Normalizable, integrable and linearizable saddle points in the Lotka-Volterra system *

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Abstract

We consider the Lotka-Volterra Equations

$$\dot{x} = x(1 + ax + by), \qquad \dot{y} = y(-\lambda + cx + dy),$$

with λ a non negative number. Our aim is to understand the mechanisms which lead to the origin being linearizable, integrable or normalizable. In the case of integrability and linearizability, there is a natural dichotomy. When the system has an invariant line other than the axes, then the system is integrable and we give necessary and sufficient conditions for linearizability in this case. When there is no such line, then the conditions for linearizability and integrability are the same. In this case we show that the monodromy groups of the separatrices play a key role. In particular for $\lambda = p/q$ with $p + q \leq 12$ and $\lambda = n/2, 2/n$ with $n \in \mathbb{N}$ the origin is linearizable if and only if the monodromy groups can be shown to be linearizable by elementary arguments. We give 4 classes of these conditions, and their duals, in terms of the parameters of the system, and conjecture that these, together with two exceptional cases of Darboux linearizability, are the only integrability mechanisms for rational values of λ.

The work on normalizability is more tentative. We give some sufficient conditions for this via monodromy groups, and give a complete classification when $\lambda = 0$. We also investigate in detail the case $\lambda = 1$, with $a + c = 0$. Much of our ideas here are based on recent work on the unfolding of the Ecalle-Voronin modulus of analytic classification [12]. In particular we give examples of "half-normalizable" systems as well as an experimental example of a "transcritical bifurcation" of the functional moduli associated to the critical point.

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1 Introduction

In the paper [2] we considered polynomial vector fields $X = X_1\frac{\partial}{\partial x} + X_2\frac{\partial}{\partial y}$ with orbitally normalizable, normalizable, integrable and linearizable singular points (definitions will be given in the next section) of saddle type,

$$\begin{aligned}\dot{x} &= X_1(x,y) = x + P(x,y) = x + o(x,y)\\ \dot{y} &= X_2(x,y) = -\lambda y + Q(x,y) = -\lambda y + o(x,y),\end{aligned} \tag{1.1}$$

with $\lambda \in \mathbb{R}^+$. The classification of such points in the analytic case was known, but we wanted to see to what extent this classification was reflected in the classes of polynomial vector fields of fixed degree. Also the literature mainly considers the problem for fixed values of λ while we were interested to see how the different strata were globally organized when λ varies.

It came as a pleasant surprise, therefore, to find out that many of our examples could be constructed within the class of quadratic systems, and that most of these could in fact be found within the class of Lotka-Volterra systems

$$\begin{aligned}\dot{x} &= x(1 + ax + by)\\ \dot{y} &= y(-\lambda + cx + dy).\end{aligned} \tag{1.2}$$

Since these systems are perhaps the easiest non-linear systems after the linear ones, with a high degree of structure, and yet are sufficiently rich to contain many interesting examples, it made sense to try to give a full classification of their normalizable, integrable and linearizable points; in particular, we hoped that understanding something of the strata of the various types of critical points in the parameter space (λ, a, b, c, d) would give us a realistic picture of what happens in the general polynomial case.

A small start was made on this in [2] building on work in [4] and was extended later by Gravel and Thibault in [6]. These works classified the integrable and linearizable points for $\lambda = 2/n$ or $n/2$ for n a positive integer. Our aim here is to give a much fuller classification, especially in the cases of integrable or linearizable critical points. We conjecture that the classification is complete, apart from two exceptional cases of Darboux integrability, and prove that it is so for all rational values of $\lambda = p/q$ with $p + q \leq 12$. The case of normalizability is much harder. We have completed the study for $\lambda = 0$, and we have indicated a line of research in the case $\lambda = 1$ and $a + c = 0$. In particular, we have been able to find examples of "half-normalizable" critical points, where only one of the two moduli of the critical point vanishes, and an experimental example of a "transcritical bifurcation", not of the stability, but of the two halves of the functional modulus associated to the critical point.

As an indication of the usefulness of these results, we show that the three conjectures of [6] can be answered in the affirmative and provide some definite answers to the questions asked in that paper.

We now indicate, in more detail, the results given in the paper and some future lines of research.

Firstly, we consider the case when the origin of (1.2) is integrable or linearizable. When λ is a rational number, necessary conditions for integrability and linearizability can be found from explicit calculations of the normal form, or the equivalent Saddle Quantities. However, the case is different from the one encountered in [4] and [6], in that the strata of integrable systems obtained can be of codimension 2 and we need to calculate as much as 3 saddle quantities to determine them. For $\lambda = p/q$ the k-th obstruction to integrability occurs in the terms of degree $k(p+q)+1$, and takes the form of a homogeneous polynomial in the coefficients of degree $k(p+q)$ and less. For practical reasons, therefore, we restricted our calculations to those cases with $p+q \leq 12$. Having obtained computationally a number of conditions, it is then necessary to show that these are sufficient. A number of such techniques were given in [2], [4] and [6] but are not sufficient for the new values of λ investigated here.

Rather than multiplying techniques to cover these new cases, however, the nice surprise was that we found we could unify all the known cases of integrable systems of (1.2) for $\lambda = p/q$ with $p+q \leq 12$ and the previous cases $\lambda = n/2$ and $\lambda = 2/n$ for $n \in \mathbb{N}$ into exactly two types of integrability condition.

The first one is given by the existence of a third invariant line. This corresponds to the condition

$$\lambda ab + (1-\lambda)ad - cd = 0. \tag{1.3}$$

If this condition is satisfied, then we can obtain a Darboux first integral, and we show in Section 5 that we can completely classify the linearizable systems in this case. In the case where λ is irrational, we give diophantine conditions on the parameters for linearizability in terms of the continuous fraction approximation to λ.

The second one covers the case when the system has no invariant line other than the separatrices $x = 0$ and $y = 0$ (that is, (1.3) fails to hold). In this case, we show in Section 4 that if the origin is integrable, then it is automatically linearizable. We then look at the monodromy groups of one of the separatrices $x = 0$ or $y = 0$, together with the monodromy of the line at infinity. Each of these lines can be considered as a copy of the Riemann sphere with three singular points on it. If one of these has trivial monodromy, and the other is linearizable, then the third critical point must also be linearizable. We can also iterate this argument. Altogether, in Section 3, we give four such elementary conditions for the origin to be integrable via monodromy groups, together with their duals, and we show that these are sufficient to account for all the cases of integrability when $p+q \leq 12$.

It seemed natural to conjecture that the two conditions above are necessary and sufficient for all rational values of λ. Interestingly, however, there are a couple of isolated examples of systems with invariant algebraic curves, giving a Darboux integrating factor. These occur for $\lambda = 8/7$ and $\lambda = 13/7$. However, we are inclined to believe that these three sorts of mechanism comprise all the forms of integrability and linearizability which occur for Lotka-Volterra equations. Indeed, these two cases are the only examples of Liouvillian integrable Lotka-Volterra equations which do not fall into our previous classification (see [1]). More insteresting, several strata of Darboux integrable systems listed in [1] are codimension 2 and lie inside our codimension 1 strata. There is little evidence of what really happens in the case when λ is irrational at the moment, but many of our strata cover cases of irrational as well as irrational values of λ.

The second part of the work considers conditions for which the origin of (1.2) is orbitally normalizable. This work is much more tentative at the moment, with parts of a more "experimental" nature. The problem is that the conditions for orbital normalizability are not given by formal calculations, but by analyzing the convergence of the formal transformation to orbital normal form.

In Section 3 we give some conditions for orbital normalizability which are given from monodromy considerations. It is natural to ask if these are the only cases that occur, but we have little concrete evidence on this point at the moment. However, we do consider two cases in detail.

The first of these is a complete study of the limiting case $\lambda = 0$. Here, the system (1.2) has a saddle-node at the origin with an analytic center manifold. This last fact allows us to exactly characterize orbitally normalizable saddle-nodes. Again, we can show that these conditions are sufficient by the two mechanisms above. However, we also give an alternative description based on recent work [12] on the unfolding of parabolic points of holomorphic maps (see also [17] and [18]). This is done in Section 6.

The idea here, is to consider the case $\lambda = 0$ as a limit of sequences of systems (1.2) with $\lambda = \frac{1}{n}$ for which the origin is integrable. When the system with $\lambda = 0$ is non orbitally normalizable, then it cannot be approached by a sequence of integrable systems. We have tried to give independent proofs of the cases that we need here, but refer the reader to [12] for more details of this topic whose field of application is quite general.

The case of orbital normalizability for $\lambda > 0$ is a lot more difficult: there is no explicit way to calculate the conditions for orbital normalizability. However, in the case where $\lambda = 1$, we can still use ideas of [12] to show that a non-orbitally normalizable point cannot be approached by a sequence of systems with integrable points for $\lambda = 1 \pm \frac{1}{n}$.

More specifically, the analytic type of the critical point depends on an equivalence class of a pair of germs of diffeomorphisms fixing the origin (ψ^0, ψ^∞), called the Martinet-Ramis modulus (or Ecalle-Voronin modulus if we classify the holonomy of a separatrix). The critical point is orbitally normalizable if and only if both ψ^0 and ψ^∞ are linear. The nonlinearity of ψ^0 (resp. ψ^∞) does not allow one to approach the limiting system by a sequence of systems integrable at the origin with $\lambda_n = 1 - \frac{1}{n}$ (resp. $\lambda_n = 1 + \frac{1}{n}$).

In Section 7, we consider the case $\lambda = 1$ with $a + c = 0$ in some detail. We show the existence of "half-orbitally normalizable" points for $\lambda = 1$ (points for which either ψ^0 or ψ^∞ is linear, i.e. for which the holonomy map is semi-iterable in Ecalle's terminology [3]), and give a (necessarily incomplete) computational view of the behavior of the various strata of critical points in the neighborhood of $\lambda = 1$ with $a + c = 0$. One of the interesting features of this investigation is the phenomenon of a "transcritical bifurcation", whereby a 1-parameter family of generically linearizable systems transforms to a family of generically non-linearizable systems. The transformation can be seen in terms of the coalescence of two fixed points of the holonomy. The dynamics near each critical point is characterized by the unfolding of exactly one of the diffeomorphisms ψ^0 or ψ^∞. These are exchanged when the two points coalesce.

2 Preliminaries

2.1 Main definitions

Definition 2.1

1. The origin of (1.1) is *integrable* (or orbitally linearizable) if there exists an analytic change of coordinates $(X,Y) = (x+o(x,y), y+o(x,y))$ in the neighborhood of the origin transforming the system into
$$\begin{aligned}\dot{X} &= Xh(X,Y)\\ \dot{Y} &= -\lambda Y h(X,Y),\end{aligned} \tag{2.1}$$
where $h = 1 + O(X,Y)$.

2. The origin of (1.1) is *linearizable* if there exists an analytic change of coordinates $(X,Y) = (x+o(x,y), y+o(x,y))$ in the neighborhood of the origin transforming the system into the linear system
$$\begin{aligned}\dot{X} &= X\\ \dot{Y} &= -\lambda Y.\end{aligned} \tag{2.2}$$

3. For $\lambda = \frac{p}{q}$, the origin of (1.1) is *normalizable* if there exists an analytic change of coordinates $(X,Y) = (x+o(x,y), y+o(x,y))$ in the neighborhood of the origin transforming the system into
$$\begin{aligned}\dot{X} &= Xh(U)\\ \dot{Y} &= -\lambda Y k(U),\end{aligned} \tag{2.3}$$
where $U = X^pY^q$ and $h(U), k(U) = 1 + O(U)$.

4. For $\lambda = \frac{p}{q}$, the origin of (1.1) is orbitally normalizable if there exists an analytic change of coordinates $(X,Y) = (x+o(x,y), y+o(x,y))$ in the neighborhood of the origin transforming the system into
$$\begin{aligned}\dot{X} &= Xh(U)l(X,Y)\\ \dot{Y} &= -\lambda Y k(U)l(X,Y),\end{aligned} \tag{2.4}$$
where $U = X^pY^q$ and $h(U), k(U) = 1 + O(U)$ and $l(X,Y) = 1 + O(X,Y)$.

The following facts are well known:

Theorem 2.2

1. *The origin of* (1.1) *is integrable if and only if the holonomy of any of its separatrices is linearizable ([14] and [16]).*

2. *For $\lambda = \frac{p}{q}$, the origin of* (1.1) *is orbitally normalizable if and only if the holonomy map of any of its separatrices can be embedded in a flow (is iterable in Ecalle's terminology [3]). Its Martinet-Ramis modulus (or the Ecalle-Voronin modulus of the holonomy map) is then given by a pair of linear diffeomorphisms.*

2.2 The Martinet-Ramis modulus of a saddle-node

We recall the definition of the Martinet-Ramis modulus of analytic classification of a saddle-node (see for instance [13]). A saddle-node is formally orbitally equivalent by means of a transformation $(x,y) \mapsto (z,w)$ to a polynomial normal form

$$\begin{aligned} \dot{x} &= x^2 \\ \dot{y} &= y(1+ax) \end{aligned} \tag{2.5}$$

(the time orientation may be reversed).

Proposition 2.3 *Any generic complex saddle-node is orbitally analytically equivalent to a germ*

$$\begin{aligned} \dot{x} &= x^2 \\ \dot{y} &= y(1+ax) + x^2R(x,y). \end{aligned} \tag{2.6}$$

The sectorial normalization theorem (proved in [7] and presented in [13], [8]) claims that germs (2.6) and (2.5) are analytically equivalent in some sector-like domains. These domains are described as follows.

Let us divide a small disk $|x| < r$ in 2 equal sectors with vertex 0: For this we take $\alpha \in (0, \pi/2)$. Then $S_1 = \{x, |x| < r, arg(x) \in (-\pi/2 + \alpha, 3\pi/2 - \alpha)\}$ and $S_2 = \{x, |x| < r, arg(x) \in (\pi/2 + \alpha, 5\pi/2 - \alpha)\}$. Let $D = \{|y| < r\}$ be a disk in the w axis, and $\tilde{S}_j = S_j \times D$, $i = 1, 2$.

Theorem 2.4 *(see [7], [13], [8]) Any germ* (2.6) *is analytically equivalent in a sector $\tilde{S}_j$ (with r small) to* (2.5)*. Moreover, the normalizing map $H_j : \tilde{S}_j \to \mathbb{C}^2$ has the following properties:*

1. *H_j preserves z:*
$$H_j(x,y) = (x, h_j(x,y)),$$
and brings (2.6) *to* (2.5)*;*

2. *h_j has asymptotic Taylor series in x with coefficients which are holomorphic functions in y:*
$$\hat{h}(x,y) = \sum_0^\infty a_k(y)x^k, \quad a_0(y) \equiv y;$$
$\hat{h}$ is the same for h_1 and h_2;

3. Maps H_j with these properties are unique.

The tuple $H = (H_1, H_2)$ is called *the normalizing atlas* of the germ (2.6). Its transition functions generate *the Martinet–Ramis modulus* of the analytic classification of complex saddle-nodes.

The (multivalued) function

$$F(x,y) = yf_a(x), \qquad f_a(x) = e^{\frac{1}{x}}x^{-a} \tag{2.7}$$

is the first integral of the germ (2.5). Hence, the function

$$F_j = h_j f_a \tag{2.8}$$

is the first integral of the germ (2.6) in $\tilde{S}_j$. In the intersection of their domains, one integral in this list is a holomorphic function of another. We will apply this on the two components of the intersection of the sectors. Denote by $S^{\pm}$ the two sectors:

$$S^{\pm} = S_1 \cap S_2 \cap \{\pm Rex > 0\} \tag{2.9}$$

By the previous remark on the first integrals, there exists a holomorphic function ψ^∞ such that in $S^+ \times D$

$$h_2 f_a = \psi^\infty(h_1 f_a).$$

As the first integrals take all values in $\mathbb{C}$ in that domain the function ψ^∞ is entire. It is in particular defined in the neighborhood of the origin. Let $\psi^\infty(u) = \psi_0 + \psi_1 u + \psi_2(u)u^2$. Then

$$h_2(z,w) = \psi_0 f_a^{-1}(z) + \psi_1 h_1(z,w) + f_a(z)\psi_2(h_1 f_a)h_1^2. \tag{2.10}$$

By statement 2 of Theorem 2.4, $h_1 \to w, h_2 \to w$ as $z \to 0$. But in $S^+, f_a \to \infty$ as $z \to 0$. Hence, ψ_0 may be arbitrary, $\psi_1 = 1$ and $\psi_2 \equiv 0$. Summarizing, we get

$$\psi^\infty(w) = w + C \tag{2.11}$$

$$h_2(z,w) = h_1(z,w) + Cf_a^{-1}(z). \tag{2.12}$$

The constant C is one component of the Martinet–Ramis modulus. What we have recovered analytically is the fact that the affine maps (translations if the linear part is the identity) are the only analytic diffeomorphisms from $\mathbb{C}$ to $\mathbb{C}$ [18].

A parallel consideration gives the relation between h_1 and h_2 in $S^- \times D$, but the result is different because the function f_a on the sector S^- behaves in an opposite way than on S^+. A second difference comes from the fact that $f_a(z)$ is multi-valued. Hence if we want to compare the value of $f_a(z)$ on S_2, call it $\overline{f}_a(z)$, with the value of $f_a(z)$ on S_1 we have, on $S_1 \cap S_2$, $\overline{f}_a(z) = \exp(-2\pi i a)f_a(z)$. Hence we have:

$$h_2\overline{f}_a = \psi^0(h_1 f_a).$$

Let $\psi^0(w) = \psi_0 + \psi_1 w + \psi_2(w)w^2$. Then an analogue to formula (2.10) holds with h_1, h_2 replaced by h_2, h_1. As before, $h_2 \to w, h_1 \to w$ as $z \to 0$ in S^-. But now $f_a \to 0, f_a^{-1} \to \infty$ as $z \to 0$ in S^-. Hence, $\psi_0 = 0, \psi_1 = \exp(-2\pi i a), \psi_2$ may be an arbitrary function. Summarizing, we get

$$\begin{aligned} h_2 &= \overline{f_a}^{-1}\psi^0(h_1 f_a) = \exp(2\pi i a) f_a^{-1}\psi^0(h_1 f_a), \\ &\quad \psi^0(0) = 0, \psi^{0\prime}(0) = \exp(-2\pi i a), \end{aligned} \tag{2.13}$$

and ψ^0 is an arbitrary holomorphic function with the above restriction. Roughly speaking, the function ψ^0 is the second component of the Martinet–Ramis modulus.

2.3 The Ecalle-Voronin modulus of a diffeomorphism and its unfolding

The holonomy map of any separatrix (resp. strong separatrix) of a vector field with a saddle point (resp. saddle-node) characterizes the equivalence class of the vector field under orbital equivalence. For a vector field of the form (1.1) the holonomy of the x-separatrix (resp. y-separatrix) has the form $y \mapsto \exp(-2i\pi\lambda)y + o(y)$ (resp. $x \mapsto \exp(\frac{-2i\pi}{\lambda})x + o(x)$). In particular as soon as $\lambda = n$ (resp. $\lambda = \frac{1}{n}$) the holonomy of the x-separatrix (resp. y-separatrix) has a multiplier equal to 1 at the origin. We will limit ourselves to this case. The generalization to arbitrary $\lambda \in \mathbb{Q}^+$ is still to be done. Similarly for $\lambda = 0$ the holonomy of the x-separatrix has a multiplier equal to 1.

We now briefly summarize some results of [12] on the unfoldings of the Ecalle-Voronin invariants of a parabolic point of a diffeomorphism and their consequences for our questions on the organization of the different strata of orbitally normalizable and integrable points.

We consider a generic parabolic point of a diffeomorphism

$$f(z) = z + z^2 + o(z^2) \tag{2.14}$$

and a generic unfolding in "prepared form"

$$f_\epsilon(z) = z + (z^2 - \epsilon)h(z, \epsilon). \tag{2.15}$$

The perspective of [12] is to compare the family f_ϵ with a "model" family, namely the time-one maps for the family of vector fields

$$\frac{z^2 - \epsilon}{1 + a(\epsilon)z}\frac{\partial}{\partial z}. \tag{2.16}$$

If μ_0 and μ_∞ are the eigenvalues at the singular points $-\sqrt{\epsilon}$ and $\sqrt{\epsilon}$ of (2.16), then we can remark that

$$\begin{aligned} a(\epsilon) &= \tfrac{1}{\mu_\infty} + \tfrac{1}{\mu_0} \\ \tfrac{1}{\sqrt{\epsilon}} &= \tfrac{1}{\mu_\infty} - \tfrac{1}{\mu_0}, \end{aligned} \tag{2.17}$$

i.e. ϵ and $a(\epsilon)$ are analytic invariants of the system (2.19). We say that the family (2.15) is "prepared" when the multipliers $\lambda_0 = \exp(2\pi i \mu_0)$ and $\lambda_\infty = \exp(2\pi i \mu_\infty)$ at the two fixed points satisfy

$$\frac{1}{\sqrt{\epsilon}} = \frac{1}{\mu_\infty} - \frac{1}{\mu_0}. \tag{2.18}$$

Then the family is compared to the model (2.16) where $a(\epsilon) = \frac{1}{\mu_\infty} - \frac{1}{\mu_0}$. Any generic family of the form (2.15) can be prepared by means of a change in z and in the parameter ϵ (see details in [12]). The parameter of a prepared family is canonical: it is an analytic invariant of the unfolding. Hence any equivalence between two prepared families must preserve this parameter.

The paper [12] describes a complete modulus of analytic classification for prepared families of the form (2.15) for values of ϵ in a small neighborhood of the origin (but [12] does not describe the moduli space). As ϵ is an analytic invariant for a prepared family it is given by a family of moduli for each fixed value of ϵ. This modulus is given by an unfolding of the Ecalle-Voronin modulus of f_0.

Description of the Ecalle-Voronin modulus for $\epsilon = 0$. This modulus is given by the *orbit space*. We consider two fundamental domains $C^\pm$ of crescent shapes as in Figure 1, which are given by two curves $l_\pm$ and their images by f_0.

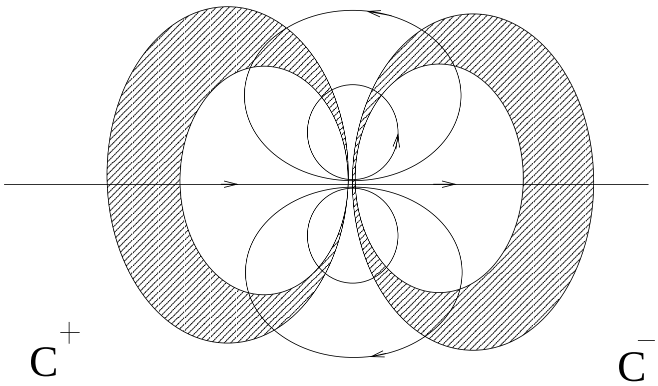


Figure 1: The Ecalle-Voronin modulus

Each orbit is represented by at most one point in each crescent, but some orbits can have representatives in the two crescents. Hence the orbit space is the union of the two crescents modulo the identification of points of the same orbit. To give this identification in an intrinsic way one remarks that the two crescents in which we identify the curves $l_\pm$ and $f(l_\pm)$ have the conformal structure of spheres $S^\pm$, with the points 0 and ∞ identified. The coordinates on the spheres are unique up to linear changes of coordinates. Then the Ecalle-Voronin modulus is the equivalence class of pairs of germs (ψ^0, ψ^∞) of analytic diffeomorphisms, where $\psi^0 : (S^-, 0) \to (S^+, 0)$ and $\psi^\infty : (S^-, \infty) \to (S^+, \infty)$ are defined respectively in the neighborhoods of 0 and ∞, under conjugation by linear changes of coordinates in the source and target space. The map f_0 is not iterable (non embedable) as soon as one of the two germs ψ^0 or ψ^∞ is nonlinear.

Definition 2.5 The map f_0 is called *semi-iterable* in Ecalle's terminology [3] if one of ψ^0 or ψ^∞ is linear.

The unfolded Ecalle-Voronin modulus. In [12] it is proved that for any sufficiently small neighborhood U of the origin in z-space there exists a small neighborhood V of the origin in parameter space ϵ such that for each $\epsilon \in V$ the orbit space is described as follows

- There exists two crescents $C_\epsilon^\pm$ with end points at the two singular points bounded by curves $l_{\pm,\epsilon}$ and their images $f_\epsilon(l_{\pm,\epsilon})$ (Figure 2).

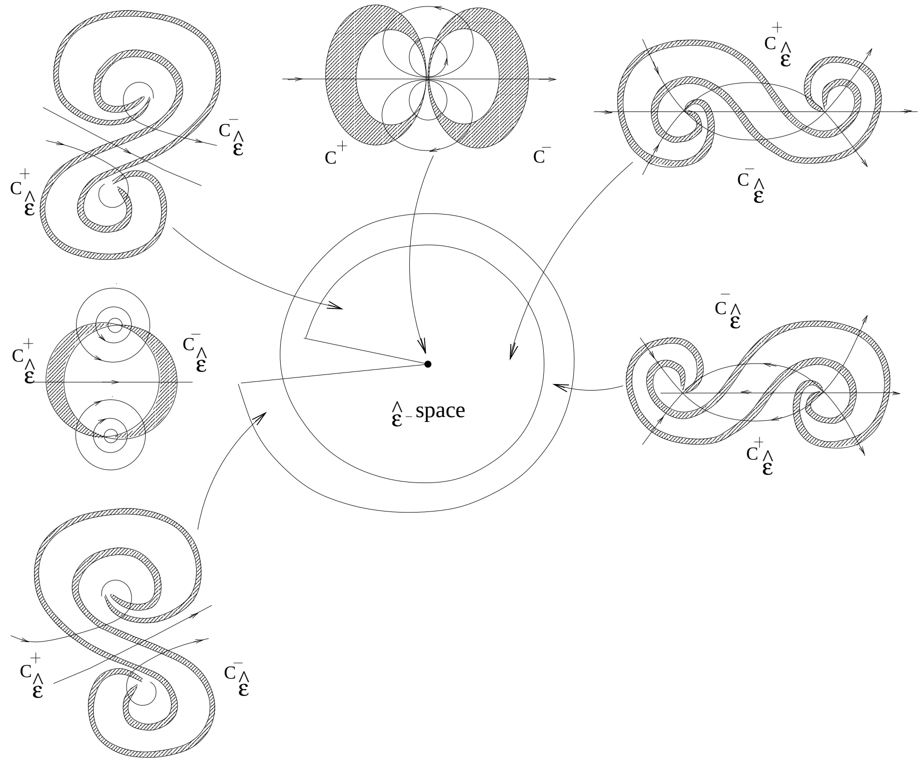


Figure 2: The modulus for the family

- The crescents $C_\epsilon^\pm$ in which we identify the curves $l_{\pm,\epsilon}$ and their images $f_\epsilon(l_{\pm,\epsilon})$ have the conformal structure of spheres $S_\epsilon^\pm$ with the singular point $\sqrt{\epsilon}$ (resp. $-\sqrt{\epsilon}$) located at ∞ (resp. 0).

- Points in the two neighborhoods of 0 and ∞ on the spheres $S_\epsilon^\pm$ are identified modulo analytic maps, $\psi_\epsilon^0, \psi_\epsilon^\infty : S_\epsilon^- \to S_\epsilon^+$, defined in the neighborhoods of 0 and

∞ respectively. These maps are obviously uniquely defined up to the choice of coordinates on the spheres. Hence it is natural to consider the equivalence classes of pairs $(\psi_\epsilon^0, \psi_\epsilon^\infty)$ under the equivalence relation:

$$(\psi_\epsilon^0, \psi_\epsilon^\infty) \sim (\tilde{\psi}_\epsilon^0, \tilde{\psi}_\epsilon^\infty) \Longleftrightarrow \exists C, C' \in \mathbb{C} \begin{cases} \tilde{\psi}_\epsilon^0(w) = C'\psi_\epsilon^0(Cw) \\ \tilde{\psi}_\epsilon^\infty(w) = C'\psi_\epsilon^\infty(Cw) \end{cases} \tag{2.19}$$

Let us denote the equivalence class of $(\psi_\epsilon^0, \psi_\epsilon^\infty)$ by $[(\psi_\epsilon^0, \psi_\epsilon^\infty)]$.

The family $\{[(\psi_\epsilon^0, \psi_\epsilon^\infty)]\}_{\epsilon \in V}$ is a complete modulus of analytic classification for the prepared family (2.15).

The dependence of the modulus on ϵ. In general the modulus does not depend continuously on $\epsilon \in V$. However given $\delta > 0$ arbitrarily small there exists V a sufficiently small neighborhood of 0 in ϵ-space such that if we limit ourselves to values of $\hat{\epsilon}$ in a sectorial neighborhood $\arg \hat{\epsilon} \in (-\pi + \delta, 3\pi - \delta)$ of the universal covering of ϵ-space which projects onto V then it is possible to take representatives of the modulus $\psi_{\hat{\epsilon}}^{0,\infty}$ which depend analytically on $\hat{\epsilon} \neq 0$ and continuously on $\hat{\epsilon}$ at $\hat{\epsilon} = 0$.

From the unfolded modulus we can deduce the dynamics near each of the fixed points by means of a renormalized return map. This dynamics is only interesting when the multiplier is on the unit circle, since in the other cases the fixed points are linearizable.

The renormalized return maps. These maps are defined on one sphere, for instance S_ϵ^-. In the neighborhood of $\pm\sqrt{\epsilon}$ which we identify to ∞ and 0 on S_ϵ^- we define return maps by iterating f_ϵ until the image is contained in S_ϵ^-: given $z \in C_\epsilon^-$ in the neighborhood of $\sqrt{\epsilon}$ (resp. $-\sqrt{\epsilon}$) and w its coordinate on S_ϵ^-, let $n \in \mathbb{N}$ be minimum such that $f^n(z) \in C_\epsilon^-$ and let $k_\epsilon^\infty(w)$ (resp. $k_\epsilon^0(w)$) be its coordinate on S_ϵ^-. Then k_ϵ^∞ (resp. k_ϵ^0) is the return map in the neighborhood of $\sqrt{\epsilon}$ (resp. $-\sqrt{\epsilon}$). These return maps are given by the composition of the maps ψ_ϵ^0 and ψ_ϵ^∞ with a global transition map $L_\epsilon : S_\epsilon^+ \to S_\epsilon^-$, the Lavaurs map. The Lavaurs map is an analytic map from $\mathbb{CP}^1$ to $\mathbb{CP}^1$ fixing 0 and ∞. Hence it is linear, yielding that the nonlinear part of the return map comes from the unfolding of the two components of the Ecalle-Voronin modulus. Let us call these two return maps $k_\epsilon^- = L_\epsilon \circ \psi_\epsilon^0$ and $k_\epsilon^+ = L_\epsilon \circ \psi_\epsilon^\infty$.

The interpretation of the renormalized return maps when the fixed points have multipliers of modulus 1. In order to be able to get conclusions on the dynamics for $\epsilon \neq 0$ from the Ecalle-Voronin invariant of the diffeomorphism for $\epsilon = 0$ we need to have continuity in ϵ. Note that the domain in $\hat{\epsilon}$-space on which ψ_ϵ^0 and ψ_ϵ^∞ are defined covers exactly once the semi-axis $\mathbb{R}^-$. On this semi-axis we have a splitting of the Ecalle-Voronin invariant, namely the dynamics of $-\sqrt{\epsilon}$ (resp. $\sqrt{\epsilon}$) is controlled by ψ_ϵ^0 (resp. ψ_ϵ^∞). Moreover all the properties of nonlinearity of $\psi^{0,\infty}$ are inherited by $\psi_\epsilon^{0,\infty}$ for small ϵ.

We now limit ourselves to discuss the dynamics near $-\sqrt{\epsilon}$. The first derivative $(k_\epsilon^0)'(0) = L_\epsilon'(0)(\psi_\epsilon^0)'(0)$ is intrinsic. If $\lambda_0 = \exp(2\pi i \mu_0)$ is the multiplier of f_ϵ at $-\sqrt{\epsilon}$ then $(k_\epsilon^0)'(0) = \exp(-2\pi i/\mu_0)$. As ψ_ϵ^0 is uniquely defined only up to linear changes of coordinates in the source and target space we can always suppose that $(\psi_\epsilon^0)'(0) = 1$ and $L_\epsilon(w) = \exp(-2\pi i/\mu_0)w$.

The following theorem is proved in [12]. It shows how the non-normalizability of the diffeomorphism f_0 implies the non-linearizability of the perturbed diffeomorphisms f_ϵ for some (but not necessarily all) small values of ϵ.

Theorem 2.6 *Let $\tau = \exp(2\pi i p/q)$, $\psi^0(w) = w + \sum_{i\geq 2} a_i w^i$ and let $g^0(w) = \tau\psi^0(w)$. If τ and ψ^0 are such that the q-th iterate of g^0, namely $(g^0)^q(w) = w + \sum_{i\geq q+1} b_i w^i$ be nonlinear, i.e. there exists $i \geq q+1$ such that $b_i \neq 0$, then there exists $N \in \mathbb{N}$ such that if $n > N$ and ϵ is such that $\mu_0 = \frac{q}{p-nq}$, then the fixed point $-\sqrt{\epsilon}$ is non linearizable.*

Similarly to study the neighborhood of $\sqrt{\epsilon}$ we localize ∞ at 0 by taking $\tilde{w} = 1/w$. From (2.17) it follows that we cannot simultaneously scale $(\psi_\epsilon^0)'(0) = 1$ and $(\psi_\epsilon^\infty)'(0) = 1$ unless $a(\epsilon) \in 2\pi i\mathbb{Z}$ ($a(\epsilon)$ is interpreted as a shift between the two singular points). We now choose the coordinates on $S_\epsilon^\pm$ such that $(\psi_\epsilon^\infty)'(0) = 1$, where $\tilde{\psi}_\epsilon^\infty$ is ψ_ϵ^∞ in the variable $\tilde{w}$.

Theorem 2.7 *Let $\tau = \exp(-2\pi i(p/q)$, $\tilde{\psi}^\infty(\tilde{w}) = \tilde{w} + \sum_{i\geq 2} c_i \tilde{w}^i$ and let $g^\infty(\tilde{w}) = \tau\psi^\infty(\tilde{w})$. If τ and $\tilde{\psi}^\infty$ are such that $(g^\infty)^q(\tilde{w}) = \tilde{w} + \sum_{i\geq q+1} d_i \tilde{w}^i$ is nonlinear, i.e. there exists $i \geq q+1$ such that $d_i \neq 0$, then there exists $N \in \mathbb{N}$ such that if $n > N$ and ϵ is such that $\mu_\infty = \frac{q}{p+nq}$, then the fixed point $\sqrt{\epsilon}$ is non linearizable.*

2.4 The link between the Ecalle-Voronin modulus and the Martinet-Ramis modulus

In the case of a saddle-node the orbit space of the holonomy map allows to define a (ramified) first integral. Indeed the orbit space is given by two copies of $\mathbb{CP}^1$ identified in the neighborhoods of 0 and ∞. Any leaf of the foliation except the center manifold intersects a section $\{y = y_0\}$ on which the holonomy map is defined. Hence it suffices to define the first integral on the section and to extend it by the value 0 on the center manifold. It is defined (in a multivalued way) by the points of $\mathbb{CP}^1$ which belong to its orbit.

3 The monodromy group of a separatrix

In this section we consider a saddle point with a separatrix given by either an invariant line or a non singular conic and give sufficient conditions for the integrability of a saddle point by looking at the monodromy group of the separatrix. We apply this to the Lotka-Volterra equations, to obtain four classes of explicit conditions which give integrable (orbitally linearizable) or normalizable critical points.

The surprising thing is that, even though these conditions on the monodromy groups are elementary, they comprise all the known cases of integrability for the Lotka-Volterra equations, except for the case where the system has an invariant straight line (covered in the next section) and two exceptional Darboux integrable cases [1, 15].

Consider the foliation on $\mathbb{CP}^2$ generated by the 1-form associated to the vector field. Let Γ be an invariant line or conic for the 1-form, and $Q_1, \dots Q_n$ be the singular points of the foliation which lie on Γ. For (1.2) we have three such lines: the two axes and the line at infinity. Clearly $\Gamma' = \Gamma \setminus \{Q_1, \dots, Q_n\}$ is isomorphic to an n-punctured sphere.

Choose a family of analytic transversals, Σ_x, through each point x in Γ', and fix a base point, P, in Γ', and an analytic parameterization z of Σ_P with $z = 0$ corresponding to the point P. For each path γ in $\pi(\Gamma', P)$, we can define a map from a neighborhood of P in Σ_P to Σ_P by lifting the path γ to the leaf of the foliation though $s \in \Sigma_P$ via the transversals Σ_x, $x \in \gamma$. Using the parameter z, this map can be identified with the germ of a diffeomorphism from $\mathbb{C}$ to itself, fixing $z = 0$. We call the set of all such diffeomorphisms $\mathrm{Diff}(\mathbb{C}, 0)$.

Clearly the map $M : \pi(\Gamma', P) \to \mathrm{Diff}(\mathbb{C}, 0)$ is in fact a group homomorphism. We denote the image of the path γ by M_γ. The *monodromy group* is the image of M. The *monodromy of one singular point* Q_i is M_γ where γ is a loop turning around Q_i exactly once in the positive direction and not containing any other singular point in its interior.

Remark 3.1

1. M_γ depends only of the homotopy type of γ in Γ'.
2. If we use a different base point P_1 then the two monodromy groups are conjugate. Likewise a different choice of transversals and their parameterizations, has the effect of conjugating the group. Thus the following notions for the monodromy of a singular point are intrinsic:
   - the monodromy of the singular point is the identity;
   - the monodromy of the singular point is linearizable;
   - the monodromy of the singular point is *normalizable*, i.e. it is the time-one map of a flow (Ecalle [3] would call it iterable). Note that the monodromy of an orbitally normalizable vector field is normalizable. This is because it is true for (2.4).

Theorem 3.2 *Consider a polynomial system with a saddle point at the origin*

$$\begin{aligned} \dot{x} &= x(1 + P(x, y)) = x(1 + O(x, y)) \\ \dot{y} &= -\lambda y + Q(x, y) = -\lambda y + o(x, y), \end{aligned} \tag{3.1}$$

where $\lambda > 0$. If all singular points of the system on the y-axis except the origin are integrable and if all of them but one have identity monodromy maps corresponding to the invariant y-axis then the origin is also integrable.

Proof. We consider the completion of the line $x = 0$ as the Riemann sphere S^1. Let $Q_1, \ldots, Q_n$ be the singular points of the system on that leaf.

Let Q_i be a point of saddle or node type. It is known that Q_i is integrable if and only if the corresponding monodromy map is linearizable (this is proved in [14] and [16] for a saddle. For a node it can easily be proved by considering the analytic normal form at the node).

Take a base point $y_0 \in S^1 \setminus \{Q_1, \ldots, Q_n\}$ and loops γ_i from y_0 winding once around the singular points Q_i in the positive sense, then γ_1 is homotopic to $\gamma_n^{-1} \circ \cdots \circ \gamma_2^{-1}$, with appropriate re-labelling of the Q_i. As a result M_{γ_1} is conjugate to $M_{\gamma_n}^{-1} \circ \cdots \circ M_{\gamma_2}^{-1}$. Since all of them are the identity except one which is linearizable then the map M_{γ_1} is linearizable. □

Corollary 3.3 *Consider a polynomial system* (1.1) *with a saddle point at the origin where* $\lambda > 0$. *If all singular points of the system on the* y*-axis except the origin and a singular point* Q *have identity monodromy maps corresponding to the invariant* y*-axis and if the point* Q *is orbitally normalizable then the origin is orbitally normalizable.*

Proof. The proof follows exactly the same lines as the ones of the preceding theorem using the fact that a point is orbitally normalizable if and only if its holonomy map is normalizable, i.e. is the time-1 map of a vector field. □

Remark 3.4 These results can clearly be applied to systems with an invariant conic in the same way. In fact, we can always arrange for the conic to be a line by a projective change of coordinates.

We apply these results to the Lotka-Volterra family (1.2). This family is invariant under

$$(x, y, t, \lambda, a, b, c, d) \mapsto (-\lambda y, -\lambda x, -\frac{t}{\lambda}, \frac{1}{\lambda}, d, c, b, a) \tag{3.2}$$

and corresponding cases under this invariance are called *dual.*

Lemma 3.5 *A node is linearizable if and only it it has two analytic separatrices.*

Proof. A node with eigenvalues λ_1, λ_2 whose quotient is in $\mathbb{R}^+$ can always be brought to normal form by an analytic change of coordinates. When $\frac{\lambda_2}{\lambda_1} \notin \mathbb{N} \cup 1/\mathbb{N}$ then the normal form is linear and the two axes are analytic separatrices. When $\frac{\lambda_2}{\lambda_1} = n \in \mathbb{N}$ the normal form is

$$\begin{aligned} \dot{x} &= \lambda_1 x \\ \dot{y} &= \lambda_2 y + \alpha x^n \end{aligned} \tag{3.3}$$

If $\alpha = 0$ then the system is linear as before and all integral curves through the origin are analytic, while if $\alpha \neq 0$ the curve $x = 0$ is the unique analytic integral curve through the origin. Similarly for $\frac{\lambda_2}{\lambda_1} \in 1/\mathbb{N}$. □

Theorem 3.6 *We consider the Lotka-Volterra system (1.2) with $\lambda > 0$. Then the origin is integrable if one of the following conditions is satisfied.*

(A_n). $\lambda + \frac{c}{a} = n$ *with* $n \in \mathbb{N}$, $2 \le n < \lambda + 1$.

(B_n). $\frac{b}{d} + \frac{1}{\lambda} = n$ *with* $n \in \mathbb{N}$, $2 \le n < \frac{1}{\lambda} + 1$.

(C_n). $\frac{c}{a} + n = 0$ *with* $n \in \mathbb{N} \cup \{0\}$ *and* $n < \lambda$ *and* $\lambda \neq n + \frac{1}{m}$ *with* $m \in \mathbb{N}$. *If* $\lambda = n + \frac{1}{m}$ *then the origin is orbitally normalizable and an additional condition is necessary for integrability. In particular* $c = 0$ *is an integrability condition unless* $\lambda = \frac{1}{m}$, *in which case the system is only orbitally normalizable and an additional integrability condition,* $b + (m-1)d = 0$ *is necessary [4].*

(D_n). $\frac{b}{d} + n = 0$ *with* $n \in \mathbb{N} \cup \{0\}$ *and* $n < \frac{1}{\lambda}$ *and* $\frac{1}{\lambda} \neq n + \frac{1}{m}$ *with* $m \in \mathbb{N}$. *If* $\frac{1}{\lambda} = n + \frac{1}{m}$ *then the origin is orbitally normalizable and an additional condition is necessary for integrability. In particular* $b = 0$ *is an integrability condition unless* $\lambda = m$, *in which case the system is only orbitally normalizable and an additional integrability condition,* $(m-1)a + c = 0$, *is necessary [4].*

$(E_{n,m})$. $\lambda + \frac{c}{a} = n$ *and* $1 - \frac{b}{d} = \frac{1}{m}$ *with* $n, m \in \mathbb{N}$, $n > 1$ *and* $0 < \frac{(c-a)(d-b)}{ad-bc} \notin \mathbb{N}$. *If the last expression is an integer, then the critical point may only be orbitally normalizable.*

$(F_{n,m})$. $\frac{1}{\lambda} + \frac{b}{d} = n$ *and* $1 - \frac{c}{a} = \frac{1}{m}$ *with* $n, m \in \mathbb{N}$, $n > 1$ *and* $0 < \frac{(c-a)(d-b)}{ad-bc} \notin \mathbb{N}$. *If the last expression is an integer, then the critical point may only be orbitally normalizable.*

$(G_{n,m})$. $\lambda + \frac{c}{a} = n$, $1 - \frac{b}{d} > 0$ *and* $\frac{ad-bc}{(c-a)(d-b)} = m$ *with* $m, n \in \mathbb{N} \setminus \{1\}$.

$(H_{n,m})$. $\frac{1}{\lambda} + \frac{b}{d} = n$, $1 - \frac{c}{a} > 0$ *and* $\frac{ad-bc}{(c-a)(d-b)} = m$ *with* $n, m \in \mathbb{N} \setminus \{1\}$.

(Note that some strata with different names may be identical for some values of λ *and of the indices. This can for instance happen with* $(E_{n,m})$ *and* $(G_{n,m'})$*.)*

Proof. To apply the previous theorem and corollary we need to calculate the Jacobian matrix and the eigenvalues at all singular points along the axes and along infinity. On each separatrix there are three critical points: the one at the origin with ratio of eigenvalues $-\lambda$, one in the finite plane, and one where the axes cross the line at infinity. The Jacobians for the finite critical points $P_1 = (-\frac{1}{a}, 0)$ (resp. $P_2 = (0, \frac{\lambda}{d})$) on the x-axis (resp. y-axis) are

$$\begin{pmatrix} -1 & -\frac{b}{a} \\ 0 & -\lambda - \frac{c}{a} \end{pmatrix} \quad \textit{resp.} \quad \begin{pmatrix} 1 + \lambda\frac{b}{d} & 0 \\ \lambda\frac{c}{d} & \lambda \end{pmatrix}. \tag{3.4}$$

showing that the monodromy of the finite critical points on the x-axis (resp. y-axis) is the identity if $\lambda + \frac{c}{a} = n$ (resp. $\frac{b}{d} + \frac{1}{\lambda} = n$) with $n \in \mathbb{N}$, $n \ge 2$.

We now study the singular points at infinity. For that purpose we first consider the chart $(u,z) = (y/x, 1/x)$ to calculate the Jacobian matrix at the intersection of the line at infinity with the x-axis, which we denote $P_x = (0,0)$. We can also calculate the Jacobian at the other critical point $P_\infty = (\frac{a-c}{d-b}, 0)$ on the line at infinity. After multiplication by z, the system becomes:

$$\begin{aligned}\dot{u} &= (c-a)u + (d-b)u^2 - (1+\lambda)uz \\ \dot{z} &= -az - buz - z^2,\end{aligned} \tag{3.5}$$

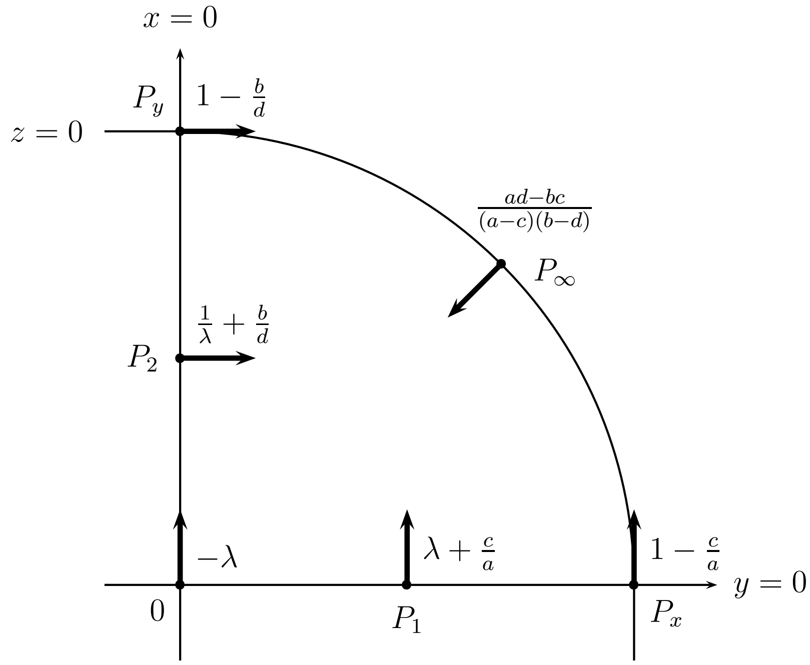
yielding the following Jacobian matrices for P_x and P_∞:

$$\begin{pmatrix} c-a & * \\ 0 & -a \end{pmatrix} \quad \textit{resp.} \quad \begin{pmatrix} -(c-a) & * \\ 0 & \frac{ad-bc}{b-d} \end{pmatrix}. \tag{3.6}$$

Similarly the chart $(v,w) = (x/y, 1/y)$ is used to study the infinite singular point P_y along the y-axis. Its Jacobian matrix is given by

$$\begin{pmatrix} b-d & * \\ 0 & -d \end{pmatrix}. \tag{3.7}$$

We can represent the ratios of eigenvalues on the diagram below, where the arrows represents the direction of the eigenvalue which is the numerator of the eigenvalue ratio.



Note that the sum of the eigenvalue ratios along any line is equal to 1. This follows from the index formula of Lins Neto [11].

We now prove the cases (A)–(H) given above. We may remove the indices when they are not necessary.

Case $(A_n)/(B_n)$: In Case (A_n), the condition implies that the monodromy of P_1 corresponding to the invariant x-axis is the identity and the critical point P_x is a node. It is always linearizable since there are two analytic separatrices. Case (B_n) is the dual of Case (A_n).

Case $(C_n)/(D_n)$: Case (C_n) is similar to Case (A), but now the monodromy at P_x is the identity corresponding to the invariant x-axis, and P_1 is a node. It is linearizable if $\lambda \neq n + \frac{1}{m}$ with $m \in \mathbb{N}$, and normalizable otherwise (the case of a resonant node). In this case the obstruction to linearizability consists of only one condition. Case (D_n) is the dual of Case (C_n).

Case $(E_{n,m})/(F_{n,m})$: Case $(E_{n,m})$ requires a double application of Theorem 3.2. The conditions imply that the monodromy of P_1 corresponding to the invariant x-axis is the identity. Thus the monodromy at the origin is conjugate to the inverse of the monodromy of P_x corresponding to the invariant x-axis. Now, this monodromy is linearizable (resp. normalizable) if and only if P_x is integrable (resp. orbitally normalizable). This is the case if and only if the monodromy of P_x corresponding to the other separatrix (in this case, the line at infinity) is linearizable (resp. normalizable). Now, the conditions given in Case $(E_{n,m})$ guarantee that the monodromy of P_y corresponding to the line at infinity is the identity. (P_y is a node with ratio of eigenvalues $m \in N$). Hence the origin is integrable (resp. orbitally normalizable) if and only if the monodromy of P_∞ is linearizable (resp. normalizable) corresponding to the line at infinity. Now the final condition in Case $(E_{n,m})$ guarantees that P_∞ is a non-resonant node, and therefore linearizable. If the condition is relaxed then the node can be resonant and we can only deduce the normalizability of the origin (unless we perform additional calculations). Case $(F_{n,m})$ is the dual of Case $(E_{n,m})$.

Case $(G_{n,m})/(H_{n,m})$: Case $(G_{n,m})$ is the same as Case (E) except that now, the monodromy of P_∞ corresponding to the line at infinity is the identity and the point P_y is a node (necessarily linearizable). Case $(H_{n,m})$ is the dual of Case $(G_{n,m})$. □

4 Integrable and linearizable systems in the Lotka-Volterra family

Before studying the integrable and linearizable Lotka-Volterra system for rational values of λ let us remark the following general phenomenon on quadratic systems. Recall that a *Darboux factor* for a polynomial vector field X of degree m is an analytic function f such that $X(f) = fL$, for some polynomial L of degree at most $m-1$, called the *cofactor* of f.

Theorem 4.1 *We consider a quadratic vector field*

$$\begin{aligned} \dot{x} &= x + P_2(x,y) = x + o(x,y) \\ \dot{y} &= -\lambda y + Q_2(x,y) = -\lambda y + o(x,y) \end{aligned} \tag{4.1}$$

with $\lambda > 0$ *for which the two separatrices of the origin are analytic Darboux factors* $F_1(x,y) = x + o(x,y)$ *and* $F_2(x,y) = y + o(x,y)$ *with respective cofactors* K_1 *and* K_2.

If K_1, K_2 and the divergence, div, are linearly independent then the origin is integrable if and only if it is linearizable.

Proof. Let us suppose that the origin is integrable. Then we have a first integral of the form $H(x,y) = F_1^\lambda F_2 \phi(x,y)$ with $\phi(x,y) = 1 + O(x,y)$ analytic. Then the function $\phi(x,y)$ is a Darboux factor whose cofactor $K_3(x,y)$ is a linear combination of K_1 and K_2. The first integral $H_1 = \ln H$ corresponds to an integrating factor $V(x,y) = F_1 F_2\, \psi(x,y)$ whose cofactor K_4 is given by the divergence plus a linear combination of K_1 and K_2. Hence F_1, F_2 and $\psi(x,y)$ have linearly independent cofactors. Moreover the cofactors of ϕ and ψ have no constant term. Hence it is possible to find functions $X = F_1 \phi^{\alpha_1} \psi^{\alpha_2}$ and $Y = F_2 \phi^{\beta_1} \psi^{\beta_2}$ with respective cofactors 1 and $-\lambda$, yielding a linearizing change of coordinates. □

Corollary 4.2 *In the Lotka-Volterra family* (1.2) *any integrable system is linearizable as soon as*

$$\lambda ab + (1-\lambda)ad - cd \neq 0. \tag{4.2}$$

This condition is equivalent to the fact that the system does not have a third invariant line, $1 + ax - dy/\lambda = 0$, or, when $a = d = 0$, an exponential factor e^{cx-by}.

Thus the question of linearizability for integrable systems need only be considered when the system has an invariant line or an exponential factor of the form above (we shall usually include this latter case with the former unless otherwise stated). This will be studied in the next section. We now limit ourselves to the problem of integrability.

We have the following conjecture.

Conjecture 4.3 The Lotka-Volterra system (1.2) with $\lambda \in \mathbb{Q}^+$ is integrable if and only if either

1. the system has a third invariant line, i.e.

$$\lambda ab + (1-\lambda)ad - cd = 0; \tag{4.3}$$

2. one of the conditions of Theorem 3.6 is satisfied;

3. or there is an invariant algebraic curve, $f = 0$.

In fact, we know from the lists given in [1, 15] that there are essentially only two cases (with $\lambda = 8/7$ and $13/7$ and their duals) where this last condition holds, which are not contained in the previous two conditions. These (after scaling) are the systems

$$\begin{aligned} \dot{x} &= x(1 - 2x + y) \\ \dot{y} &= y(-\tfrac{8}{7} + 4x + y) \end{aligned} \tag{4.4}$$

with invariant cubic

$$F(x,y) = 1372xy(3x - y) - 1764xy - 63y - 72 = 0, \tag{4.5}$$

and

$$\begin{array}{rcl} \dot{x} & = & x(1-2x+y) \\ \dot{y} & = & y(-\frac{13}{7}+4x+y) \end{array} \tag{4.6}$$

with the invariant quartic

$$F(x,y) = 343x^2y(3x-y) - 588x^2y + 21xy + 18x - 9 = 0, \tag{4.7}$$

together with their duals.

Theorem 4.4 *The conjecture is proved for $\lambda = \frac{p}{q}$ with $p+q \leq 12$ and all $\lambda = \frac{n}{2}$ and $\lambda = \frac{2}{n}$ for $n \in \mathbb{N}$. Moreover for $\lambda = \frac{n-1}{n}$ the systems satisfying $b+d=0$ are orbitally normalizable. For $\lambda = \frac{3}{7}$ the systems satisfying $b+2d=0$ are orbitally normalizable.*

Proof. The proof consists in calculating the saddle quantities and checking that they vanish only either for (4.3) or under one of the conditions of Theorem 3.6 (see calculations below and [4, 6]). The process allows to find some orbitally normalizable systems. We can of course, using duality, limit ourselves to $\lambda \leq 1$. The conditions for $\lambda = 1/n$ and $2/n$ for $n \in \mathbb{N}$ were given in [4] and [6] respectively. In these cases we can prove that the list of conditions is necessary and sufficient by a counting argument: it is easy to prove that the first two saddle quantities cannot vanish elsewhere than the known sufficient conditions.

	$\lambda = \frac{1}{n},\ n \in \mathbb{N}$
invariant line	$ab + (n-1)ad - ncd = 0$
(D_0)	$b = 0$ if $n > 1$, $b = 0 = c$ if $n = 1$
$(D_m)_{1 \leq m \leq n-2}$	$md + b = 0$

	$\lambda = \frac{2}{n},\ \frac{n-1}{2} \in \mathbb{N}$
invariant line	$2ab + (n-2)ad - ncd = 0$
$(B_k)_{2 \leq k \leq \frac{n+1}{2}}$	$2b + (n-2k)d = 0$
$(D_q)_{0 \leq q \leq \frac{n-3}{2}}$	$b + qd = 0$

In all cases where $\lambda \notin 1/\mathbb{N}$, then $b=0$ and $c=0$ are strata of integrable systems by (D_0) (resp. (C_0)). We also have the stratum $\lambda ab + (1-\lambda)ad - cd = 0$ which corresponds to a system with a third invariant line. The other explicit strata of integrable systems for the following values of λ: $\lambda = \frac{3}{4}, \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, \frac{4}{5}, \frac{4}{7}, \frac{5}{6}, \frac{5}{7}$ appear in Table 1. Each case is named by the condition in the theorem yielding the integrability. $\square$

	$\lambda = \frac{3}{4}$		$\lambda = \frac{3}{5}$
(B_2)	$3b - 2d = 0$	(B_2)	$3b - d = 0$
(D_1)	$4a + 13c = b + d = 0$	(D_1)	$b + d = 0$
$(E_{2,2})$	$5a - 4c = 2b - d = 0$	$(E_{2,2})$	$7a - 5c = d - 2b$
$(H_{3,2})$	$a + c = 3b - 5d = 0$	$(F_{3,2})$	$a - 2c = 4d - 3b$

	$\lambda=\frac{3}{7}$		$\lambda=\frac{3}{8}$
(B_2)	$3b+d=0$	(B_2)	$3b+2d=0$
(B_3)	$3b-2d=0$	(B_3)	$3b-d=0$
(D_1)	$b+d=0$	(D_1)	$b+d=0$
(D_2)	$2a+11c=b+2d=0$	(D_2)	$b+2d=0$
$(E_{2,2})$	$11a-7c=2b-d=0$	$(E_{2,2})$	$13a-8c=2b-d=0$
$(H_{4,2})$	$a+c=3b-5d=0$	$(F_{4,2})$	$a-2c=3b-4d=0$

	$\lambda=\frac{4}{5}$		$\lambda=\frac{4}{7}$
(B_2)	$4b-3d=0$	(B_2)	$4b-d=0$
(D_1)	$6a^2+33ac+43c^2=b+d=0$	(D_1)	$b+d=0$
$(E_{2,2})$	$6a-5c=2b-d=0$	$(E_{2,2})$	$10a-7c=2b-d=0$
$(E_{2,3})$	$6a-5c=3b-2d=0$	$(F_{3,2})$	$a-2c=4b-5d=0$
$(H_{3,2})$	$2a+c=4b-7d=0$	$(F_{3,3})$	$2a-3c=4b-5d=0$
		$(G_{2,3})$	$10a-7c=b+2d=0$

	$\lambda=\frac{5}{6}$
(B_2)	$5b-4d=0$
(D_1)	$288a^3+2128a^2c+5013ac^2+3798c^3=b+d=0$
$(E_{2,2})$	$7a-6c=2b-d=0$
$(E_{2,3})$	$7a-6c=3b-2d=0$
$(E_{2,4})$	$7a-6c=4b-3d=0$
$(H_{3,2})$	$3a+c=5b-9d=0$

	$\lambda=\frac{5}{7}$
(B_2)	$5b-3d=0$
(D_1)	$b+d=0$
$(E_{2,2})$	$9a-7c=2b-d=0$
$(E_{2,3})$	$9a-7c=3b-2d=0$
$(F_{3,2})$	$a-2c=5b-8d=0$
$(G_{2,3})$	$9a-7c=3b-d=0$
$(H_{3,2})$	$a+2c=5b-8d=0$

Table 1

The following conjecture may be simpler to verify than Conjecture 4.3

Conjecture 4.5 Except for the case $\lambda=7/8$ the Lotka-Volterra system (1.2) with $\lambda=\frac{m}{m+1}$, $m>2$ is integrable if and only one of the following conditions is satisfied:

$$\begin{array}{llll} b=0, & c=0, & mab+ad-(m+1)cd=0, & \\ (B_2), & (D_1), & (E_{2,2})-(E_{2,m-1}), & (H_{3,2}). \end{array} \tag{4.8}$$

Proposition 4.6 *1. Conjecture* 4.5 *is true for* $m = 2, 3, 4, 5, 6$.

2. For $\lambda = 7/8$ *the system is integrable if and only if either one of the conditions of* (4.8)*, or the additional condition* $2a + d = 4b - c = 0$ *is satisfied. The later case corresponds to a Darboux integrable system with an additional cubic curve dual to* (4.5)*.*

Proof. For $m = 2, 3, 4, 5$ the proof follows from Table 1. For $\lambda = 7/8$ the necessity comes from the calculation of the saddle quantities. The sufficiency of the additional case comes from the fact that the system is dual to (4.4). □

5 Linearizable Lotka-Volterra systems with an invariant line

We now consider the Lotka-Volterra system (1.2) with an invariant line i.e.

$$\lambda ab + (1 - \lambda)ad - cd = 0. \tag{5.1}$$

is satisfied. In this case the origin is always integrable (it has a Darboux first integral), but Theorem 4.1 does not apply; however, we are able to give a complete characterization of the conditions under which the origin is linearizable. The case $a = d = 0$ is treated in Proposition 5.4.

We first consider the case when λ is rational.

Theorem 5.1 *The origin of* (1.2) *with* $\lambda = \frac{p}{q}$ *and with an invariant line (i.e.* (5.1) *is satisfied) is linearizable if and only if one of the following conditions is satisfied:*

1. $b = d = 0$;

2. $a = c = 0$;

3. $k = (1 - c/a)/(1 + \lambda) = (1 - b/d)/(1 + 1/\lambda)$ *takes one of the values* $k = 0, 1/(p + q), \ldots, (p + q - 1)/(p + q)$.

Proof. In the case $b = 0$, the system (1.2) is integrable if and only if it is linearizable, since the first equation is linearizable by the substitution $X = x/(1 + ax)$. From (5.1), we have $d = 0$ or $c = (1 - \lambda)a$, the first condition gives $b = d = 0$, and the second gives either $a = c = 0$ which is the dual case to $b = d = 0$, $1 - c/a = \lambda$. This latter case, is just the third condition above with $k = p/(p + q)$. The case $c = 0$ follows dually.

Thus, we can assume that $b, c \neq 0$. Without loss of generality we shall take $b = c = 1$ by an appropriate scaling; then the system has an invariant line if and only if

$$ad(\lambda - 1) + d - \lambda a = 0. \tag{5.2}$$

It is easy to show that the invariant line in the system (1.2) with (5.1) can be written as

$$f = 1 + l = 1 + ax - dy/\lambda = 0 \tag{5.3}$$

with cofactor $ax + dy$.

As mentioned before, if either a or d are zero then both are zero and the system is integrable but not linearizable (see Proposition 5.4 below).

The system has a first integral ϕ of the form

$$\phi = x^\lambda y f^C, \tag{5.4}$$

where

$$C = -(a\lambda + 1)/a = -(\lambda + d)/d. \tag{5.5}$$

The change of coordinates

$$X = xf^r, \qquad Y = yf^r, \tag{5.6}$$

where $r = C/(1+\lambda)$ brings ϕ to $\phi = X^\lambda Y$. Using these new coordinates, we find that

$$\dot{X} = XM, \qquad \dot{Y} = -\lambda Y M, \tag{5.7}$$

where

$$M = 1 + ax(1+r) + y(1+rd) = 1 + (1+r)l, \tag{5.8}$$

the last equality following from (5.5). Now, from [2] the system is linearizable if and only if there is a function h such that $\dot{h} = 1 - M$, or alternatively,

$$Xh_X - \lambda Y h_Y = 1/M - 1. \tag{5.9}$$

If $1/M - 1 = \sum_{k_1+k_2>0} \beta_{k_1k_2} X^{k_1} Y^{k_2}$ and $h(X,Y) = \sum_{k_1+k_2>0} \gamma_{k_1k_2} X^{k_1} Y^{k_2}$ then solving (5.9) is equivalent to

$$(k_1 - \lambda k_2)\gamma_{k_1k_2} = \beta_{k_1k_2}. \tag{5.10}$$

The rest of the section is devoted to giving an explicit expression for $1/M - 1$ in terms of X and Y. For λ rational there will be an obstruction to the existence of a solution of (5.9) if all $\beta_{k_1k_2}$ are nonzero.

First of all, we take $L = aX - dY/\lambda$. Then

$$f^{r+1} = f^r(1 + ax - dy/\lambda) = f^r + L, \tag{5.11}$$

whence

$$l(1+l)^r = L. \tag{5.12}$$

Thus, there is an expansion of l as a power series in L, and hence an expansion of $1/M$ in a power series in L. We let

$$\frac{1}{M} - 1 = \sum_{i>0} \alpha_i L^i, \tag{5.13}$$

Clearly, to calculate the coefficient of $X^{k_1}Y^{k_2}$ in $1/M$ it is sufficient to calculate the coefficient $\alpha_{k_1+k_2}$ of $L^{k_1+k_2}$ in the expansion above and multiply by $(-1)^{k_2}a^{k_1}(d/\lambda)^{k_2}\binom{k_1+k_2}{k_1}$.

Differentiating (5.12) logarithmically, we find that

$$\frac{1+(r+1)l}{l(1+l)}\frac{dl}{dL}=\frac{1}{L}. \tag{5.14}$$

So that

$$\frac{1}{M}=\frac{L\,dl/dL}{l(1+l)}. \tag{5.15}$$

We now change variable and use $t=l/(1+l)$ so that (5.12) gives

$$t=L(1-t)^{r+1}, \tag{5.16}$$

and (5.15) becomes

$$\frac{1}{M}=\frac{L}{t}\frac{dt}{dL}. \tag{5.17}$$

If we take

$$k=r+1, \tag{5.18}$$

then (5.16) and Lemma 5.7 below gives the explicit expansion

$$(1-t)^m=\sum_{j=0}^{\infty}(-1)^jP(j,m)L^j, \tag{5.19}$$

where

$$P(j,m)=\frac{m(m+jk-1)!}{j!(m+j(k-1))!}. \tag{5.20}$$

Now, using (5.16) to expand the right hand side of (5.17) we get

$$\frac{1}{M}=L(1/L+k\frac{d}{dL}\ln(1-t)). \tag{5.21}$$

From (5.13) then

$$k\ln(1-t)=\sum_{i>0}\frac{\alpha_i}{i}L^i. \tag{5.22}$$

If $k=0$ then all the α_i are zero so we have a linearizable critical point - so we assume that $k\neq 0$ from now on.

Now, from (5.16)

$$ln(1-t)=L(1-t)^k+L^2(1-t)^{2k}/2+L^3(1-t)^{3k}/3+\cdots, \tag{5.23}$$

and so

$$\frac{\alpha_i}{ik}=\sum_{j=1}^{i}(-1)^{i-j}\frac{P(i-j,jk)}{j}=\sum_{j=0}^{i-1}(-1)^j\frac{P(j,(i-j)k)}{i-j}. \tag{5.24}$$

This last expression can be expanded to give

$$\begin{array}{rcl} \frac{\alpha_i}{ik} & = & \frac{1}{i}\sum_{j=0}^{i-1}(-1)^j\binom{ik}{j} \\ & = & \frac{1}{i}\left[1+\sum_{j=1}^{i-1}(-1)^j\left(\binom{ik-1}{j-1}+\binom{ik-1}{j}\right)\right] \\ & = & \frac{1}{i}(-1)^{i-1}\binom{ik-1}{i-1} \\ & = & (-1)^{i-1}\frac{\binom{ik}{i}}{ik}. \end{array} \tag{5.25}$$

So we have

$$\alpha_i = (-1)^{i-1}\binom{ik}{i}. \tag{5.26}$$

Finally, since $\lambda = p/q$ is rational, if the system is linearizable, then we need the coefficient of X^pY^q in $1/M$ to be zero in order to solve for that term in the equation for h above (5.9). That is $\binom{(p+q)k}{(p+q)}$ must vanish, and therefore $k = 0, 1/(p+q), \dots, (p+q-1)/(p+q)$.

Conversely, if k takes one of these values, then given any positive integer m, the coefficient of $X^{pm}Y^{qm}$ in $1/M$ is given by $\binom{(p+q)km}{(p+q)m}$ which also vanishes. So we can solve explicitly for h in (5.9) for each positive integer m, and it is clear that h will converge in this case. $\square$

To study the case λ irrational we need a few preliminaries.

Definition 5.2

1. A series $\sum a_jx^j$ with radius of convergence $R \neq 0, \infty$ is of *geometric type* if

$$\liminf_{n\to\infty} |a_n|^{\frac{1}{n}} > 0.$$

   In particular all a_n are nonzero for n sufficiently large.

2. A series $\sum a_{ij}x^iy^j$ is of geometric type if

$$\liminf_{n\to\infty} \min_{i+j=n} |a_{ij}|^{\frac{1}{n}} > 0.$$

   In particular all a_{ij} are nonzero for $i+j$ sufficiently large.

3. For λ a positive irrational number we introduce the expansion of λ in continuous fraction. This yields a sequence of approximations of λ by means of $\frac{p_n}{q_n}$. λ is a *Cremer number* if and only if

$$\limsup_{n\to\infty} \frac{1}{q_n}\log q_{n+1} = +\infty. \tag{5.27}$$

The following fact is well known.

Proposition 5.3 *A positive irrational number λ is a Cremer number if and only if for any series $\sum a_{ij}x^iy^j$ of geometric type, the series $\sum \frac{a_{ij}}{i-\lambda j}x^iy^j$ is divergent.*

Proof. Indeed it is known that

$$\frac{1}{q_n+q_{n+1}} < |q_n\lambda - p_n| < \frac{1}{q_{n+1}}$$

(see for instance [20], although this is a standard estimate for continuous fractions.) Then

$$q_{n+1} < \left|\frac{1}{q_n\lambda - p_n}\right| < 2q_{n+1}.$$

The condition (5.27) is equivalent to

$$\limsup_{n\to\infty} q_{n+1}^{1/q_n} = +\infty$$

from which the conclusion follows. □

In the following proposition, we consider the case where $a = d = 0$. If $b = 0$ or $c = 0$ then, as in the proof of Theorem 5.1, the system must be linearizable, so it is sufficient to take the case where both b and c are non-zero, and scale them so that $b = c = 1$.

Proposition 5.4 *The system*

$$\begin{aligned} \dot{x} &= x(1+y) \\ \dot{y} &= y(-\lambda + x) \end{aligned} \tag{5.28}$$

is integrable but not linearizable for all rational λ and for all irrational λ which are Cremer numbers.

Proof. The system has the first integral $H(x,y) = x^\lambda y e^{-x+y}$. We take the change of coordinates $(X,Y) = (xe^{-\frac{x}{\lambda}}, ye^y)$. It transforms the system into

$$\begin{aligned} \dot{X} &= X(1+y)(1-\tfrac{x}{\lambda}) = XM(X,Y) \\ \dot{y} &= -\lambda Y(1+y)(1-\tfrac{x}{\lambda}) = -\lambda Y M(X,Y) \end{aligned} \tag{5.29}$$

We need to show that the series $N(X,Y) = \frac{1}{M(X,Y)} - 1$ is of geometric type.

From the inversion formula of Lemma 5.7 below we have that

$$y = \sum_{j\geq 1} (-1)^{j-1}\frac{j^{j-2}}{(j-1)!}Y^j. \tag{5.30}$$

It follows from this that $\frac{1}{1+y}$ is a series in Y of geometric type. Similarly $x = \sum_{j\geq 1} \frac{j^{j-2}}{(j-1)!\lambda^{j-1}}X^j$. Again $\frac{1}{1-\frac{x}{\lambda}}$ is of geometric type. Then $N(X,Y)$ is of geometric type, yielding either an obstruction to linearizability if λ is rational or divergence of the linearizing series if λ is a Cremer number. □

The following theorem is proved in [2]

Theorem 5.5 *If λ is not Cremer number then any critical point of an analytic vector field with ratio of eigenvalues $-\lambda$ is linearizable as soon as it is integrable.*

For λ irrational, we have the following theorem

Theorem 5.6 *Let $\lambda \in \mathbb{R}^+ \setminus \mathbb{Q}$, and let $\{p_k/q_k\}$ be the sequence of approximants of λ derived from its continued fraction expansion. Under Condition* (5.1) *the origin of* (1.2) *is linearizable if and only if one of the following conditions is satisfied*

1. *λ is not a Cremer number;*
2. *λ is a Cremer number and, if k is defined by*

$$k = \frac{1-\frac{c}{a}}{1+\lambda} = \frac{1-\frac{b}{d}}{1+\frac{1}{\lambda}}, \tag{5.31}$$

then $0 \leq k < 1$ and there exists a sequence $s_n \in \mathbb{N}$ and a number R, such that

$$|k(1+\lambda)q_n - s_n| < R^{q_n}|\lambda q_n - p_n|.$$

In particular any number $k \in [0,1)$ for which $k(1+\lambda) = \alpha\lambda + \beta$ for $\alpha, \beta \in \mathbb{Z}$ satisfies the condition above. If k is irrational, then k must be also Cremer.

Proof. The proof of the theorem starts as the proof of Theorem 5.1: we must find a function h satisfying (5.9). In this case a formal solution always exists, given by (5.10). We need only to study when it is convergent. If we control the growth of the terms in $x^{p_n}y^{q_n}$ where $\frac{p_n}{q_n}$ is an approximant of λ derived from its continued fraction expansion so that the subseries of these terms is convergent then the whole series of h will be convergent. We need to look at the behaviour of the terms $\frac{1}{\lambda q_n - p_n}\binom{k(p_n+q_n)}{p_n+q_n}$ for $n \to \infty$. We can only expect convergence if the numerator has a small factor, i.e. $k \in [0,1)$. The only factor which matters is $\frac{k(p_n+q_n)-s_n}{\lambda q_n - p_n}$ with $0 \leq s_n < p_n + q_n$. The subseries is convergent if and only if there exists $R > 0$, a sequence $\{s_n\}$ such that

$$|k(p_n+q_n) - s_n| < R^{q_n}|\lambda q_n - p_n|. \tag{5.32}$$

But

$$k(p_n+q_n) - s_n = k(p_n - \lambda q_n) + k(1+\lambda)q_n - s_n,$$

yielding

$$\begin{aligned} |k(p_n+q_n) - s_n| &\leq k|p_n - \lambda q_n| + |k(1+\lambda)q_n - s_n| \\ |k(1+\lambda)q_n - s_n| &\leq k|p_n - \lambda q_n| + |k(p_n+q_n) - s_n|. \end{aligned} \tag{5.33}$$

Hence (5.32) is satisfied for some $R > 0$ if and only if there exists $R' > 0$ such that

$$|k(1+\lambda)q_n - s_n| < R'^{q_n}|\lambda q_n - p_n|$$

Let $k \in [0,1)$ satisfy $k(1+\lambda) = \alpha\lambda + \beta$ for $\alpha\, \beta \in \mathbb{Z}$. Then

$$|k(1+\lambda)q_n - (\alpha p_n + \beta q_n)| \leq |\alpha(\lambda q_n - p_n)|,$$

so k satisfies the condition above. The last statement follows directly from (5.32). □

Question. For a fixed Cremer number λ, the numbers k of the form $k(1+\lambda) = \alpha\lambda + \beta$ for $\alpha, \beta \in \mathbb{Z}$, except for $k = 0$, are also Cremer numbers and form a countable set. Are there other values of $k \in [0,1)$ satisfying the conditions of the theorem?

Lemma 5.7 *1. Consider the equation $s = z(1+s)^n$ about $z = 0$. Then $s = \sum_{n\geq 1} a_j z^j$ where*

$$a_j = \frac{\binom{jn}{j-1}}{j}.$$

We also have

$$(1+s)^m = \sum_{j\geq 0} P(j,m) z^j$$

where

$$P(j,m) = \frac{m(m+jn-1)!}{j!(m+j(n-1))!} = \begin{cases} 1 & j = 0 \\ \frac{m}{j}\binom{m+jn-1}{j-1} & j \geq 1. \end{cases} \tag{5.34}$$

2. Consider the equation $x = Xe^x$ about $X = 0$. Then $x = \sum_{n\geq 1} b_j X^j$, where

$$b_j = \frac{j^{j-1}}{j!} = \frac{j^{j-2}}{(j-1)!}.$$

Proof. We seek s as a power series in z. Such a power series must exist by the implicit function theorem. The coefficient of z^j in s can be deduced by an iterative procedure and will be a polynomial in n of degree $j-1$ by induction. If we let $s = x/n$ and $z = X/n$ then we get

$$x = X(1 + x/n)^n. \tag{5.35}$$

Taking the limit as $n \to \infty$ in $x(X) = ns(X/n)$, we get the coefficients of the solution of

$$x = Xe^x.$$

Now let

$$M = 1 + s, \tag{5.36}$$

then

$$M = 1 + zM^n.$$

We let $P(j,m)$ be the coefficient of z^j in M^m. Again, we can show that $P(j,m)$ is a polynomial in m of degree j. If we can find the values of $P(j,m)$ when m is an integer, we therefore will know $P(j,m)$ for all m.

We want to show that (5.34) is satisfied by induction on j. The case $j = 0$ is trivially true. From above,

$$M^m = MM^{m-1} = M^{m-1} + zM^{m+n-1},$$

and so

$$P(j,m) = P(j,m-1) + P(j-1,m+n-1).$$

By our inductive hypothesis, (5.34) holds for $j-1$, so that

$$P(j,m) = P(j,m-1) + \frac{(m+n-1)(m+nj-2)!}{(j-1)!(m+n(j-1))!}.$$

We now prove that (5.34) holds for j by induction on m. Namely, $P(j,0) = 0$ for $j > 0$, and

$$\begin{aligned} P(j,m) &= \frac{(m-1)(m+jn-2)!}{j!(m+j(n-1)-1)!} + \frac{(m+n-1)(m+jn-2)!}{(j-1)!(m+j(n-1))!} \\ &= \frac{(m+jn-2)!}{j!(m+j(n-1))!}((m-1)(m+j(n-1)) + j(m+n-1)) \\ &= \frac{m(m+jn-1)!}{j!(m+j(n-1))!}. \end{aligned}$$

Finally, to obtain the second result, we consider (5.35) and take the limit as n tends to infinity as above. The coefficient of X^j in x is $P(j,1)/n^{j-1}$, and so we have for $j > 0$,

$$\begin{aligned} \lim_{n\to\infty} P(j,1)/n^{j-1} &= \lim_{n\to\infty} \tfrac{1}{j!} j(j-1/n)\cdots(j-(j-2)/n) \\ &= \frac{j^{j-2}}{(j-1)!}. \end{aligned} \tag{5.37}$$

□

6 The Lotka-Volterra system with $\lambda = 0$

We consider (1.2) with $\lambda = 0$ and characterize the integrable and orbitally normalizable points.

Theorem 6.1 *The origin of* (1.2) *with* $\lambda = 0$ *is integrable if* $d = 0$*. When* $d \neq 0$ *the origin is orbitally normalizable if and only if one of the following conditions is satisfied:*

I. $b + md = 0$ *for* $m \in \mathbb{N} \cup \{0\}$*;*

II. $a = c$.

Proof. If $d = 0$, then dividing the vector field by x yields a vector field with a non singular point at the origin. Hence there exists a local analytic first integral.

We now consider the case $d \neq 0$ and we scale $d = 1$, i.e. we study the system:

$$\begin{array}{rcl} \dot{x} & = & x(1 + ax + by) \\ \dot{y} & = & y(cx + y) \end{array} \tag{6.1}$$

The necessity of the conditions I and II comes from calculating normalizing transformations and finding necessary conditions for convergence.

We prove the sufficiency of I and II in two ways, one way in the spirit of the monodromy arguments developed above, and the second using the general theory of the unfolding of non orbitally normalizable points [12]. Roughly the idea of the second proof is that points which are not orbitally normalizable cannot be approached by sequences of integrable points for which the hyperbolicity ratio is the inverse of an integer. We give a proof of the special case used here. In the next section, we shall see how similar techniques can be applied to a much more difficult problem.

Let us now start with the necessity. The origin is orbitally normalizable if we can find an analytic change of coordinates $(x_1, y_1) = (x + o(x, y), y + o(x, y))$ bringing the system to the normal form:

$$\begin{array}{rcl} \dot{x_1} & = & x_1(1 + By_1) \\ \dot{y_1} & = & y_1^2. \end{array} \tag{6.2}$$

We first let

$$Y = \begin{cases} y(1 + ax)^{-\frac{c}{a}} & a \neq 0, \\ ye^{-cx} & a = 0. \end{cases} \tag{6.3}$$

This transforms the system into

$$\begin{array}{rcl} \dot{x} & = & \begin{cases} x\frac{(1+ax)^{2-\frac{c}{a}}}{1+(a-bc)x} + bxY\frac{1+ax}{1+(a-bc)x} & a \neq 0 \\ \frac{x(1+bYe^{cx})e^{-cx}}{1-bcx} & a = 0 \end{cases} \\ & = & x(1 + bY) + x^2(a + (b-1)c + b^2cY) + o(x^2) \\ \dot{Y} & = & Y^2. \end{array} \tag{6.4}$$

Hence $B = b$.

To remove the terms in x^2 in $\dot{x}$ we use a change of coordinate of the form $x = X + X^2 \sum_{i \geq 0} a_i Y^i$. Identifying terms in X^2 yields the following equations:

$$\begin{array}{l} a_0 = a + (b-1)c \\ a_1 = -ba_0 + b^2c = b(c-a) \\ a_{i+1} = -(i+b)a_i. \end{array} \tag{6.5}$$

The transformation is obviously convergent if and only if one of the conditions I or II is satisfied.

Let us now show that these conditions are sufficient. Our first proof ties these results together with the sufficient conditions considered in Sections 3 and 4.

Case I: We consider the monodromy on the y-axis. The conditions imply that the ratio of eigenvalues at P_y is a positive integer. The two other critical points on the y-axis have converged to give the saddle node. The y-axis is in fact the center manifold of the saddle-node, which therefore has a trivial monodromy. However, the monodromy of the center manifold does not determine the analytic classification of the saddle-node in general, and so we need to argue in a little more detail.

The formal normal form of the saddle-node is given by (6.2) with $B = b$, and therefore the modulus of the saddle node is given by two functions ψ^0 and ψ^∞ as described in Section 2. From [8], we know that for a saddle-node with an analytic center manifold, ψ^∞ is just the identity map, and the monodromy around the center manifold is given by the equivalence class of $\psi^\infty \circ (\psi^0)^{-1}$ under conjugacy. Hence ψ^0 is the identity yielding that the saddle-node must be normalizable.

In the particular case $b = 0$ an explicit normalizing change of coordinates can be found by composing the change of variables (6.3) to the form (2.8), which is this case is

$$\dot{x} = x(1 + a(x)), \qquad \dot{Y} = Y^2,$$

where $a(x)$ is an some analytic function of x with $a(0) = 0$, with a change of variables $X = X(x)$.

Case II: If $a = c$ then we have an exponential factor $D = exp((1 + ax)/y)$, and a first integral of the system can be constructed explicitly: $x^d y^{-b} D^{-1}$.

Our second proof is of much wider applicability. We have here a particular case of a saddle-node which has an analytic center manifold. We will show in Theorem 6.2 below that a non orbitally normalizable saddle-node with a center manifold cannot be approached by integrable saddles with hyperbolicity ratios given by the inverse of an integer.

In our case the systems of case I with $b = -m$ are approached by a sequence of integrable systems of the form (1.2) with parameters $(\lambda_n, a_n, b_n, c_n, d_n) = (\frac{1}{n}, a, -m, c, 1)$ (systems of type B_{n-m}). Similarly the systems of case II with $a = c$, $d = 1$ are approached by a sequence of integrable systems of the form (1.2) with parameters $(\lambda_n, a_n, b_n, c_n, d_n) = (\frac{1}{n}, a, b, a + \frac{a(b-1)}{n}, 1)$ (systems with an invariant line). Thus, both of the limiting cases must be normalizable. $\square$

We will apply a similar line of argument to analyze the harder case of the normalizability of a resonant saddle in the next section. The theorem which we present below could be deduced from Theorem 2.6 which has been proved by geometric methods. In fact it is just a particular case of it. The geometric methods are far superior as they allow to deal with the general case, while it is would be difficult to generalize the proof below which studies the normalizing series. However Theorem 6.2 and its proof, together with the examples of this paper present the way the general phenomena of Theorems 2.6 and 2.7 were discovered. This is why we decide to present it.

Theorem 6.2 *We consider an analytic system with a saddle-node at the origin and analytic center manifold:*

$$\begin{array}{rcl} \dot{x} & = & x^2 \\ \dot{y} & = & y(1+ax)+\sum_{k\geq 2} f_k(x)y^k. \end{array} \tag{6.6}$$

for which the normalizing change of coordinates is divergent, and an unfolding for which $y=0$ is invariant. By a change of coordinate depending analytically on $\epsilon \neq 0$ and continuously of ϵ near $\epsilon = 0$ we can suppose that the unfolding has the form:

$$\begin{array}{rcl} \dot{x} & = & x^2-\epsilon \\ \dot{y} & = & y(1+a(\epsilon)x)+\sum_{k\geq 2} f_k(x,\epsilon)y^k, \end{array} \tag{6.7}$$

with $a(0) = a$ and $f_k(x,0) = f_k(x)$ for $k \geq 2$. Then

1. *$\exists N_0 \in \mathbb{N}$ such that $\forall n > N_0$, if ϵ is such that the saddle point located at $(-\sqrt{\epsilon}, 0)$ has hyperbolicity ratio $\frac{1}{n}$, then this saddle point is not integrable.*

2. *More precisely, suppose that for $\epsilon = 0$ the normalizing change of coordinates has the form $y = Y + \sum_{k\geq 2} g_k(x)Y^k$, with $g_k(x)$ analytic for $k < q$ and $g_q(x)$ the sum of a divergent Borel-summable series (see for instance [13] for a definition), then $\exists N_1 \in \mathbb{N}$ such that $\forall n > N_1$, if ϵ is such that the saddle point located at $(-\sqrt{\epsilon}, 0)$ has hyperbolicity ratio $\frac{q-1}{n}$, then this saddle point is not integrable.*

Proof. The first part follows as soon as we can show that the strong separatrices depend analytically on $\epsilon \neq 0$ and continuously of ϵ near $\epsilon = 0$. This has been studied by Glutsyuk [5] in a cone in the ϵ-space but his proof is valid in a full neighborhood of the origin. Indeed Glutsyuk first shows that the family can be brought by the preparation theorem to the form

$$\begin{array}{rcl} \dot{x} & = & x^2-\epsilon+yR(x,y,\epsilon) \\ \dot{y} & = & y(1+q(x,y,\epsilon)). \end{array} \tag{6.8}$$

For $\epsilon \neq 0$ the strong separatrices of the singular points $(\pm\sqrt{\epsilon}, 0)$ are analytic curves $x = F_\epsilon^\pm(y)$ depending analytically on ϵ. Glutsyuk has shown that the graphs of $F_\epsilon^\pm$ are defined on neighborhoods of zero whose size is independent of ϵ. One first performs the change $x_1 = x - F_\epsilon^+(y)$ which straightens the separatrix of $(\sqrt{\epsilon}, 0)$. Let $x_1 = \overline{F}_\epsilon^-(y)$ be the equation of the separatrix of $(-\sqrt{\epsilon}, 0)$. The next transformation is $x_2 = -2x_1\frac{\sqrt{\epsilon}}{\overline{F}_\epsilon^-(y)}$ which preserves the first separatrix and straightens the second one. Making a translation in x_2 and scaling in x_2 yields the form (6.7) with the required dependence on the parameter. Changing $x_2 = x$ we now have a system.

$$\begin{array}{rcl} \dot{x} & = & x^2-\epsilon \\ \dot{y} & = & y(1+g(x,y,\epsilon)). \end{array} \tag{6.9}$$

We now let $g(x,y,\epsilon) = g_1(x,\epsilon)+O(y)$. By Kostov's theorem [10], a change of coordinate in x allows us to bring $\frac{1+g_1(x,\epsilon)}{x^2-\epsilon}dx$ to the form $\frac{1+a(\epsilon)\overline{x}}{\overline{x}^2-\epsilon}d\overline{x}$. Applying this to the system

(6.9) divided by $1+g_1(x,\epsilon)$ and then multiplying the system obtained by $1+a(\epsilon)\overline{x}$ yields the result.

We prove (2) which implies (1). We perform an analytic change of coordinate $y = Y + \sum_{k=2}^{q-1} h_k(x)Y^k$ so as to bring the system for $\epsilon = 0$ to the form

$$\begin{array}{rcl} \dot{x} & = & x^2 \\ \dot{Y} & = & Y(1+ax) + \sum_{k\geq q} \overline{f}_k(x)Y^k. \end{array} \tag{6.10}$$

Moreover, using if necessary an additional change of coordinate of the form $Y = Y_1 + bY_1^q$, we can suppose that $\overline{f}_q(0) = 0$ (which is the form we need to be able to use the formula given in the appendix).

Applying this change of coordinate to the full family (6.7) we work with a family of the form:

$$\begin{array}{rcl} \dot{x} & = & x^2 - \epsilon \\ \dot{Y} & = & Y(1+a(\epsilon)x) + \sqrt{\epsilon}\sum_{k=2}^{q-1} \overline{f}_k(x,\epsilon)Y^k + \sum_{k\geq q} \overline{f}_k(x,\epsilon)Y^k. \end{array} \tag{6.11}$$

The saddle point P is located at $(-\sqrt{\epsilon}, 0)$ and has eigenvalues $(-2\sqrt{\epsilon}, 1 - a\sqrt{\epsilon})$. The hyperbolicity ratio is of the form $\frac{q-1}{n}$ if $\sqrt{\epsilon} = \frac{q-1}{a(\epsilon)(q-1)+2n}$ (when a is complex we choose the root with positive real part).

We now localize the system at P by means of $x_1 = x + \sqrt{\epsilon}$ and use rescaling in x and t so that the system has the form:

$$\begin{array}{rcl} \dot{x_1} & = & x_1(x_1 - 2\sqrt{\epsilon}) \\ \dot{Y} & = & Y(1+\alpha(\epsilon)x_1) + \sqrt{\epsilon}\sum_{k=2}^{q-1} \tilde{f}_k(x_1,\epsilon)Y^k + \sum_{k\geq q} \tilde{f}_k(x_1,\epsilon)Y^k, \end{array} \tag{6.12}$$

with $\alpha(\epsilon) = a(\epsilon) + O(\sqrt{\epsilon})$ so that $\alpha(0) = a$. The family is now "prepared". The preparation process has not changed significantly the radius of convergence of the f_k.

From now on we consider the family in this form. To simplify the notation we remove the tildes and do not always write the dependence of α over ϵ (but we remember that $\alpha(0) = a$), so we work with the system:

$$\begin{array}{rcl} \dot{x} & = & x(x - 2\sqrt{\epsilon}) \\ \dot{y} & = & y(1+\alpha(\epsilon)x) + \sqrt{\epsilon}\sum_{k=2}^{q-1} f_k(x,\epsilon)y^k + \sum_{k\geq q} f_k(x,\epsilon)y^k. \end{array} \tag{6.13}$$

Remark that in all these changes the new function $f_q(x,0)$ is the same as $\overline{f}_q(x)$ in (6.10). Suppose that $\overline{f}_q(x) = \sum_{n\geq 1} a_n x^n$. Our hypothesis is

$$C = \sum_{l\geq 1} c_l = \sum_{l\geq 1} \frac{a_l(1-q)^{a(q-1)+l-1}}{\Gamma(a(q-1)+l)} \neq 0 \tag{6.14}$$

(see Appendix).

Let $f_q(x,\epsilon) = \sum_{l\geq 0} b_l(\epsilon)x^l$. Then obviously $b_l(0) = a_l$ and $b_0(0) = 0$. We need to study how the $b_l(\epsilon)$ vary.

We have that $f_q(x,\epsilon)$ is an analytic function in x which depends continuously on ϵ. Suppose that $|f_q(x,\epsilon)| \le M$ on $|(x,\sqrt{\epsilon})| < \delta$. Then $|b_l(\epsilon)| \le \frac{M}{\delta^l}$ by the Cauchy integral formula for $|\sqrt{\epsilon}| < \delta$. We can also choose δ such that $|b_0(\epsilon)| < \eta$ for $0 < \eta \ll |C|$.

The series (6.14) is absolutely convergent and majorized by the absolute convergent series $\sum_{l\ge 1} \frac{M}{\delta^l} \left| \frac{(1-q)^{a(q-1)+l}}{\Gamma(a(q-1)+l)} \right|$.

We first consider the case $f_k(x,\epsilon) \equiv 0$ for $k < q$ (this covers in particular the case $q = 2$).

$$\begin{array}{rcl} \dot{x} & = & x(x-2\sqrt{\epsilon}) \\ \dot{y} & = & y(1+\alpha(\epsilon)x) + \sum_{k\ge q} f_k(x,\epsilon)y^k. \end{array} \tag{6.15}$$

Later we will adapt the proof to cover the general case. We will show that for $\sqrt{\epsilon} = \frac{q-1}{2n}$ (i.e. the origin is a saddle point with hyperbolicity ratio $\frac{q-1}{2n}$ in (6.15)), then the equation has a nonzero resonant monomial of the form $x^n y^q$ as soon as n is sufficiently large.

To bring (6.15) to normal form we consider a change of coordinates of the form $y = Y + g_q(x)Y^q + O(Y^{q+1})$. We then compose it with an analytic change of coordinate $X = x + o(x)$ linearizing $\dot{x} = \frac{x(x-2\sqrt{\epsilon})}{1+\alpha(\epsilon)x}$ [10]. Note that the latter does not destroy the work done first so that the existence of a nonzero resonant term of the form $x^n Y^q$ can be seen from the Y^q terms in the change of coordinates $y \mapsto Y$ only. Thus $g_k(x)$ satisfies the differential equation:

$$x(x-2\sqrt{\epsilon})g_q'(x) + (q-1)(1+\alpha(\epsilon)x)g_q(x) = f_q(x,\epsilon). \tag{6.16}$$

This linear equation has a solution (we write $\alpha = \alpha(\epsilon)$):

$$\begin{array}{rcl} g_q(x) & = & x^{\frac{q-1}{2\sqrt{\epsilon}}}(x-2\sqrt{\epsilon})^{-(q-1)(\alpha+\frac{1}{2\sqrt{\epsilon}})} \\ & & \displaystyle\int x^{-1-\frac{q-1}{2\sqrt{\epsilon}}}(x-2\sqrt{\epsilon})^{(q-1)(\alpha+\frac{1}{2\sqrt{\epsilon}})-1} f_q(x,\epsilon)dx \end{array} \tag{6.17}$$

(by the integral we mean the primitive with no free term.) We show that $g_q(x)$ has a term in $\ln x$, i.e. that the expansion of the integrand has a term in x^{-1} as soon as $\sqrt{\epsilon} = \frac{q-1}{2n}$ with ϵ sufficiently small, i.e. n sufficiently large. Indeed, in this case, if $f_q(x,\epsilon) = \sum_{l\ge 0} b_l(\epsilon)x^l$, the integrand has the form

$$I(x) = x^{-n-1}(x - \frac{q-1}{n})^{\alpha(q-1)+n-1} \sum_{l\ge 0} b_l(\epsilon)x^l. \tag{6.18}$$

The coefficient of x^{-1} in the integrand is:

$$D(\epsilon) = \sum_{l=0}^{n} b_l(\epsilon) \left(\frac{1-q}{n}\right)^{\alpha(q-1)+l-1} \binom{\alpha(q-1)+n-1}{n-l}, \tag{6.19}$$

which we must show to be nonzero as soon as ϵ is sufficiently small (n sufficiently large).

Recall, that for any $\eta > 0$ we can choose δ such that $|b_0(\epsilon)| < \eta$. We shall also choose $N_1 > 0$ such that

$$\begin{array}{l} \left|C - \sum_{l \le N_1} \frac{a_l(1-q)^{a(q-1)+l-1}}{\Gamma(a(q-1)+l)}\right| < \eta \\ \sum_{l>N_1} \left|\frac{M(1-q)^{a(q-1)+l-1}}{\delta^l \Gamma(a(q-1)+l)}\right| < \eta. \end{array} \tag{6.20}$$

(Note that the latter is majorizing the remainder of the series.) Let us write (6.19) the coefficient of the resonant monomial calculated above as $\sum_{l=1}^n d_l$. We first choose N_1 sufficiently large so that for $l > N_1$ we have $|d_l| < 2\left|\frac{M(1-q)^{a(q-1)+l-1}}{\delta^l \Gamma(a(q-1)+l)}\right|$. For that given N_1 we show that we can take n sufficiently large so that the d_l will be close to c_l for $l \le N_1$. The proof then follows from (6.20) and the hypothesis that $C \neq 0$.

We have

$$\begin{aligned} d_l(\epsilon) &= b_l(\epsilon)\left(\tfrac{1-q}{n}\right)^{\alpha(q-1)+l-1}\binom{\alpha(q-1)+n-1}{n-l} \\ &= b_l(\epsilon)\left(\tfrac{1-q}{n}\right)^{\alpha(q-1)+l-1}\frac{\Gamma(\alpha(q-1)+n)}{\Gamma(\alpha(q-1)+l)(n-l)!}. \end{aligned} \tag{6.21}$$

Thus

$$|d_l(\epsilon)| \le \frac{M}{\delta^l}\left|\left(\frac{1-q}{n}\right)^{\alpha(q-1)+l-1}\frac{\Gamma(\alpha(q-1)+n)}{\Gamma(\alpha(q-1)+l)(n-l)!}\right|. \tag{6.22}$$

It hence suffices to show that

$$\left|(1-q)^{(\alpha-a)(q-1)}\right|\left|\frac{\Gamma(\alpha(q-1)+n)}{n^{\alpha(q-1)+l-1}(n-l)!}\,\frac{\Gamma(a(q-1)+l)}{\Gamma(\alpha(q-1)+l)}\right| < 2 \tag{6.23}$$

for n large and $N_1 < l \le n$. This follows for sufficiently small η by using the following properties of the Gamma function:

- For $n \in \mathbb{N}$ large we have the asymptotic behavior: $\Gamma(n+a) \sim n^a\Gamma(n)$;
- $\Gamma(z+1) = z\Gamma(z)$;
- For $n \in \mathbb{N}, \Gamma(n+1) = n!$.

In particular, we have

$$\frac{\Gamma(\alpha(q-1)+n)}{n^{\alpha(q-1)+l-1}(n-l)!} \sim \frac{n^{\alpha(q-1)}(n-1)!}{n^{\alpha(q-1)+l-1}(n-l)!} = \frac{(n-1)\dots(n-l+1)}{n^{l-1}}. \tag{6.24}$$

This implies $d_l(\epsilon) < 2|c_l|$ for $l > N_1$ and hence $\sum_{l>N_1} d_l(\epsilon) < 2\eta$.

We now consider the case $0 \le l \le N_1$. First $d_0(\epsilon)$ is small. Then for $l \ge 1$

$$\frac{d_l(\epsilon)}{c_l} = \frac{b_l(\epsilon)}{a_l}\,\frac{\Gamma(\alpha(q-1)+n)}{n^{\alpha(q-1)+l-1}(n-l)!}\,\frac{\Gamma(a(q-1)+l)(1-q)^{(\alpha-a)(q-1)}}{\Gamma(\alpha(q-1)+l)}. \tag{6.25}$$

For ϵ sufficiently small, i.e. n sufficiently large, all factors tend to 1. Indeed $\alpha(\epsilon)$ satisfies $\alpha(0) = a$ and the asymptotics of the second factor is given above.

We now consider the case where $f_k \not\equiv 0$ for $k < q$. The additional difficulty in this case is that there may occur resonant terms of lower degree when $(q-1, n) \neq 1$. So already here the method with series become more difficult to handle. We therefore made the choice of finishing the proof by means of the geometric methods of [12] to give the reader a taste of them and demonstrate their power in tackling quite difficult problems.

The geometric proof goes via the holonomy as explained in Section 2. To show that the saddle $-\sqrt{\epsilon}$ is not linearizable is the same as proving that its holonomy is not linearizable, which in turn is equivalent to showing that the renormalized return map k_ϵ^- near $-\sqrt{\epsilon}$ is not linearizable. This map has a wild behavior for ϵ small. However it can be written $k_\epsilon^- = L_\epsilon \circ \psi_\epsilon^0$, where L_ϵ, the linear Lavaurs map, is the wild part while ψ_ϵ^0 depends continuously on ϵ for ϵ in a sector around the $\mathbb{R}^+$-axis. If we suppose $(\psi_\epsilon^0)'(0) = 1$ we are now limiting ourselves to values of ϵ for which the Lavaurs map has the form $L_\epsilon(w) = \exp(-2\pi i \frac{n}{q-1})w$. So we must show that the map $k_\epsilon^-(w) = \exp(-2\pi i \frac{n}{q-1})\psi_\epsilon^0(w)$ is not linearizable, which is the same as showing that its $(q-1)$-th iterate $(k_\epsilon^-)^{q-1}$ is nonlinear. But the map is a small perturbation of the map $\kappa_\epsilon = \exp(-2\pi i \frac{n}{q-1})\psi_0^0$. As

$$\psi_0^0(w) = w + Cw^q + o(w^q) \tag{6.26}$$

with $C \neq 0$, the map $\kappa_\epsilon = \exp(-2\pi i \frac{n}{q-1})\psi_0^0$ is not linearizable since the monomial w^q is resonant. This implies that $(\kappa_\epsilon)^{q-1}$ is nonlinear. Hence $(k_\epsilon^-)^{q-1}$ is nonlinear.

The last thing which needs an explanation is the special form of ψ_0^0 in (6.26). This comes from the definition of the Martinet-Ramis modulus by means of comparison of two first integrals defined in two sectors. These first integrals are of the form (2.8) where y_j, $j = 1, 2$, are normalizing changes of coordinates in the two sectors. From the form of the system, the y_j have the same expansion $y_j = y + h_q(x)y^q + o(y^q)$, with $h_q(x)$ Borel-summable except in the direction $\mathbb{R}^-$. The form of ψ_0^0 follows. $\square$

7 Limit phenomena

We now come to the study of the limit phenomena, which, although at a preliminary and experimental level, is probably the most interesting part of our paper.

We discuss a particular case, namely we take the strata (C_1) of Theorem 3.6 with $n = 1$. Scaling $b = c = 1$ we consider the system

$$\begin{aligned} \dot{x} &= x(1 - x + y) \\ \dot{y} &= y(-\lambda + x + dy). \end{aligned} \tag{7.1}$$

By Theorem 3.6 the system is integrable as soon as $\lambda > 1$ and $\lambda \neq 1 + \frac{1}{m}$ with $m \in \mathbb{N}$. If $\lambda = 1 + \frac{1}{m}$ then the system is orbitally normalizable and an additional condition is necessary for linearizability. We also know that the system is non integrable for $\lambda = 1$ except when $d = -1$ (in that case the system has a line of zeros).

We explore the additional conditions making the system integrable for $\lambda = 1+\frac{1}{m}$ and how these points accumulate as $m \to \infty$, i.e. $\lambda \to 1$. The points show a remarkable structure (Figure 3) and are organized in a countable number of regular sequences, each sequence accumulating to a limit point on $\lambda = 1$. We are interested in these limit points on $\lambda = 1$ as they are the "organizing centers" for the structure: they organize the region $\lambda > 1$. Among these limit points the point $d = -1$ is too degenerate to be interesting as it corresponds to a line of singular points. We are particularly interested in the other points which are at least "half-normalizable" (These correspond to the points for which the monodromy map is *half-iterable* in Ecalle's terminology [3], i.e. ψ^∞ is linear).

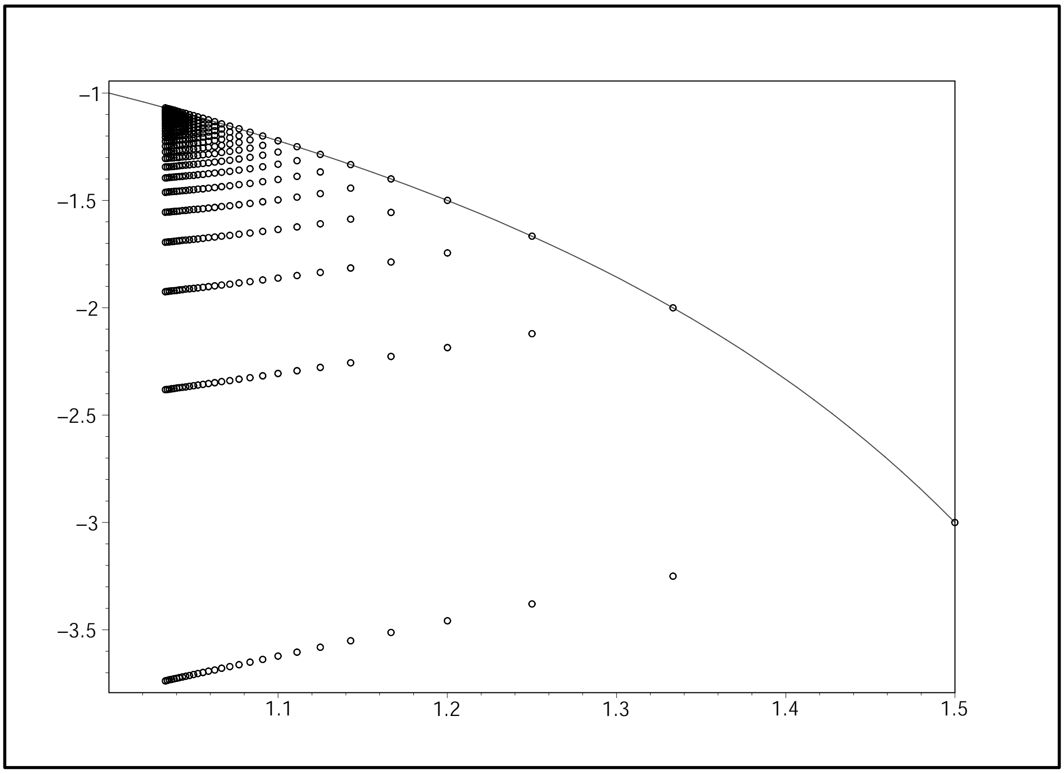


Figure 3: The values of d for which the system (7.1) is integrable for $\lambda = 1 + 1/n$. The system is integrable on the curve $d = \frac{\lambda}{\lambda-2}$ where it has an invariant line.

We also explore (using Reduce) the conditions under which the system (7.1) is integrable for $\lambda = \frac{p}{q} < 1$. Although we could not complete the calculations for very large p and q as the second saddle quantities are needed to determine the integrability conditions for the origin, a pattern seems to be observed: integrable points converge to the points $d = \pm 1$ on $\lambda = 1$ (Figure 4).

Here again $d = -1$ is too degenerate to be really interesting. We focus on $d = 1$ and show that it is half-normalizable on the side opposite to the previous one, i.e. ψ^0

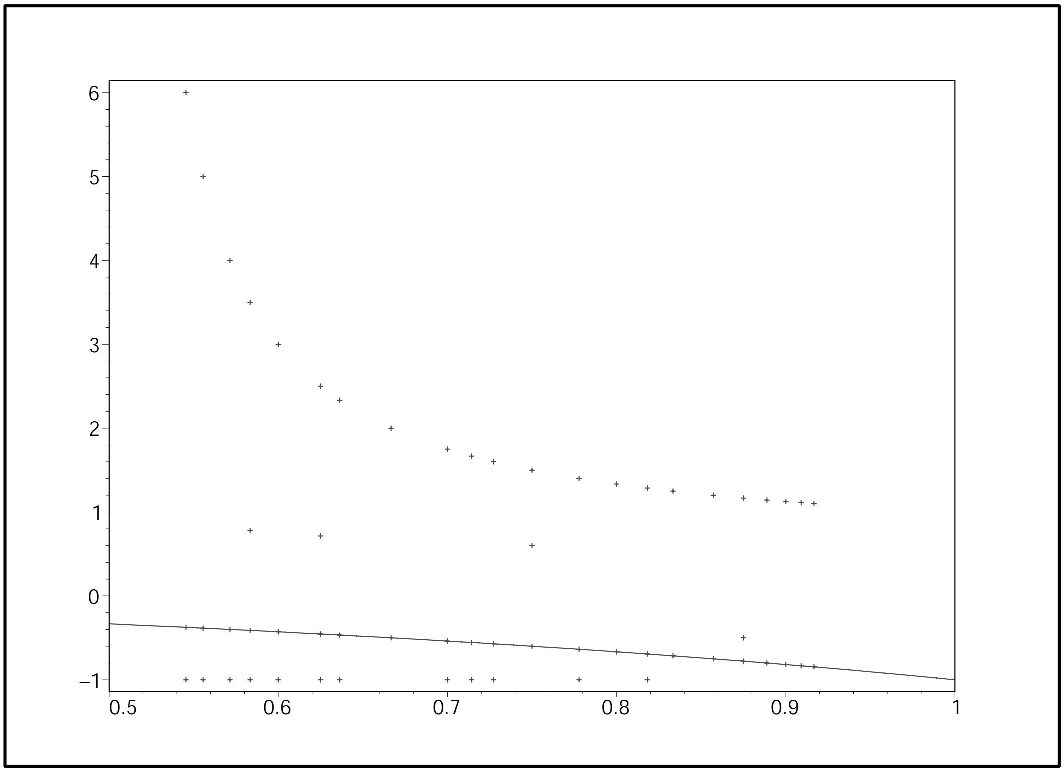


Figure 4: The values of d for which the system (7.1) is integrable for $\lambda = p/q \in (1/2, 1)$, $q \leq 12$. The system is integrable on the curve $d = \frac{\lambda}{\lambda - 2}$ where it has an invariant line. The sporadic points are: i) the Darboux integrable system for $\lambda = \frac{7}{8}$, ii) $(H_{3.2})$ for $\lambda = \frac{3}{4}$, iii) $(H_{3,3})$ for $\lambda = \frac{5}{8}$, iv) $(H_{3,4})$ for $\lambda = \frac{7}{12}$.

is linear. Let us now make this more precise. [The slightly unusual point at $\lambda = 7/8$ in (Figure 4) is just one of the two 'sporadic' cases mentioned before.]

For $\lambda = 1$ all points of system (7.1) with $d \neq -1$ are nonintegrable with non-vanishing first saddle quantity. We say that they are "not integrable to first order". Their analytic classification is given by the Ecalle-Voronin invariant $[\psi^0, \psi^\infty]$) described in Section 2.3.

Definition 7.1 We call the point *half-normalizable* if the holonomy is semi-iterable, i.e. one of the diffeomorphisms (ψ^0, ψ^∞) is linear.

It was explained in Section 2.3 how the two diffeomorphisms ψ^0 and ψ^∞ control the two sides of $\lambda = 1$ when we perturb $\lambda = 1$ to $\lambda \in \mathbb{R}^+$. When for instance ψ^0 (resp. ψ^∞) is nonlinear for a value of $d = d_0$ then the system for $\lambda = 1$ cannot be approached by integrable saddles when $\lambda = 1 - \frac{1}{n}$ (resp. $\lambda = 1 + \frac{1}{n}$) (see [12] for the general theory.) We will encounter here half-normalizable points of the two types: $d = 1$ is of the first type while the limit points of sequences on the right in Figure 3 are of the second type.

Proposition 7.2 *The system* (7.1) *with* $d = 1$ *and* $\lambda = 1$ *has a half-normalizable point at the origin. It cannot be approached by integrable saddles with* $\lambda = 1 + \frac{1}{n}$. *On the other hand it lies on* $d = \frac{\lambda}{2\lambda-1}$, *all points of which are integrable except when* $\lambda = 1 + \frac{1}{n}$. *In the latter case the points are normalizable.*

Proof. Let us look at the 3 singular points on the y-axis and their monodromy. When $\lambda = 1$, the monodromy of the point $(0,1)$ is the identity. The point P_y is a saddle-node with an analytic center manifold. Hence it is at least half-normalizable. Let us put it to normal form. The change of coordinates $(v,z) = (\frac{1}{y}, \frac{x}{y})$ brings (7.1) (after multiplication by z) to

$$\begin{aligned} \dot{v} &= -2v^2 + 2vz \\ \dot{z} &= -z - vz + z^2. \end{aligned} \tag{7.2}$$

We make the change of coordinate $V = -\frac{2v}{(1-z)^2}$. This brings the system to the form

$$\begin{aligned} \dot{V} &= V^2(1 - z + z^2) \\ \dot{z} &= -z + \tfrac{1}{2}Vz(1-z)^2 + z^2. \end{aligned} \tag{7.3}$$

Scaling of time yields the system

$$\begin{aligned} \dot{V} &= V^2 \\ \dot{z} &= -z(1 - \tfrac{1}{2}V) - \tfrac{1}{2}Vz^2 + o(z^2). \end{aligned} \tag{7.4}$$

The point P_y is not orbitally normalizable (the change of coordinate $z = Z + f(V)Z^2$ removing the terms in z^2 is divergent). Moreover by Theorem 9.1 of the appendix this implies the non-integrability of the saddle points with hyperbolicity ratios $\frac{1}{n}$ where n is sufficiently large. This implies the non-linearizability of the monodromy map when its multiplier is of the form $\exp(-\frac{2\pi i}{n})$. Hence if $(\overline{\psi}^0, \overline{\psi}^\infty)$ is the Ecalle-Voronin modulus of the monodromy map we have that $\overline{\psi}^\infty$ is linear while $\overline{\psi}^0$ is not. The monodromy of the origin is the inverse of that of P_y. Hence if (ψ^0, ψ^∞) is its Ecalle-Voronin modulus we have that ψ^0 is linear and ψ^∞ is not. The nonlinearity of ψ^∞ controls the non-linearizability of the monodromy map when the multiplier is of the form $\exp(\frac{2\pi i}{n})$. The latter corresponds to the non-integrability of the saddle point at the origin of (7.1) when $\lambda = 1 + \frac{1}{n}$.

We must now show that for all points of the curve $d = \frac{\lambda}{2\lambda-1}$ the origin is integrable except when $\lambda = 1 + \frac{1}{n}$ where it is only orbitally normalizable. Indeed we have $\frac{b}{d} = 2 - \frac{1}{\lambda}$ which means that the origin is in stratum (B_2) for $\lambda < 1$. For $\lambda > 1$ we need to remark that the system (7.1) is in (C_1). Hence all points are integrable except possibly for $\lambda = 1 + \frac{1}{n}$. To show the normalizability of the origin when $\lambda = 1 + \frac{1}{n}$ explicitly, we can use the additional condition $d = \frac{\lambda}{2\lambda-1}$ to show that the system has the following invariant conic which was first found by Chavarriga:

$$F(x,y) = \left(1 - \frac{y}{2\lambda - 1}\right)^2 - \frac{2xy}{(1-\lambda)(1-2\lambda)} = 0.$$

This conic yields an integrating factor

$$V(x,y) = x^{\frac{2\lambda-1}{\lambda-1}} y^{\frac{\lambda}{\lambda-1}} F^{-\frac{\lambda+1}{2(\lambda-1)}}.$$

As proved in [2] this yields the integrability of the origin except when the two exponents of the factors x and y in $V(x,y)$ are integers greater than 1, in which case the point is only orbitally normalizable. This is the case precisely when $\lambda = 1 + \frac{1}{n}$. □

Remark 7.3 Chavarriga found 5 strata of codimension 2 in which the system has an integrating factor found with the help of an invariant conic. As for the case appearing above these 5 conditions (of codimension 2) are covered by the codimension 1 strata of Theorem 3.6. In all cases two strata are necessary to cover the condition.

Let us now discuss the limit points of sequences of integrable points for $\lambda = 1 + \frac{1}{n}$.

These limit points of sequences of integrable points for $\lambda = 1+\frac{1}{n}$ are half-normalizable with ψ^∞ being linear. Here we are in a special case as we know that the origin is orbitally normalizable (half-normalizable) for $\lambda = 1$ if and only if the saddle-node $(1,0)$ of system (7.1) is orbitally normalizable (half-normalizable). In the particular case of the saddle-node it is classified by a pair of diffeomorphisms one of which can be taken as a Möbius transformation $z \mapsto \frac{z}{1+C(d)z}$, with $C(d)$ an analytic function of d (when localized at 0, i.e. a translation when localized at ∞). Hence the vanishing of $C(d)$ guarantees the triviality of one diffeomorphism. We claim that the limit points of the sequences of integrable systems for $\lambda = 1 + \frac{1}{m}$ are precisely the zeros of $C(d)$. But we could not push the calculations to an end in this case. The fact that these zeros accumulate to $d = -1$ is no contradiction with the analyticity of $C(d)$. Indeed the first saddle quantity vanishes at $d = -1$ and the theory of Ecalle-Voronin or Martinet-Ramis applies for a fixed order of non-integrability only.

The phenomemon observed here is a kind of "transcritical bifurcation". Let us describe it in more detail.

The "transcritical bifurcation": Except for $d = -1$ the monodromy M_0 of the origin has a generic parabolic point (i.e. a double fixed point) with multiplier 1. The limit points at $\lambda = 1$ in Figure 3 are *semi-normalizable*: If (ψ^0, ψ^∞) is the Ecalle-Voronin modulus of M_0, then we have

$$\begin{cases} \psi^\infty \text{ linear} \\ \psi^0 \text{ non linear.} \end{cases} \tag{7.5}$$

At $\lambda = 1$ we have a *transcritical bifurcation.* Indeed M_0 has two fixed points for $\lambda \neq 1$ and a double fixed point for $\lambda = 1$. The first fixed point is the origin and the second fixed point corresponds to an invariant manifold [9]. These two points "pass through each other" at $\lambda = 1$ as is usual in a transcritical bifurcation (Figure 5):

- For $\lambda < 1$, ψ^∞ controls the invariant manifold while ψ^0 controls the origin. Hence the origin is necessarily non trivial by Theorem 2.6 for $\lambda = 1 - 1/m$ with m large, while the invariant manifold may or may not be trivial.

- For $\lambda > 1$, ψ^∞ controls the origin which can hence be integrable if $d(\lambda)$ is well chosen, while ψ^0 controls the invariant manifold which is necessarily nontrivial for $\lambda = 1 + 1/m$ by Theorem 2.6.

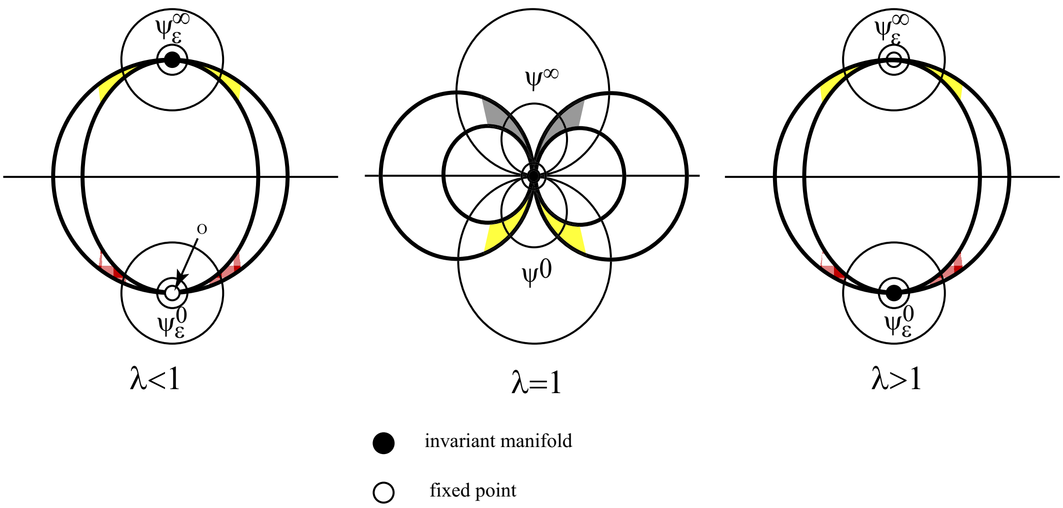


Figure 5: The transcritical bifurcation

8 Conclusion and further directions

Although it has not been possible to complete the classification of all the cases of linearizable and integrable Lotka-Volterra equations, we feel that the pattern of results is fairly clear: that apart from some exceptional cases with Darboux factors, the majority of cases on integrability come from the existence of an extra invariant line or from the elementary monodromy arguments of Section 3. Also, we have the conviction that the Lotka-Volterra systems are a very natural choice for examining some of the harder phenomena of moduli and their deformations — for example finding explicit conditions for normalizable and half-normalizable critical points; transcritical bifurcations and so on.

There are several other questions which follow on from here. First, to understand how the integrable points given by monodromy arguments relate to the types of integrability encountered in larger classes of polynomial systems: that is, the existence of Darboux first integrals, or of a blow-down to a node [6]. For instance we think that the blow-down to a node is always a particular case of our method with the monodromy group of the separatrix. In particular, since the monodromy has such a simple form, what are the global consequences of this? Another topic of interest is to examine in more detail what happens for irrational values of λ, or for rational values, but with normalizable rather than integrable points.

A third direction is to generalize the concept of monodromy to the t-monodromy of Voronin [19]. This should allow us to apply similar arguments to linearizability problems which we have applied to integrability problems here without the use of Theorem 4.1.

As a final application of our results, we consider the conjectures and questions given at the end of [6]. In many cases, the results here allow a fairly complete treatment of these. The conditions for integrability/linearizability given in (A_n) and (B_n) in Section 3 are called "Theorem D" in [6].

1. *Conjecture 1: Theorem D can be applied to the system*

$$\dot{x} = x(1 + ax + by), \qquad \dot{y} = y(-\lambda + cx + dy) + fx^2.$$

   The integrability in the case (B_n) follows directly along the lines of the proof given in Section 3, since the x-axis is still an invariant line. It is not clear that (A_n) should hold in this case. Linearizability follows from Theorem 4.1 .

2. *Conjecture 2: The origin of* (1.2) *with* $a = d = 0$ *is integrable, but not linearizable for all rational* λ*.* This is basically Proposition 3.

3. *Conjecture 3: There exists a point in parameter space such that the origin of* (1.2) *with* $\lambda = n + 1/q$ *is normalizable, but not integrable.* It is easy to give examples of this in the class (C_n) above. For example, take $\lambda = 3/2$, $a = 3/2$, $d = 3/4$ and $b = c = 1$.

4. *Question 1: Can Theorem D be generalized to the case* $\lambda \in \mathbb{R}$*?* This is just cases (A_n) and (B_n) of Section 3.

5. *Question 2: Is there any point* (a, b, c, d) *that is not integrable for all* $\lambda \in \mathbb{Q}^+$*?* Such an example is given in [2].

6. *Question 3: What happens when* $p + q$ *tends to infinity? Will the varieties of integrable and linearizable systems remain in a particular area, or will they be distributed randomly or uniformly over the parameter space.* This is a very interesting question: from our results, it would seem that the integrable/linearizable systems are indeed restricted to certain regions of parameter space (for example, $c/a < 1$ in Case (A_n)), but that over this subspace there is some uniformity (for example Theorem 7) rather than randomness.

7. *Question 4: How many saddle quantities are required to determine the sufficient conditions for integrability for a given* $\lambda \in \mathbb{Q}$*?* In all the calculations we have done, the maximum required seems to be three.

9 Appendix

Theorem 9.1 *We consider an analytic system* (6.10) *with a saddle-node at the origin and analytic center manifold. Suppose that the normalizing change of coordinates* $y = Y + \sum_{k\geq q} g_k(x)Y^k$*, is such that* $g_q(x)$ *is divergent, then the conjugacy class of* (6.6) *is characterized by* $(1, a, [(\psi^0, \psi^\infty)])$ *where*

- *1 is the codimension of the saddle-node;*
- a *is the formal invariant;*
- $[(\psi^0, \psi^\infty)]$ *is an equivalence class of pairs of germs of analytic diffeomorphisms, where* ψ^∞ *is the identity (the linearity reflects the analyticity of the center manifold) and* $\psi^0(z) = e^{-2\pi ia}z + Cz^q + o(z^q)$ *with* $C \neq 0$.

(Note that a *can be recovered from* ψ^0 *and* ψ^∞*. So the conjugacy class is characterized by* $(1, [(\psi^0, \psi^\infty)])$*only.) Moreover if* $f_q(x) = \sum_{l\geq 1} a_l x^l$ *then* C *is given in* (6.14).

Proof. By a change of coordinate $y = y_1 + by_1^n$ with appropriate b we can of course suppose that $f_q(0) = 0$. The method is similar to the one used in [13]. We look for a formal change of coordinate $y = Y + \sum_{n\geq q} g_n(x)Y^n$ bringing the system to the normal form

$$\begin{array}{rcl} \dot{x} & = & x^2 \\ \dot{Y} & = & Y(1+ax). \end{array} \tag{9.1}$$

The change of coordinate is Borel-summable (1-summable) in all directions except the negative real axis. It is the composition of the successive changes of coordinates $y = y_q, \dots, y_n = y_{n+1} + h_n(x)y_{n+1}^n$ removing the y_n^n terms of the system. $h_q(x)$ must satisfy the linear differential equation

$$x^2 h_q'(x) + (q-1)(1+ax)h_q(x) - f_q(x) = 0, \tag{9.2}$$

$h_q(0) = 0$, with solution

$$h_q(x) = x^{a(q-1)} e^{\frac{q-1}{x}} \int_0^x e^{-\frac{q-1}{t}} t^{a(q-1)-2} f_q(t)dt \tag{9.3}$$

The solution is Borel-summable (1-summable) except in the direction $\mathbb{R}^-$. This yields to a local solution $h_q(x)$ defined for $|x| < r$ and $\arg x \in (-\frac{3\pi}{2} + \epsilon, \frac{3\pi}{2} - \epsilon)$ where $0 < \epsilon < \frac{\pi}{2}$. The normalized system has a first integral:

$$\begin{array}{rcl} H(x,y) & = & Yx^{-a}e^{\frac{1}{x}} \\ & = & (y - h_q(x)y^q + \dots)x^{-a}e^{\frac{1}{x}}. \end{array} \tag{9.4}$$

We are interested to the transformation between the two determinations of H in the region $\Re x < 0$. Let H_+ and H_- be these two determinations corresponding to the respective two determinations of h_q^+ and h_q^- of h_q in the region $\Re x < 0$. Then

$$H_+ = e^{-2\pi ia}H_- + (k_q^+(x) - k_q^-(x))H_-^q + o(H_-^q) = \psi_q(H_-), \tag{9.5}$$

where

$$k_q^{\pm}(x) = x^{(q-1)a} e^{-\frac{q-1}{x}} h_q^{\pm}(x). \tag{9.6}$$

The coefficient C we are looking for is given by

$$C = x^{(q-1)a} e^{-\frac{q-1}{x}} (h_q^+(x) - h_q^-(x)) = k_q^+(x) - k_q^-(x). \tag{9.7}$$

It is calculated as in [13]. We let

$$l_q(x) = \int_0^x e^{-\frac{q-1}{t}} t^{a(q-1)-2} \overline{f}_q(t)dt. \tag{9.8}$$

Then $C = l_q^+(x) - l_q^-(x)$, where again $l_q^{\pm}(x)$ are the two determinations of $l_q(x)$ in the region $\Re x < 0$. We make the change of variable $X = -\frac{x}{q-1}$, $T = -\frac{t}{q-1}$. Then

$$l_q(x) = (1-q)^{a(q-1)-1} \int_0^X e^{\frac{1}{T}} T^{a(q-1)-2} \overline{f}_q(T(1-q))dT. \tag{9.9}$$

The change of variable $\frac{1}{T} - \frac{1}{X} = -\frac{\xi}{X}$ yields

$$\begin{aligned} l_q(x) &= -(1-q)^{a(q-1)-1} X^{a(q-1)} e^{\frac{1}{X}} \textstyle\int_0^\infty \overline{f}_q(\frac{X(1-q)}{1-\xi})(1-\xi)^{-(q-1)a} e^{-\frac{\xi}{X}} \frac{d\xi}{X} \\ &= -e^{\frac{1}{X}} \textstyle\sum_{k\geq 1} a_k (1-q)^{a(q-1)+k-1} X^{a(q-1)+k} \int_0^{-\infty} (1-\xi)^{-(q-1)a-k} e^{\frac{-\xi}{X}} \frac{d\xi}{X}. \end{aligned} \tag{9.10}$$

The two determinations are obtained by taking integration along two half-lines D^+ and D^- obtained by making $\mathbb{R}^+$ rotate in the positive (resp. negative) direction. We need to calculate the difference as $D^{\pm}$ approach $\Re X > 0$.

We use Lemma 5.1 of [13] which proves that

$$I(\alpha; X) = \int_{D^+ - D_-} (1-\xi)^{\alpha} e^{-\frac{\xi}{X}} \frac{d\xi}{X} = -\frac{2i\pi}{\Gamma(-\alpha) X^{\alpha} e^{-\frac{1}{X}}}. \tag{9.11}$$

Hence

$$C = l_q^+(X) - l_q^-(X) = -2i\pi \sum_{k\geq 1} \frac{a_k (1-q)^{(q-1)a+k-1}}{\Gamma(a(q-1)+k)}. \tag{9.12}$$

□

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