)}\left({\delta_{23}}^2(1+{\delta_{12}}^2)a_{(23)} - {\delta_{12}}^2(1+{\delta_{23}}^2)a_{(12)}\right). \end{array} \tag{69}$$

Then, for $\epsilon \in S_\cap$, the compatibility relations of Theorem 3.6 may be replaced by the following conditions:

$$a_{(12)}a_{(23)} - a_{(123)} - a_{(132)} = 0 \tag{70}$$

and

- *for $\lambda_{1,0}(\epsilon) - \lambda_{2,0}(\epsilon) \not\equiv \lambda_{2,0}(\epsilon) - \lambda_{3,0}(\epsilon)$,*

$$\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{C} \equiv 0, \tag{71}$$

- *for* $\lambda_{1,0}(\epsilon) - \lambda_{2,0}(\epsilon) \equiv \lambda_{2,0}(\epsilon) - \lambda_{3,0}(\epsilon)$,

$$\mathcal{A} \equiv -\mathcal{B}, \quad \mathcal{C} \equiv 0, \tag{72}$$

with $\lambda_{j,q}(\epsilon)$ *as in Notation 3.15.*

We postpone the proof.

Corollary 3.19. *The analyticity at* $\epsilon = 0$ *of all the invariant products* $\hat{a}_\sigma$ *is not necessary for realizability. Not all systems with a nonresonant irregular singular point of Poincaré rank* 1 *have an unfolding with unfolded Stokes matrices depending analytically on* ϵ*.*

Proof. While for fixed $\epsilon = 0$, any pair of Stokes matrices is realizable, the conditions of Theorem 3.18 imply that the Stokes matrices at $\epsilon = 0$ must have a special form in order that the invariant products $\hat{a}_\sigma$ be analytic at $\epsilon = 0$. A particular example is constructed as follows: consider a system $x^2y' = A(x)y$ realizing the Stokes matrices with all coefficients equal to 1. Then, when $\epsilon = 0$, we have $a_{(12)} = a_{(23)} = a_{(123)} = a_{(132)} = 1$, which does not satisfy (70). Hence, the unfolding $(x^2 - \epsilon)y' = A(x)y$ cannot satisfy the conditions of Theorem 3.18. □

The following proposition is purely computational and will be used to prove Theorem 3.18.

Proposition 3.20. *In the generic case* $n = 3$*, there exists an invertible diagonal matrix* $\bar{Q}$ *such that relation (34) is satisfied if and only if the following relations are satisfied*

$$\begin{aligned}(\bar{a}_{(12)} - \tilde{a}_{(12)})(-1 + \tilde{\Delta}_{23,L})(-1 + \tilde{\Delta}_{13,R}) \\ = \tilde{\Delta}_{23,L}\left((\tilde{q} + \tilde{a}_{(123)} - \bar{a}_{(123)}) - \tilde{\Delta}_{12,R}(\bar{q} + \bar{a}_{(132)} - \tilde{a}_{(132)})\right),\end{aligned} \tag{73}$$

$$\begin{aligned}(\bar{a}_{(23)} - \tilde{a}_{(23)})(-1 + \tilde{\Delta}_{12,R})(-1 + \tilde{\Delta}_{13,L}) \\ = \tilde{\Delta}_{12,L}\left(\tilde{\Delta}_{23,L}(\tilde{q} + \tilde{a}_{(132)} - \bar{a}_{(132)}) - (\bar{q} + \bar{a}_{(123)} - \tilde{a}_{(123)})\right),\end{aligned} \tag{74}$$

$$\begin{aligned}(\bar{a}_{(13)} - \tilde{a}_{(13)})(-1 + \tilde{\Delta}_{12,L})(-1 + \tilde{\Delta}_{23,R}) \\ = -\tilde{\Delta}_{23,R}\left(\tilde{q} + \tilde{\Delta}_{12,L}(\tilde{a}_{(123)} - \bar{a}_{(123)})\right) + \left(\tilde{\Delta}_{12,L}\bar{q} + \bar{a}_{(132)} - \tilde{a}_{(132)}\right),\end{aligned} \tag{75}$$

$$\begin{aligned}&\left(\tilde{q} + \tilde{a}_{(123)} - \bar{a}_{(123)} - \tilde{\Delta}_{12,R}(\bar{q} + \bar{a}_{(132)} - \tilde{a}_{(132)})\right)(-1 + \tilde{\Delta}_{13,L})(-1 + \tilde{\Delta}_{23,R}) \\ &= \left(\tilde{q} + \tilde{a}_{(132)} - \bar{a}_{(132)} - \tilde{\Delta}_{12,L}(\bar{q} + \bar{a}_{(123)} - \tilde{a}_{(123)})\right)(-1 + \tilde{\Delta}_{23,L})(-1 + \tilde{\Delta}_{13,R}),\end{aligned} \tag{76}$$

$$\begin{aligned}&\tilde{\Delta}_{13,L}\left(\tilde{a}_{(12)}(\tilde{a}_{(132)} + \tilde{a}_{(123)}{\delta_{32}}^2 + \tilde{\Delta}_{23,R}\tilde{q}) - \bar{a}_{(23)}(\bar{a}_{(123)} + \bar{a}_{(132)}{\delta_{21}}^2 + \tilde{\Delta}_{12,R}\bar{q})\right) \\ &\quad = \left(\bar{a}_{(132)} + \tilde{\Delta}_{12,L}(\bar{q} + \bar{a}_{(123)})\right)(-1 + \tilde{\Delta}_{23,L})(-1 + \tilde{\Delta}_{13,R}) \\ &\qquad - \left(\tilde{a}_{(132)} + \tilde{\Delta}_{23,R}(\tilde{q} + \tilde{a}_{(123)})\right)(-1 + \tilde{\Delta}_{13,L})(-1 + \tilde{\Delta}_{12,R}) \\ &\qquad + \tilde{\Delta}_{23,L}\bar{a}_{(13)}\bar{a}_{(23)}(-1 + \tilde{\Delta}_{12,L})(-1 + \tilde{\Delta}_{12,R}) \\ &\qquad - \tilde{\Delta}_{12,L}\tilde{a}_{(12)}\tilde{a}_{(13)}(-1 + \tilde{\Delta}_{23,L})(-1 + \tilde{\Delta}_{23,R}),\end{aligned} \tag{77}$$

and the elements of each of the following pairs vanish simultaneously at the same order on $S_\cap \cup \{0\}$

(78)

$$\begin{array}{l}
\bar{c}_{12}(-1+\bar{\Delta}_{31,R})(-1+\bar{\Delta}_{32,L})+\bar{\Delta}_{32,R}\bar{c}_{32}(\bar{c}_{13}+\bar{\Delta}_{21,R}(\bar{c}_{12}\bar{c}_{23}-\bar{c}_{13})) \quad \text{and} \quad \tilde{c}_{12},\\
\bar{c}_{21}(-1+\bar{\Delta}_{32,R})(-1+\bar{\Delta}_{31,L})+\bar{\Delta}_{32,R}\bar{c}_{23}(\bar{c}_{31}+\bar{\Delta}_{21,L}(\bar{c}_{21}\bar{c}_{32}-\bar{c}_{31})) \quad \text{and} \quad \tilde{c}_{21},\\
\bar{c}_{23} \quad \text{and} \quad \tilde{c}_{23}(-1+\tilde{\Delta}_{12,L})(-1+\tilde{\Delta}_{13,R})+\tilde{\Delta}_{12,L}\tilde{c}_{21}(\tilde{c}_{13}+\tilde{\Delta}_{23,R}(\tilde{c}_{23}\tilde{c}_{12}-\tilde{c}_{13})),\\
\bar{c}_{32} \quad \text{and} \quad \tilde{c}_{32}(-1+\tilde{\Delta}_{13,L})(-1+\tilde{\Delta}_{12,R})+\tilde{\Delta}_{12,L}\tilde{c}_{12}(\tilde{c}_{31}+\tilde{\Delta}_{23,L}(\tilde{c}_{32}\tilde{c}_{21}-\tilde{c}_{31})),\\
\bar{c}_{13}+\bar{\Delta}_{21,R}(\bar{c}_{12}\bar{c}_{23}-\bar{c}_{13}) \quad \text{and} \quad \tilde{c}_{13}+\tilde{\Delta}_{23,R}(\tilde{c}_{12}\tilde{c}_{23}-\tilde{c}_{13}),\\
\bar{c}_{31}+\bar{\Delta}_{21,L}(\bar{c}_{21}\bar{c}_{32}-\bar{c}_{31}) \quad \text{and} \quad \tilde{c}_{31}+\tilde{\Delta}_{23,L}(\tilde{c}_{21}\tilde{c}_{32}-\tilde{c}_{31}).
\end{array}$$

Proof. Relations (78) come from Theorem 3.6 and Remark 3.2, using (23), (13) and (25) for the replacements.

The set of four compatibility relations from Theorem 3.6 may be replaced by a set of five relations, by Theorem 3.9. We use the equivalent set obtained with $\sigma \in S_3$ and $\sigma \neq (1)$, thus having a set of equations $\{eqn_\sigma = 0\}$, each of them indexed by the corresponding $\sigma \in S_3$. We use the simplified set of equivalent equations $eqn_{(132)}$ and $eqn_\sigma - eqn_{(132)} = 0$ for $\sigma \neq (132)$. Relations (73) to (77) are then obtained with the required replacements. □

Proof of Theorem 3.18. By Proposition 3.20, the compatibility relations of Theorem 3.6 may be replaced by the relations (73) to (77). Let us take the products $\hat{a}_\sigma$ analytic in ϵ. Relation (73) is satisfied if and only if (70) is satisfied. Replacing (70) into relations (74) to (77) and using (66) yields the only relation

$$\mathcal{A}\tilde{\Delta}_{12}+\mathcal{B}\tilde{\Delta}_{23}+\mathcal{C}\tilde{\Delta}_{12}\tilde{\Delta}_{23}+\mathcal{B}\tilde{\Delta}_{12}{}^2\tilde{\Delta}_{23}+\mathcal{A}\tilde{\Delta}_{12}\tilde{\Delta}_{23}{}^2\equiv 0, \tag{79}$$

with $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ given by (69).

We will now study further the identity (79). We will prove that it is satisfied if and only (71) or (72). The relation $\lambda_{1,0}(\epsilon)-\lambda_{2,0}(\epsilon)\equiv\lambda_{2,0}(\epsilon)-\lambda_{3,0}(\epsilon)$, is equivalent to $\hat{\Delta}_{12}=\hat{\Delta}_{23}$. In this case, if the conditions (72) are satisfied, it is clear that (79) is also satisfied. Conversely, suppose that (79) is satisfied. It has the form

$$\mathcal{A}x+\mathcal{B}y+\mathcal{C}xy+\mathcal{B}x^2y+\mathcal{A}xy^2\equiv 0$$

for $x=\tilde{\Delta}_{12}$, $y=\tilde{\Delta}_{23}$. Let us look at the different cases.

- If $\lambda_{1,0}(0)-\lambda_{2,0}(0)\neq\lambda_{2,0}(0)-\lambda_{3,0}(0)$, then there is a direction in ϵ-space where x is flatter that y ($x\prec y$) or the contrary. Let us consider the case where $x\prec y$. Then we need have $\mathcal{A}\equiv 0$ and $\mathcal{B}y+\mathcal{C}xy+\mathcal{B}x^2y\equiv 0$. For the same reason we need have $\mathcal{B}\equiv 0$ and $Cxy\equiv 0$, which yields (71). In the case $y\prec x$, we need have $\mathcal{B}\equiv 0$ and $x(\mathcal{A}+\mathcal{C}y+\mathcal{A}y^2)\equiv 0$. Hence $\mathcal{A}+\mathcal{C}y+\mathcal{A}y^2\equiv 0$, which implies $\mathcal{A}\equiv 0$, and we get the same conclusion as before.
- In the case $\lambda_{1,0}(0)-\lambda_{2,0}(0)=\lambda_{2,0}(0)-\lambda_{3,0}(0)$, then we have $\mathcal{A}\frac{x}{y}+\mathcal{B}+\mathcal{C}x+\mathcal{B}x^2+\mathcal{A}xy\equiv 0$. But, $\frac{x}{y}=\exp\left(\frac{c(\epsilon)}{\sqrt{\epsilon}}\right)$ for $c(\epsilon)$ analytic in ϵ such that $c(0)=0$. It follows that $\frac{x}{y}$ is an analytic function of $\sqrt{\epsilon}$ bounded away from 0 for ϵ small. Hence, we need to have $\mathcal{A}\frac{x}{y}+\mathcal{B}\equiv 0$ and $\mathcal{C}x+\mathcal{B}x^2+\mathcal{A}yx\equiv 0$. The relation $\mathcal{C}+\mathcal{B}x+\mathcal{A}y\equiv 0$ gives $C\equiv 0$. If $\lambda_{1,0}(\epsilon)-\lambda_{2,0}(\epsilon)$ is not identically equal to $\lambda_{2,0}(\epsilon)-\lambda_{3,0}(\epsilon)$, we conclude $\mathcal{A}\equiv\mathcal{B}\equiv 0$. If $\lambda_{1,0}(\epsilon)-\lambda_{2,0}(\epsilon)\equiv\lambda_{2,0}(\epsilon)-\lambda_{3,0}(\epsilon)$, then this gives the equations $\mathcal{A}+\mathcal{B}\equiv 0$.

□

Let us now study in detail the systems of equations (71) and (72).

Proposition 3.21. *In the generic case where* $\lambda_{1,0}(\epsilon) - \lambda_{2,0}(\epsilon) \not\equiv \lambda_{2,0}(\epsilon) - \lambda_{3,0}(\epsilon)$*, the system* (71) *is equivalent to the system of three equations*

$$\mathcal{S}_1 = a_{(123)}(1 - {\delta_{12}}^2) + a_{(12)}(a_{(13)}{\delta_{12}}^2 - a_{(23)}) = 0, \tag{80}$$

$$\mathcal{S}_2 = a_{(123)}(1 - {\delta_{23}}^2) + a_{(23)}(a_{(12)} - a_{(13)}){\delta_{23}}^2 = 0, \tag{81}$$

$$\begin{aligned}\mathcal{S}_3 = a_{(12)}a_{(13)}{\delta_{12}}^2({\delta_{23}}^2 - 1) + a_{(13)}a_{(23)}{\delta_{23}}^2({\delta_{12}}^2 - 1)\\ + a_{(12)}a_{(23)}(1 - {\delta_{12}}^2{\delta_{23}}^2) = 0.\end{aligned} \tag{82}$$

The two equations (80) *and* (82) *(respectively* (81) *and* (82)*) are sufficient when* ${\delta_{12}}^2 \not\equiv 1$ *(respectively* ${\delta_{23}}^2 \not\equiv 1$*). The equation* (82) *yields that* $(a_{(12)}, a_{(13)}, a_{(23)})$ *lie in a quadric in* $(a_{(12)}, a_{(13)}, a_{(23)})$*-space, while* (80) *or* (81) *yields that* $a_{(123)}$ *is uniquely determined by* $(a_{(12)}, a_{(13)}, a_{(23)})$*, except at simultaneous zeroes of* ${\delta_{12}}^2 - 1$ *and* ${\delta_{23}}^2 - 1$*.*

If both ${\delta_{12}}^2 \equiv 1$ *and* ${\delta_{23}}^2 \equiv 1$ *(this includes the case* $\Lambda_1(\epsilon) \equiv 0$*), then the system reduces to one of the following:*

$$a_{12} \equiv a_{23} \equiv a_{13}, \tag{83a}$$

$$a_{12} \equiv a_{23} \equiv 0, \tag{83b}$$

$$a_{13} \equiv a_{23} \equiv 0, \tag{83c}$$

$$a_{12} \equiv a_{13} \equiv 0. \tag{83d}$$

Proof. It suffices to note that the right hand sides of (80), (81) and (82) form a Groebner basis for the polynomials appearing in equations (71) and that the system formed by (80) and (82) is equivalent to the one formed by (81) and (82) under the hypotheses ${\delta_{12}}^2 \not\equiv 1$ and ${\delta_{23}}^2 \not\equiv 1$. □

Proposition 3.22. *When* $\lambda_{1,0}(\epsilon) - \lambda_{2,0}(\epsilon) \equiv \lambda_{2,0}(\epsilon) - \lambda_{3,0}(\epsilon)$*, the system of equations (72) is equivalent to a system of two equations which are linear in* $a_{(123)}$*:*

$$\mathcal{F}a_{(123)} + \mathcal{G} = 0, \tag{84}$$

$$\mathcal{K}a_{(123)} + \mathcal{L} = 0, \tag{85}$$

where

$$\begin{cases}\mathcal{F} = (1 - \delta_{12}\delta_{23})(\delta_{12} + \delta_{23}),\\ \mathcal{G} = \delta_{23}(a_{(12)}(a_{(13)}\delta_{12}^2 - a_{(23)}) + (a_{(12)} - a_{(13)})a_{(23)}\delta_{12}\delta_{23},\\ \mathcal{K} = a_{(12)}\delta_{12}^2(1 - \delta_{23}^2) + a_{(23)}\delta_{23}^2(1 - \delta_{12}^2) + 2(\delta_{12}^2\delta_{23}^2 - 1),\\ \mathcal{L} = a_{(12)}a_{(23)}[1 - \delta_{12}^2\delta_{23}^2 - a_{(23)}\delta_{23}^2 + a_{(12)}\delta_{12}^2\delta_{23}^2]\\ \qquad -a_{(12)}a_{(13)}\delta_{12}^2(1 + \delta_{23}^2) + a_{(13)}a_{(23)}\delta_{23}^2(1 + \delta_{12}^2).\end{cases} \tag{86}$$

In the generic case where $\mathcal{F}, \mathcal{G}, \mathcal{H}$ *and* $\mathcal{K}$ *do not vanish, this is in turn equivalent to the system of the two equations* (84) *and* $\mathcal{M} = 0$*, where*

$$\mathcal{M} = \mathcal{F}\mathcal{K} - \mathcal{G}\mathcal{H} = \mathcal{S}_3\mathcal{N}, \tag{87}$$

for $\mathcal{S}_3$ *given in* (82) *and*

$$\mathcal{N} = a_{(12)}\delta_{12}^2\delta_{23} - a_{(23)}\delta_{12}\delta_{23}^2 + (\delta_{12} - \delta_{23})(1 - \delta_{12}\delta_{23}). \tag{88}$$

When $\delta_{12} \equiv \delta_{23} \equiv 1$, which includes the case $\Lambda_1(\epsilon) \equiv 0$, then the system reduces to

$$a_{12} \equiv a_{23}, \tag{89a}$$

$$a_{13} \equiv a_{23} \equiv 0, \tag{89b}$$

$$a_{12} \equiv a_{13} \equiv 0. \tag{89c}$$

Propositions 3.21 and 3.22 give the generic cases together with the particular cases where $\Lambda_1(\epsilon) \equiv 0$. This case could occur naturally in some systems for symmetry reasons. There remains a number of particular cases that can be derived as well. We have omitted them since their statement was quite tedious.

Perspective. *Generically the sufficient conditions for having invariants depending analytically on ϵ describe a set of codimension 3, and hence of dimension 2. When we started this investigation, we were not sure to expect any invariant products to be analytic in ϵ, except in some trivial decomposable cases. However, there are such cases which fill a codimension 3 variety, and most of them do not correspond to a decomposable system. The conditions obtained ensure that a geometric explanation similar to the one in Remark 3.14 holds here too. Such an explanation is a form of "symmetry" with respect to the two singular points: the geometric description of the system is essentially the same from the point of view of one singular point and from the point of view of the other. What is remarkable is that the condition must hold on the system itself for $\epsilon = 0$, and not only on the unfolding. Hence, the Stokes matrices (at $\epsilon = 0$) of an irregular singular point of Poincaré rank 1 already "know" if an analytic unfolding of its Stokes matrices will be realizable. Understanding geometrically this "symmetry" at the limit $\epsilon = 0$ is certainly a very interesting question.*

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