ORGANIZING CENTER FOR THE BIFURCATION ANALYSIS OF A GENERALIZED GAUSE MODEL WITH PREY HARVESTING AND HOLLING RESPONSE FUNCTION OF TYPE III*

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Abstract. The present note is an addendum to the paper [1] which presented a study of a generalized Gause model with prey harvesting and a generalized Holling response function of type III: $p(x) = \frac{mx^2}{ax^2+bx+1}$. Complete bifurcation diagrams were proposed, but some parts were conjectural. An organizing center for the bifurcation diagram was given by a nilpotent point of saddle type lying on an invariant line and of codimension greater than or equal to 3. This point was of codimension 3 when $b \neq 0$, and was conjectured to be of infinite codimension when $b = 0$. This conjecture was in line with a second conjecture that the Hopf bifurcation of order 2 degenerates to a Hopf bifurcation of infinite codimension when $b = 0$. In this note we prove these two conjectures.

Keywords. Generalized Gause model with prey harvesting, Holling response function of type III, Hopf bifurcation, nilpotent saddle bifurcation.

1. Introduction

The model discussed in [1] is the system:

$$\begin{cases} \dot{x} = rx(1-\frac{x}{k}) - \frac{mx^2y}{ax^2+bx+1} - h_1, \\ \dot{y} = y(-d + \frac{cmx^2}{ax^2+bx+1}), \\ x \geq 0, y \geq 0, \end{cases} \tag{1.1}$$

where the eight parameters: r, k, m, a, c, d, h_1 are strictly positive and $b \geq 0$. The response function

$$p(x) = \frac{mx^2}{ax^2+bx+1} \tag{1.2}$$

is called Holling (resp. generalized Holling) of type III when $b = 0$ (resp. $b \neq 0$).

Through the following linear transformation and time scaling

$$(X, Y, T) = \left(\frac{1}{k}x, \frac{1}{ck}y, cmk^2t\right),$$

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the number of parameters was reduced to five: the simplified system that was considered is the following

$$\begin{cases} \dot{x} = \rho x(1-x) - y\frac{x^2}{\alpha x^2+\beta x+1} - \lambda, \\ \dot{y} = y(-\delta + \frac{x^2}{\alpha x^2+\beta x+1}), \\ x \geq 0, y \geq 0, \end{cases} \tag{1.3}$$

with parameters

$$(\rho, \alpha, \beta, \delta, \lambda) = \left(\frac{r}{cmk^2}, ak^2, bk, \frac{d}{cmk^2}, \frac{h_1}{cmk^3}\right). \tag{1.4}$$

For $\beta \geq 0$ and small λ, the system has two singular points C and D on the positive invariant x-axis. When λ grows to $\frac{\rho}{4}$, the two singular points merge in a saddle-node and disappear. Also, the system has at most singular point, E, in the first quadrant, which is of anti-saddle type (node, focus or center). When the point E enters or exists the first quadrant, it does so by merging with either C or D in a transcritical bifurcation. The three points C, D and E can all merge together in a nilpotent point of saddle type located at $(\frac{1}{2}, 0)$. This occurs when

$$\begin{cases} \lambda = \frac{\rho}{4}, \\ \delta = \frac{1}{\alpha+2\beta+4}. \end{cases}$$

In the generic case, namely when

$$B_2 = \alpha\beta + 6\alpha - \beta^2 - 8\beta - 24 \neq 0,$$

the codimension is 2 (instead of 3) because of the presence of an invariant line through the point. When $B = 0$, the nilpotent point has codimension greater than 2. The corresponding bifurcation is the organizing center of the bifurcation diagram. In [1], the codimension was shown to be 3 when $\beta > 0$, and conjectured to be infinite when $\beta = 0$. For $\beta > 0$, this bifurcation locus is the termination of the locus of surface of Hopf bifurcation of order 2 at the point E. When $\beta = 0$, it was conjectured that the Hopf bifurcation was of infinite order as soon as its order was greater than 1. In order words, it was conjectured that E was a center as soon as it had two pure imaginary eigenvalues and the first Lyapunov constant (or first coefficient of the Hopf bifurcation) vanishes.

Here we prove these two conjectures. Our method of proof is quite original. It consists in finding an analytic transformation of the system bringing it to a time reversible system. To show that the transformed system is indeed reversible, we compute its power series. We use an encyclopedia of integer sequences to guess the general term, and Mathematica to provide the sum of the series. Once, we have conjectured the exact form of the system, a straightforward computation going in the inverse direction allows to prove that this conjectured form is the exact form of the system: indeed, substituting the inverse change of coordinate in the conjectured form provides the original system.

2. Proof of the conjectures

2.1. **Proof of the first conjecture.**

Theorem 2.1. *We consider the system* (1.3) *with* $\beta = 0$ *under parameter values for which there exists a singular point* $E = (x_0, y_0)$ *inside the first quadrant. If the system has a Hopf bifurcation of order greater than* 1 *at* E*, then* E *is a center.*

PROOF. If $E = (x_0, y_0)$ is a singular point of (1.3) with $y_0 \neq 0$, then

$$\begin{cases} x_0^2(1-\alpha\delta) - 1 = 0, \\ y_0 = \frac{(\rho x_0(1-x_0)-\lambda)(1+\alpha x_0^2)}{x_0^2}. \end{cases} \tag{2.1}$$

and we use (2.1) to eliminate δ and y_0. We divide the system by $\frac{x^2}{\alpha x^2+1}$ and we localize it at (x_0, y_0) by $(x_1, y_1) = (x - x_0, y - y_0)$. The system now has the form

$$\begin{aligned} \dot{x_1} &= -y_1 - f(x_1), \\ \dot{y_1} &= g(x_1) + l(x_1)y_1, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} f(x_1) =& -\frac{1}{x_0^2(x_0+x_1)^2} x_1(-2\lambda x_0 + \rho x_0^2 - \alpha\rho x_0^4 + 2\alpha\rho x_0^5 - \lambda x_1 + \rho x_0 x_1 \\ & - 2\alpha\rho x_0^3 x_1 + 5\alpha\rho x_0^4 x_1 - \alpha\rho x_0^2 x_1^2 + 4\alpha\rho x_0^3 x_1^2 + \alpha\rho x_0^2 x_1^3), \\ g(x_1) =& \frac{-(\lambda - \rho x_0 + \rho x_0^2)x_1(2x_0+x_1)}{x_0^2(x_0+x_1)^2}, \\ l(x_1) =& \frac{x_1(2x_0+x_1)}{(1+\alpha x_0^2)(x_0+x_1)^2}. \end{aligned} \tag{2.3}$$

The Jacobian matrix at the origin is given by $\begin{pmatrix} -f'(0) & -1 \\ g'(0) & 0 \end{pmatrix}$. The determinant is positive under the conditions $x_0, y_0 > 0$ (details in [1]). Hence, we have a Hopf bifurcation if $f'(0) = 0$ which yields

$$-2\lambda + \rho x_0 - \alpha\rho x_0^3 + 2\alpha\rho x_0^4 = 0. \tag{2.4}$$

This allows to eliminate λ. The order of the Hopf bifurcation is greater than 1 if the first Lyapunov constant, $L(1)$, vanishes. The formula of $L(1)$ for arbitrary β was computed in [1]:

$$\begin{aligned} L(1) =& \rho^2 x_0^2(1-2x_0)(\alpha x_0^2 + \beta x_0 + 1)^2[(\beta^3 + 2\alpha\beta - \alpha\beta^2)x_0^4 \\ & + (6\beta^2 - 6\alpha\beta)x_0^3 + (6\beta - 6\alpha)x_0^2 + 4\beta x_0 + 6]. \end{aligned} \tag{2.5}$$

For $\beta = 0$, this yields

$$L(1) = 0 \Leftrightarrow (1-2x_0)(\alpha x_0^2 + 1)(1 - \alpha x_0^2) = 0. \tag{2.6}$$

We have $\alpha > 0$. Also, $1 - 2x_0 = 0$ is excluded, since otherwise $y_0 = 0$. Hence, we need have

$$1 - \alpha x_0^2 = 0, \tag{2.7}$$

and we are left with the two parameters ρ et x_0 in the system

$$\begin{aligned} \dot{x_1} &= -y_1 - f(x_1) = -y_1 - x_1^2 h(x_1) \\ \dot{y_1} &= g(x_1) + l(x_1)y_1, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} g(x_1) &= -\frac{(-\rho x_0 + 2\rho x_0^2)x_1(2x_0+x_1)}{x_0^2(x_0+x_1)^2}, \\ l(x_1) &= \frac{x_1(2x_0+x_1)}{2(x_0+x_1)^2}, \\ h(x_1) &= \frac{\rho(-x_0 + 4x_0^2 - x_1 + 4x_0x_1 + x_1^2)}{x_0^2(x_0+x_1)^2}. \end{aligned} \tag{2.9}$$

This suggests to try a change of coordinates $(x_1, y_1) \longmapsto (X, y_1)$ and time scaling which would bring the first equation to

$$\dot{X} = -y_1 - X^2. \tag{2.10}$$

So we let

$$\begin{cases} X = x_1\sqrt{h(x_1)} = H(x_1), \\ T = \frac{t}{k(X)}, \text{ où } k(X) = (H^{-1})'(X) = \frac{1}{H'(H^{-1}(X))}. \end{cases} \tag{2.11}$$

The transformed system becomes

$$\begin{aligned} \frac{dX}{dT} &= H'(x_1)\dot{x_1}(H^{-1})'(X) = \dot{x_1} = -y_1 - X^2, \\ \frac{dy_1}{dT} &= (g(x_1) + l(x_1)y_1)(H^{-1})'(X) = \frac{(g(H^{-1}(X)) + l(H^{-1}(X))y_1)}{H'(H^{-1}(X))}, \end{aligned} \tag{2.12}$$

which we write under the form

$$\begin{cases} \frac{dX}{dT} = -y_1 - X^2, \\ \frac{dy_1}{dT} = m(X) + n(X)y_1. \end{cases} \tag{2.13}$$

Experimentally, using Mathematica, we see that $m(X)$ and $n(X)$ look like odd functions of X, and we are going to prove that they are indeed odd functions of X, from which the result will follow. Since

$$m(X) = \frac{2(\rho x_0 - 2\rho x_0^2)}{x_0^2} n(X) = \frac{2\rho(1 - 2x_0)}{x_0} n(X), \tag{2.14}$$

it suffices to study $n(X)$ sont des fonctions impaires. Pour cela, nous faisons calculer le développement en série de $m(X)$ et $n(X)$ par Mathematica. The power series expansion of $n(X)$ has the form

$$\begin{aligned} n(X) = {} & \frac{x_0^2}{\rho(-1+4x_0)} X - \frac{2x_0^4}{\rho^2(-1+4x_0)^3} X^3 + \frac{6x_0^6}{\rho^3(-1+4x_0)^5} X^5 \\ & - \frac{20x_0^8}{\rho^4(-1+4x_0)^7} X^7 + \frac{70x_0^{10}y_1}{\rho^5(-1+4x_0)^9} X^9 + O(X^{11}) \end{aligned} \tag{2.15}$$

An encyclopedia of integer sequences yields that $1, 2, 6, 20, 70, \ldots$ are the first terms of the sequence $\left\{\frac{(2n)!}{(n!)^2}\right\}_{n\geq 0}$, yielding the conjecture

$$n(X) = \left(\frac{x_0^2 X}{\rho(-1+4x_0)}\right) \sum_{n=0}^{\infty} \left((-1)^n \frac{(2n)!}{(n!)^2} \left(\frac{x_0^2 X^2}{\rho(-1+4x_0)^2}\right)^n\right). \tag{2.16}$$

Mathematica provides a sum for this sequence, namely

$$n(X) = \frac{X x_0^2}{\rho(-1+4x_0)\sqrt{\frac{\rho - 8\rho x_0 + 16\rho x_0^2 + 4X^2 x_0^2}{\rho(-1+4x_0)^2}}}, \tag{2.17}$$

To prove that this is indeed the right formula for $n(X)$, it suffices to substitute

$$X = x_1\sqrt{h(x_1)} = H(x_1), \tag{2.18}$$

and to check that

$$\frac{n(H(x_1))}{H'(x_1)} = l(x_1), \tag{2.19}$$

From (2.16) we conclude that $n(X)$ is odd, and hence that E is a center $\square$

2.2. **Proof of the second conjecture.** For $\beta = 0$, $\rho = 4\lambda$ and $\delta = \frac{1}{\alpha+4}$, the three singular points C, D and E merge together in the triple point $(\frac{1}{2}, 0)$. This point is a nilpotent saddle with an invariant line, $y = 0$, through it. This suggests to use a normal form respecting the invariant line. Because of the invariant line, the corresponding bifurcation has codimension 2 (the two conditions are to have two zero eigenvalues), and such a normal form can be taken as

$$\begin{cases} \dot{X} = Y - X^2, \\ \dot{Y} = Y(\sum_{n\geq 1} B_n X^n), \end{cases} \tag{2.20}$$

with $B_1 > 0$. We have codimension greater than 2 if $B_2 = 0$. This occurs for $\alpha = 4$ (see [1]).

Theorem 2.2. *For $\beta = 0$, $\rho = 4\lambda$, $\delta = \frac{1}{8}$ and $\alpha = 4$, then the normal form* (2.20) *of* (1.3) *at $(\frac{1}{2}, 0)$ is time-reversible, namely $B_{2j} = 0$ for all $j \geq 1$.*

Proof. It is not necessary to make an independent proof of this theorem. The considered bifurcation is the endpoint for $\rho = 4\lambda$ of the surface of Hopf bifurcation of order greater than 1 .

Indeed, under the conditions, we get the limit values $x_0 = \frac{1}{2}$ and $y_0 = 0$, and we have $f'(0) = g'(0) = 0$ in (2.3). Also the limit condition for the Hopf bifurcation to have order greater than 1 namely

$$L(1) = 0 \Leftrightarrow 1 - \alpha x_0^2 = 0, \tag{2.21}$$

yields $\alpha = 4$ when $x_0 = \frac{1}{2}$.

The translation $(x_1, y_1) = (x - \frac{1}{2}, y)$ brings the system to the same form (2.2) with

$$\begin{aligned} g(x_1) &\equiv 0, \\ l(x_1) &= \frac{2x_1(1+x_1)}{(1+2x_1)^2}, \\ h(x_1) &= \frac{32\lambda(1+2x_1+2x_1^2)}{(1+2x_1)^2}. \end{aligned} \tag{2.22}$$

We take the same change of coordinate and time scaling (2.11), and we obtain the system under the form

$$\begin{cases} \dfrac{dX}{dT} = -y_1 - X^2, \\ \dfrac{dy_1}{dT} = n(X)y_1. \end{cases} \tag{2.23}$$

The function $n(X)$ is obtained by substituting $x_0 = \frac{1}{2}$ in (2.17)

$$n(X) = \frac{X}{4\rho\sqrt{1 + \frac{X^2}{\rho}}}. \tag{2.24}$$

Since it is odd, the result follows. □

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References

[1] R. Etoua and C. Rousseau, *Bifurcation Analysis of a Generalized Gause Model with Prey Harvesting and a Generalized Holling Response Function of Type III*, J. Differential Equations, **249** (2010), 2316–2356.

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