Finite cyclicity of nilpotent graphics of pp-type surrounding a center *

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Abstract

This paper is part of the DRR-program of [4] to prove the finiteness part of Hilbert's 16th problem for quadratic vector fields by showing the finite cyclicity of 121 graphics. In this paper we prove the finite cyclicity of 4 graphics passing through a triple nilpotent point of elliptic type surrounding a center, namely the graphics (H_7^1), (F_{7a}^1), (H_{11}^3) and (I_{6a}^1). These four graphics are of pp-type, in the sense that they join two parabolic sectors of the nilpotent point. The exact cyclicity is 2 for (H_7^1) and (H_{11}^3). The graphics (F_{7a}^1) and $I_{6a}^1)$ occur in continuous families. Their exact cyclicity is 2 except for a discrete subset of such graphics. The method can be applied to most other graphics of the DRR-program [4] through a triple nilpotent point and surrounding a center.

This paper is dedicated to Freddy Dumortier, our estimated collaborator and dear friend, with whom we started the DRR-program in 1990.

1 Introduction

In the paper [4], together with Freddy Dumortier we presented a program (called the DRR-program to prove the finiteness part of Hilbert's 16th problem. The DRR-program consisted in the proof of the finite cyclicity of 121 graphics among the family of quadratic systems. This paper is a contribution to this program. Indeed we show the finite cyclicity of the graphics (H_7^1), (F_{7a}^1), (H_{11}^3) and (I_{6a}^1) on the Poincaré sphere (see Figure 1 and Figure 2 for the common blow-up of the singularity). These graphics surround a center and for this reason we call them *center graphics*.

Definition 1.1 A graphic of an analytic vector field $\mathcal{X}$ is called a *center graphic* if for any tubular neighborhood W of the graphic, there exists an analytic deformation $\mathcal{X}_\varepsilon$ depending on a finite number of parameters such that $\mathcal{X}_0 = \mathcal{X}$ and for each $\eta > 0$ there exists ε with $|\varepsilon| < \eta$ such that $\mathcal{X}_\varepsilon$ has an annulus of periodic solutions inside W.

The corresponding generic graphics (when the system has a focus) have been shown to have finite cyclicity in [6]. These graphics are called pp-graphics in [6] and [7], to mean that they join two parabolic sectors of the nilpotent point of multiplicity three and elliptic type.

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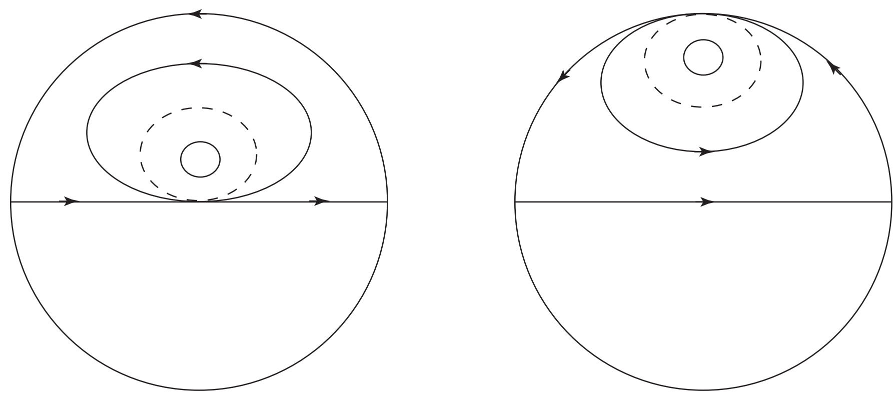


(a) The family of graphics (F^1_{7a}) ending in the hemicycle (H^1_7)

(b) The family of graphics (I^1_{6a}) ending in the hemicycle (H^3_{11})

Figure 1: The families of graphics we study here do not include the inner graphic which is the only one having a return map. The surrounding circle is the equator of the Poincaré sphere.

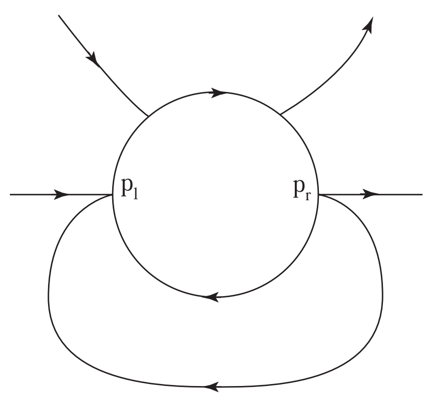
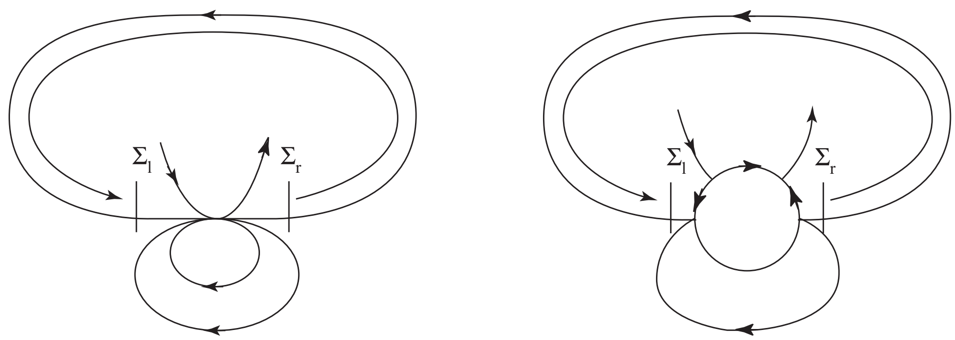


Figure 2: Common blow-up of the nilpotent singularity

In the paper [7] it is shown that such a graphic has cyclicity $\leq n$ if it satisfies the following genericity condition: the n-th derivative of the regular transition map along the graphic (see Figure 3) is nonzero.

Genericity conditions involving higher order derivatives are typical for graphics occurring in continuous families of graphics, as is the case with the pp-graphics considered here and in [7] and [6]. The ingredient of the proof was the blow-up of the family unfolding the nilpotent point. In the paper [6] it was shown that this genericity condition held for the pp-graphics of the DRR-program for quadratic systems ([4]) when the system was not integrable. We now show that the Bautin trick can be used to transform the proof of finite cyclicity in the generic case into a proof of finite cyclicity for the similar center pp-graphics. The proof works easily because the Bautin ideal is radical in the cases considered here. Indeed we can decompose a



(a) A pp-graphic (b) The graphic in the blow-up

Figure 3: The regular transition for a pp-graphic

displacement map $\delta_\varepsilon(z)$ in some center ideal obtaining some form

$$\delta_\varepsilon(z) = \sum_{j=1}^{n} \tilde{\varepsilon}_j h_j(z,\varepsilon) \tag{1.1}$$

with $n \geq 3$, where $\{\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \tilde{\varepsilon}_n\}$ generate the center ideal and the function $h_j(z,\varepsilon)$ have strictly increasing order in z at $z = z_0$. For all but a discrete subset of graphics, and also for the hemicycle graphics (H_7^1) and H_{11}^3), n is equal to 2 yielding an exact cyclicity of 2.

In principle it could be difficult to verify genericity conditions involving higher order derivatives of regular transitions: indeed the calculations of such higher order derivatives is quite involved. Moreover the regular transitions are defined on sections parallel to the axes in normalizing coordinates. The change of coordinates to normalizing coordinates should be taken into account when calculating these derivatives. In [6] a trick proved in [1] was used to minimize the calculations, which consisted in remarking that the sections in normalizing coordinates could be taken analytic and parameterized by analytic coordinates. Thus, as soon as the regular transition was nonlinear at one point on the section (corresponding to one graphic), it was nonlinear at all points of the section. It hence sufficed to verify the genericity condition near the limit graphic given by the hemicycle, where it followed from the fact that the hyperbolicity ratios were different from 1 and the regular transition along an invariant line of the limiting graphic formed by the hemicycle had a nonzero second derivative. The same kind of trick is used in this paper to make the division (1.1), thus allowing to reduce the calculations to a minimum. More involved calculations with Abelian integrals could permit to transform the finite cyclicity argument into an exact cyclicity argument for the center graphics.

We believe that the same kind of argument should allow to treat the other center graphics in the DRR-program of [4], once the finite cyclicity of the corresponding generic graphics is done (see discussion in Section 8).

2 Normal forms for quadratic families unfolding a graphic with a triple nilpotent point and surrounding a center

Such a triple nilpotent singular point can be of saddle or elliptic type.

2.1 The case of a nilpotent point in the finite plane

Proposition 2.1 *A quadratic system with a nilpotent point of saddle or elliptic type in the finite plane and a point of center type can always be brought to the form*

$$\begin{aligned} \dot{x} &= y + a_0x^2 - y^2 \\ \dot{y} &= xy, \end{aligned} \tag{2.1}$$

with $a_0 \in \mathbb{R}$ (see Figure 1(a)). The point is of saddle type if $a_0 < 0$ and of elliptic type if $a_0 > 0$. The system is part of the stratum of reversible centers.

The case $a_0 = -\frac{1}{2}$ corresponds to the particular case of a system both Hamiltonian and reversible.

The case $a_0 = 1$ corresponds to the particular case of a reversible system with a triple invariant line. Hence it is at the intersection of two strata.

The hemicycle surrounding the center has a return map if $a_0 \geq \frac{1}{2}$. So the graphics (H_7^1) and (F_{7a}^1) correspond to $0 < a_0 < \frac{1}{2}$.

Proof. If a quadratic system has a triple nilpotent point at the origin of saddle or elliptic type then it has an invariant line which we can always suppose to be the line $y = 0$. Then necessarily the linear part has the form $y\frac{\partial}{\partial x}$. Modulo a scaling in x we can suppose that the second equation has the form $\dot{y} = xy + by^2$. A change of coordinate $X = x + by$ allows to bring it to the form $\dot{y} = Xy$. Then the singular point is on the line $X = 0$. A scaling in y allows to bring the singular point at $(0, 1)$. The system then has the form (calling the coordinates (x, y)):

$$\begin{aligned} \dot{x} &= y + a_0x^2 + cxy - y^2 \\ \dot{y} &= xy. \end{aligned} \tag{2.2}$$

The divergence vanishes at $(0, 1)$ if and only if $c = 0$. □

Proposition 2.2 *In the case $a_0 \neq 1$ the general (5-parameter) quadratic perturbation of (2.1) can be brought by an affine change of coordinate and time scaling depending analytically on the parameters to the form*

$$\begin{aligned} \dot{x} &= y + ax^2 - y^2 + \varepsilon_4xy + \varepsilon_1 \\ \dot{y} &= xy + \varepsilon_2 + \varepsilon_3y, \end{aligned} \tag{2.3}$$

where

$$\varepsilon_0 = a - a_0$$

is a small parameter.

In the neighborhood of $a_0 = -\frac{1}{2}$ the system is Hamiltonian under the conditions:

$$\begin{cases} a + \frac{1}{2} = 0 \\ \varepsilon_3 = 0 \\ \varepsilon_4 = 0, \end{cases} \tag{2.4}$$

and reversible under the conditions:

$$\begin{cases} \varepsilon_2 = 0 \\ \varepsilon_3 = 0 \\ \varepsilon_4 = 0. \end{cases} \tag{2.5}$$

Except in the case $a \in \{0, \frac{1}{2}\}$ *the first integral is given by*

$$H = x^{-2a}\left(y^2 + \frac{2(1-a)}{2a-1}y + (1-a)x^2 + \frac{\varepsilon_1(1-a)}{a}\right). \tag{2.6}$$

PROOF. The general quadratic perturbation is of the form

$$\begin{aligned} \dot{x} &= y + a_0x^2 - y^2 + \textstyle\sum_{0\le i+j\le 2} a_{ij}x^iy^j \\ \dot{y} &= xy + \textstyle\sum_{0\le i+j\le 2} b_{ij}x^iy^j. \end{aligned} \tag{2.7}$$

We consider a change of coordinate

$$(x, y) = (X + \delta_1 Y + \delta_3, \delta_2 X + Y + \delta_4) \tag{2.8}$$

for the family, which reduces to the identity for $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$. Such a change of coordinates brings (2.7) to the form

$$\begin{aligned} \dot{X} &= Y + bX^2 - Y^2 + \textstyle\sum_{0\le i+j\le 2} A_{ij}X^iY^j \\ \dot{Y} &= XY + \textstyle\sum_{0\le i+j\le 2} B_{ij}X^iY^j. \end{aligned} \tag{2.9}$$

We consider the function $(\delta_1, \delta_2, \delta_3, \delta_4, a_{ij}, b_{ij}) \mapsto (A_{00}, A_{10}, A_{11}, B_{00}, B_{10}, B_{01}, B_{20}, B_{02})$. The Jacobian at $a_{ij} = b_{ij} = 0$, namely

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2a_0 & 0 \\ 2a_0 - 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 - a_0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \tag{2.10}$$

has rank 4. Hence we can solve any set of 4 equations of the form $A_{ij} = 0$ or $B_{ij} = 0$ by the implicit function theorem, as long as they correspond to 4 linearly independent lines of the matrix. In particular we can never get rid of the constant term in $\dot{Y}$. The X^2 term in $\dot{Y}$ can only be removed when $a_0 \neq 1$, i.e. we have a simple singular point at infinity in the direction of the X-axis. When $a = 1$ we have a triple point at infinity in that direction, thus explaining the obstruction to remove this term. We choose to solve $A_{10} = B_{10} = B_{20} = B_{02} = 0$. Using scalings in (X, Y, t) allows to take $A_{01} = A_{02} = B_{11} = 0$. □

2.2 The case of a nilpotent point at infinity

The difference with the previous case is that the condition of having a triple nilpotent point at infinity is of codimension 2, since the equator is invariant.

Proposition 2.3 *A quadratic system with a nilpotent point of saddle or elliptic type at infinity, an invariant line and a point of center type can always be brought to the form*

$$\begin{aligned} \dot{x} &= 1 - y + A_0x^2 \\ \dot{y} &= xy, \end{aligned} \tag{2.11}$$

with $A_0 \in \mathbb{R}$ (see Figure 1(b)). The point is of saddle type if $A_0 > 1$ and of elliptic type if $A_0 < 1$. The system is part of the stratum of reversible centers. When the point is of elliptic type, the hemicycle surrounding the center has a return map if $A_0 \leq \frac{1}{2}$. So the graphics (H_{11}^3) and (I_{6a}^1) correspond to $\frac{1}{2} < A_0 < 1$.

Proof. If a quadratic system has a triple nilpotent point at infinity of saddle or elliptic type we can always suppose that it is located along the y-axis. We can also suppose that the other infinite point is along the x-axis, that the invariant line is the line $y = 0$ and that the center is above this line. Then necessarily the quadratic part has the form $(Ax^2 + Bxy)\frac{\partial}{\partial x} + (Cxy + Dy^2)\frac{\partial}{\partial y}$. In order that the singular point at infinity has two zero eigenvalues we get $B = D = 0$. It is triple if $C \neq 0$. A scaling in x allows to take $C = 1$ and the system has the form

$$\begin{array}{rcl} \dot{x} & = & Ax^2 + \alpha + \beta x + \gamma y \\ \dot{y} & = & xy + \delta y. \end{array} \tag{2.12}$$

A translation in x allows to bring the center on the y-axis and a scaling in y allows to suppose it is located at $(0,1)$. Hence $\delta = 0$ and $\alpha = -\gamma$. The divergence vanishes at $(0,1)$ if $\beta = 0$. As necessarily $\alpha \neq 0$ (since otherwise $x = 0$ is a line of singular points) a simultaneous scaling in (x,t) allows to take $\alpha = 1$. $\square$

Proposition 2.4 *In the case $A_0 \neq \frac{1}{2}, 1$, the general (5-parameter) quadratic perturbation of (2.11) can be brought by an affine change of coordinate and time scaling depending analytically on the parameters to the form*

$$\begin{array}{rcl} \dot{x} & = & 1 - y + Ax^2 + \varepsilon_1 y^2 + \varepsilon_3 xy \\ \dot{y} & = & xy + \varepsilon_2 + \varepsilon_3 y^2 + \varepsilon_4 y, \end{array} \tag{2.13}$$

where $A - A_0$ is a small parameter.

The system is integrable under the condition $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0$. It has an invariant line under the condition $\varepsilon_2 = 0$. The first integral is given by

$$H(x,y) = y^{-2A}\left(y + \frac{1-2A}{2}x^2 + \frac{1-2A}{2A} + \varepsilon_1 \frac{1-2A}{2(A-1)}y^2\right). \tag{2.14}$$

The two parameters $(\varepsilon_1, \varepsilon_3)$ unfold the nilpotent point at infinity.

Proof. The general quadratic perturbation is of the form

$$\begin{array}{rcl} \dot{x} & = & 1 - y + Ax^2 + \sum_{0 \leq i+j \leq 2} a_{ij}x^i y^j \\ \dot{y} & = & xy + \sum_{0 \leq i+j \leq 2} b_{ij}x^i y^j. \end{array} \tag{2.15}$$

We consider a change of coordinate

$$(x,y) = (X + \delta_1 Y + \delta_3, \delta_2 X + Y + \delta_4) \tag{2.16}$$

for the family, which reduces to the identity for $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$. Such a change of coordinates brings (2.15) to the form

$$\begin{array}{rcl} \dot{X} & = & 1 - y + A_0 x^2 + \sum_{0 \leq i+j \leq 2} A_{ij} X^i Y^j \\ \dot{Y} & = & XY + \sum_{0 \leq i+j \leq 2} B_{ij} X^i Y^j. \end{array} \tag{2.17}$$

We consider the function $(\delta_1, \delta_2, \delta_3, \delta_4, a_{ij}, b_{ij}) \mapsto (A_{10}, A_{11}, A_{02}, B_{00}, B_{10}, B_{01}, B_{20}, B_{02})$. The Jacobian at $a_{ij} = b_{ij} = 0$, namely

$$\begin{pmatrix} 0 & 2A_0 & -1 & 0 \\ 2A_0 - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 - A_0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.18}$$

has rank 4. We choose to solve $A_{10} = B_{10} = B_{20} = A_{11} - B_{02} = 0$. Using scalings in (X, Y, t) allows to take $A_{00} = A_{01} = B_{11} = 0$. □

The following calculation will be needed to derive the result.

Lemma 2.5 *The family* (2.13) *localized in the neighborhood of the singular point at infinity in the direction of the y-axis by means of $(v, w) = (\frac{x}{y}, \frac{1}{y})$ is given by*

$$\begin{array}{rcl} \dot{v} & = & -w + (A-1)v^2 + w^2 + \varepsilon_1 - \varepsilon_4 vw - \varepsilon_2 vw^2 \\ \dot{w} & = & -vw - \varepsilon_3 w - \varepsilon_4 w^2 - \varepsilon_2 w^3. \end{array} \tag{2.19}$$

3 Blowing up

In this section, we want to explain how to blow up the nilpotent singularity $p = (0,0) \in \mathbb{R}^2$ for the vector field unfolding $\mathcal{X}_\varepsilon$ defined by the differential equation (2.3).

The parameter set

$$\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_0), \tag{3.1}$$

includes the local parameter

$$\varepsilon_0 = a - a_0$$

where a_0 is fixed in $]0, \frac{1}{2}[$; we will use a as well as ε_0 chosen small enough so that $a \in]0, \frac{1}{2}[$.

We refer to [3] and [9] for the details about this technique. We will discuss in great detail the case of (2.3) and then give briefly the adjustments for (2.13). For instance we replace $\varepsilon_0 = a - a_0$ by $\varepsilon_0 = A - A_0$ in (3.1).

3.1 The blow-up of the family (2.3)

Taking into account the quasi-homogeneity properties of (2.3), it is natural to choose the following formula for the blowing up :

$$\varepsilon_1 = \nu^2 \bar{\varepsilon}_1, \quad \varepsilon_2 = \nu^3 \bar{\varepsilon}_2, \quad \varepsilon_3 = \nu \bar{\varepsilon}_3 \tag{3.2}$$

and also:

$$x = r\bar{x}, \quad y = r^2 \bar{y}, \quad \nu = r\rho. \tag{3.3}$$

Here,

$$\bar{\varepsilon} = (\bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3) \in \mathbb{S}^2_{PA} \approx \mathbb{S}^2, \tag{3.4}$$

where $\mathbb{S}^2_{PA} \approx \mathbb{S}^2$ is a parameter-sphere (i.e. $\{\bar\varepsilon_1^2+\bar\varepsilon_2^2+\bar\varepsilon_3^2=1\}$), $(\bar x,\bar y,\rho)\in\mathbb{S}^2_{PH}\approx\mathbb{S}^2$, a phase-sphere (i.e. $\{\bar x^2+y^2+\rho^2=1\}$) and ν is a small positive parameter (we will write: $\nu\in(\mathbb{R}^+,0)$). Let us notice that we do not blow up the parameters $\varepsilon_4,\varepsilon_0\in(\mathbb{R},0)$. As a consequence, the blown-up space $\mathcal{E}$ is a neighborhood of the 4-dimensional manifold $\mathbb{S}^2_{PH}\times\mathbb{S}^2_{PA}\times\{(0,0)\}$ in $\mathbb{S}^2_{PH}\times\mathbb{S}^2_{PA}\times\mathbb{R}^+\times\mathbb{R}^2$. We will denote by $\pi\ :\mathcal{E}\to\mathbb{R}^7$, the blowing-up map defined by the formulas (3.2), (3.3) and $\{\varepsilon_4=\varepsilon_4,\varepsilon_0=\varepsilon_0\}$. This map π has a phase-component $\pi_{PH}:(\bar x,\bar y,r)\to(x,y)$ and a parameter-component $\pi_P:(\bar\varepsilon_1,\bar\varepsilon_2,\bar\varepsilon_3,\varepsilon_4,\varepsilon_0,r,\rho)\to\varepsilon$. We define the blown-up vector field $\bar{\mathcal{X}}=\frac{1}{\nu}\hat{\mathcal{X}}$, where $\hat{\mathcal{X}}$ is the lift of the family $\mathcal{X}_\varepsilon$ (considered as a vector field in $\mathbb{R}^7$), in the blown-up space $\mathcal{E}$.

Let us consider one limit periodic set Γ of $\mathcal{X}_0$, of type (F^1_{7a}), that we want to study. Γ is union of the singular point p with a regular orbit which has p as ω and α limits. This type of limit periodic set is called a graphic. In the case of a limit periodic set of type (H^1_7), the regular orbit is replaced by the union of three regular orbits and two opposite saddle points on the equator. One regular orbit is given by the upper half of the equator, while the two others are given by the positive and negative x-axis. Let $\Sigma_l=\{-X\}\times[-Y,Y]$ and $\Sigma_r=\{X\}\times[-Y,Y]$. We choose $X>0,Y>0$ such that Σ_l,Σ_r are sections transverse to Γ and contained in $\pi(\mathcal{E})$. Let $y_0\in]-Y,Y[$ be the point which corresponds to the intersection of Γ with Σ_l (or Σ_r by symmetry). We will denote by $y\to R_\varepsilon(y)$ the transition map of X_ε from Σ_l to Σ_r following the flow backwards. This map is defined on $[-Y_0,Y_0]$ where $0<Y_0<Y$ and ε is near 0. We will denote by $y\to T_\varepsilon(y)$ the transition map of $\mathcal{X}_\varepsilon$ from Σ_l to Σ_r following the flow forwards. This map is not always defined. For instance some singular points can be in the way and forbid the transition. Also, due to a strong deviation, all orbits from points in Σ_l may arrive below or above Σ_r.

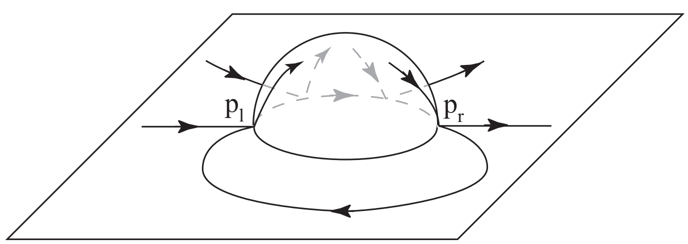


Figure 4: The critical locus $\{r\rho=0\}$ in the blow-up

Recall that for each $\bar\varepsilon\in\mathbb{S}^2$ the critical locus $C_{\bar\varepsilon}$ (Figure 4) is the union of a half-sphere $H\mathbb{S}_{\bar\varepsilon}\subset\mathbb{S}^2_{PH}\times\{\bar\varepsilon\}$, corresponding to $\rho\geq0$, and of the 2-dimensional blown-up space PH of the phase space, which is parameterized by $((\bar x,\bar y),\rho)\in\mathbb{S}^1\times\mathbb{R}^+$. These two parts are glued along their boundaries, defined in each of them by $\{\rho=0\}$. To obtain the cyclicity of Γ we have to consider all the graphics of the blown-up field $\bar{\mathcal{X}}$ which exist in $\mathcal{E}$ and are blown down in Γ (this means that they are sent onto Γ by the map π). Such a graphic exists in $C_{\bar\varepsilon}$ if a regular orbit $\gamma_{\bar\varepsilon}$ connects in $H\mathbb{S}_{\bar\varepsilon}$ the point $p_l=((-1,0),0)\in\mathbb{S}^1\times\mathbb{R}^+$ with the point $p_r=((1,0),0)\in\mathbb{S}^1\times\mathbb{R}^+$. We see below that this is the case for $\bar\varepsilon_2=0$. We also have the limit cases where there is an additional saddle-node on $\gamma_{\bar\varepsilon}$.

To study this connection $\gamma_{\bar\varepsilon}$ we work in the parameter directional chart defined by $\{\rho=1\}$. In this chart $r\equiv\nu$, and we will use this parameter ν. Also in this chart one chooses $(\bar x,\bar y)\in\bar D$, with $\bar D$ an arbitrarily large disk, $(\bar\varepsilon_1,\bar\varepsilon_3)\in\mathbb{S}^1$ and $\varepsilon_2,\varepsilon_4,\varepsilon_0,\nu$ near 0. As a consequence, the blown-up field is given in this chart by a vector field family $\bar{\mathcal{X}}_{\bar{\lambda}}$ in the phase space $\bar{D}$ and parameter:

$$\bar{\lambda} = ((\bar{\varepsilon}_1, \bar{\varepsilon}_3), \bar{\varepsilon}_2, \varepsilon_4, \varepsilon_0, \nu) \in \mathbb{S}^1 \times (\mathbb{R}, 0)^3 \times (\mathbb{R}^+, 0). \tag{3.5}$$

This vector field family $\bar{\mathcal{X}}_{\bar{\lambda}}$ is given by:

$$\begin{array}{rcll} \dot{\bar{x}} & = & \bar{y} + (a_0 + \varepsilon_0)\bar{x}^2 + \bar{\varepsilon}_1 & -\nu^2\bar{y}^2 + \nu\varepsilon_4\bar{x}\bar{y} \\ \dot{\bar{y}} & = & \bar{x}\bar{y} + \bar{\varepsilon}_3\bar{y} & +\bar{\varepsilon}_2, \end{array} \tag{3.6}$$

where we have pushed on the right the perturbative terms.

If $\bar{\varepsilon}_2 = 0$, the axis $\{\bar{y} = 0\}$ is invariant. Moreover, for $\bar{\varepsilon}_1 = 0$, the vector field $\bar{\mathcal{X}}_{(\bar{\varepsilon},0,0)}$ has a saddle-node singular point at $(\bar{x}, \bar{y}) = (0, 0)$ which bifurcates into two singular points for $\bar{\varepsilon}_1 < 0$. On the contrary, one has a connection from p_l and p_r when $\bar{\varepsilon}_2 = 0$ and $\bar{\varepsilon}_1 > 0$ (the points p_l, p_r are located on the circle at infinity in the phase plane $(\bar{x}, \bar{y})$). Inversely, if $\bar{\varepsilon}_2 \neq 0$, the field is transverse to the axis $\{\bar{y} = 0\}$. This prevents the possibility of a connection between p_l, p_r for $\bar{\varepsilon}_1 < 0$ or $\bar{\varepsilon}_2 \neq 0$. Finally, a connection exists if and only if $\bar{\varepsilon}_2 = 0$ and $\bar{\varepsilon}_1 > 0$. But we need also study the limiting case $\bar{\varepsilon}_1 = 0$ which can (and does) create limit cycles by perturbation.

As we are interested to the bifurcation of limit cycles, we will restrict $(\bar{\varepsilon}_1, \bar{\varepsilon}_3)$ to the interval $\overline{E} = \{(\bar{\varepsilon}_1, \bar{\varepsilon}_3) \in \mathbb{S}^1 \mid \bar{\varepsilon}_1 \geq 0\} \subset \mathbb{S}^1$ and we will call its interior $E = \{(\bar{\varepsilon}_1, \bar{\varepsilon}_3) \in \mathbb{S}^1 \mid \bar{\varepsilon}_1 > 0\} \subset \mathbb{S}^1$. $\overline{E}$ (resp. E) is parameterized by $\bar{\varepsilon}_3 \in [-1, +1]$ (resp. $\bar{\varepsilon}_3 \in]-1, +1[$), which we will identify with $\overline{E}$ (resp. E). Along E and $\overline{E}$, $\bar{\varepsilon}_1$ is function of $\bar{\varepsilon}_3$: $\bar{\varepsilon}_1(\bar{\varepsilon}_3) = \sqrt{1 - \bar{\varepsilon}_3^2}$. We will also write

$$\bar{\mu} = (\bar{\varepsilon}_3, \bar{\varepsilon}_2, \varepsilon_4) \in E \times (\mathbb{R}, 0) \times (\mathbb{R}, 0) \tag{3.7}$$

and we will identify $\bar{\lambda}$ with $(\bar{\mu}, \varepsilon_0, \nu)$ in $E \times (\mathbb{R}, 0)^3 \times (\mathbb{R}^+, 0)$. For convenience we will also introduce the parameter

$$\mu = (\bar{\mu}, \varepsilon_0) = (\bar{\varepsilon}_3, \bar{\varepsilon}_2, \varepsilon_4, \varepsilon_0) \in E \times (\mathbb{R}, 0)^3. \tag{3.8}$$

We will note $\gamma_{(\bar{\varepsilon}_1(\bar{\varepsilon}_3),0,\bar{\varepsilon}_3)}$ by $\gamma_{\bar{\varepsilon}_3}$. For each $\bar{\varepsilon}_3 \in E$, a unique limit periodic set $\Gamma_{\bar{\varepsilon}_3}$ is then defined in $C_{(\bar{\varepsilon}_1(\bar{\varepsilon}_3),0,\bar{\varepsilon}_3)}$: $\Gamma_{\bar{\varepsilon}_3} = \tilde{\Gamma} \cup \gamma_{\bar{\varepsilon}_3}$, where $\tilde{\Gamma}$ which connects p_l and p_r in PH, is the strict lift of Γ by the blowing-up.

To study the transition above $\gamma_{\bar{\varepsilon}_3}$, one chooses two sections $\Sigma'_l = \{-\bar{X}\} \times [-\bar{Y}, \bar{Y}]$ and $\Sigma'_r = \{\bar{X}\} \times [-\bar{Y}, \bar{Y}]$, transverse to $\bar{\mathcal{X}}_{\bar{\lambda}}$ for $\bar{\varepsilon}_2 = 0$ with a constant $\bar{X}$ taken large enough. Now, for $\bar{\varepsilon}_1 > 0$ and $\bar{\varepsilon}_2, \varepsilon_4, \varepsilon_0, \nu$ small enough, a regular transition map $\bar{T}_{\bar{\lambda}}(\bar{y})$ along the flow of $\bar{\mathcal{X}}_{\bar{\lambda}}$, is defined from a subsection $\Sigma''_l \subset \Sigma'_l$ into Σ'_r (one can choose $\Sigma''_l = \{-\bar{X}\} \times [-\tilde{Y}, \tilde{Y}]$ for some $\tilde{Y} < \bar{Y}$).

Now, taking $\tilde{Y}, \bar{Y}$ large enough, one can define a transition map $D^l_{\bar{\lambda}}(y) : \Sigma_l \to \Sigma'_l$ along the flow of $\bar{\mathcal{X}}_{\bar{\lambda}}$ near p_l and a transition map $D^r_{\bar{\lambda}}(y) : \Sigma_r \to \Sigma'_r$ along the flow of $-\bar{\mathcal{X}}_{\bar{\lambda}}$ near p_r. The transition map $T_\varepsilon = T_{\pi_P(\bar{\lambda})}$ is then given as the composition: $T_\varepsilon = (D^r_{\bar{\lambda}})^{-1} \circ \bar{T}_{\bar{\lambda}} \circ D^l_{\bar{\lambda}}$. We will see that the *natural* parameter for studying the finite cyclicity is $\bar{\lambda}$ rather than ε. So we want to consider T_ε as a function of $\bar{\lambda}$ and by abuse of notation we will use the same notation $T_\varepsilon = T_{\bar{\lambda}}$. In the next section we will use this composition to obtain a good presentation of $T_{\bar{\lambda}}$.

Remark 3.1 There exist three limit periodic sets in the blown-up vector field corresponding to

(i) $\bar{\varepsilon}_2 = 0$, $\varepsilon_1 > 0$;

(ii) $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = 0$, $\bar{\varepsilon}_3 = +1$;

(iii) $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = 0$, $\bar{\varepsilon}_3 = -1$.

The graphics in $\pi^{-1}(\Gamma) \subset \mathcal{E}$ are the graphics $\Gamma_{\bar{\varepsilon}_3}$ for $\bar{\varepsilon}_3 \in [-1,+1]$. As a consequence, one has that:

$$\mathrm{Cycl}(\mathcal{X}_\varepsilon, \Gamma) = \mathrm{Sup}\{\mathrm{Cycl}(\bar{\mathcal{X}}, \Gamma_{\bar{\varepsilon}_3}) \mid \bar{\varepsilon}_3 \in [-1,+1]\} \tag{3.9}$$

Then, to prove the finite cyclicity of our graphic we need to prove the finite cyclicity of each of these limit periodic sets. The limit periodic sets (ii) and (iii) have been shown in [7] to have cyclicity 1. The proof of [7] made no use of the genericity condition and is still valid here. So we only need to study the finite cyclicity of a graphic of type (i).

3.2 The blow-up of (2.13).

We consider here the blow-up of the family unfolding the nilpotent singular point. The family (2.13) localized at this point appears in (2.19). Here, only the parameters ε_1 and ε_3 unfold the nilpotent singularity. So they are the only parameters we blow-up. We replace (3.2) by

$$\varepsilon_1 = \nu^2 \bar{\varepsilon}_1, \quad \varepsilon_3 = \nu \bar{\varepsilon}_3. \tag{3.10}$$

As the nilpotent point appears at infinity the phase variables are now (v, z) in (2.19) we let

$$v = r\bar{v}, \quad w = r^2 \bar{w}, \quad \nu = r\rho. \tag{3.11}$$

4 Presentation of the transition maps for the graphics (F^1_{7a}) and (H^1_7)

In this section, we will restrict to $(\bar{\varepsilon}_1, \bar{\varepsilon}_3) \in \mathbb{S}^1$ with $\bar{\varepsilon}_1 > 0$, which, because of Remark 3.1, is the only case to discuss. We can parameterize this arc of circle E by $\bar{\varepsilon}_3 \in]-1,+1[$ and we will consider $\bar{\varepsilon}_1$ as a function of $\bar{\varepsilon}_3 : \bar{\varepsilon}_1(\bar{\varepsilon}_3) = \sqrt{1-\bar{\varepsilon}_3^2}$.

4.1 Presentation of $\bar{T}_{\bar{\lambda}}$ for the graphic (F^1_{7a})

This maps goes from a section $\Sigma_l = \{\bar{x} = -\bar{X}\}$ to the symmetric section $\Sigma_r = \{\bar{x} = \bar{X}\}$ over the blow-up sphere in the family chart $\rho = 1$.

Proposition 4.1 *For $\bar{\varepsilon}_3 \in E$ the transition map $\bar{T}_{\bar{\lambda}}$ is analytic in $(\bar{y}, \bar{\lambda})$ and has the following presentation:*

$$\bar{T}_{\bar{\lambda}}(\bar{y}) = \bar{y} + \alpha(\bar{\lambda}) + \beta(\bar{\lambda})\bar{y} + \bar{y}^2(\bar{\varepsilon}_2\Phi_2(\bar{y}, \bar{\lambda}) + \bar{\varepsilon}_3\Phi_3(\bar{y}, \bar{\lambda}) + \nu\varepsilon_4\Phi_4(\bar{y}, \bar{\lambda})) \tag{4.1}$$

with $\alpha(\bar{\lambda}) = c_2(\bar{\lambda})\bar{\varepsilon}_2$, $\beta(\bar{\lambda}) = c_3(\bar{\lambda})\frac{\bar{\varepsilon}_3}{\sqrt{\bar{\varepsilon}_1}} + O(\bar{\varepsilon}_2)$. The function c_2 is analytic and strictly positive in $E \times (\mathbb{R}, 0)^3 \times (\mathbb{R}^+, 0)$ and the function c_3 is analytic and strictly positive in $\bar{E} \times (\mathbb{R}, 0)^3 \times (\mathbb{R}^+, 0)$ (in particular it is strictly positive and analytic at $\bar{\varepsilon}_3 = \pm 1$).

Proof. Let us fix $(\bar\varepsilon_1, \bar\varepsilon_3) \in E$ as above. We observe that the term $\nu\varepsilon_4$ enters as a parameter in the equation (3.6). Moreover the phase portrait of (3.6) is invariant by the symmetry $(\bar x, \bar y) \to (-\bar x, \bar y)$ when $\bar\varepsilon_2 = \bar\varepsilon_3 = \nu\varepsilon_4 = 0$. As the sections Σ_l' and Σ_r' are exchanged by this symmetry, one has that $\bar T_{\bar\lambda}(\bar y) \equiv \bar y$ when $\bar\varepsilon_2 = \bar\varepsilon_3 = \nu\varepsilon_4 = 0$. It follows that $\bar T_{\bar\lambda}(\bar y)$ has the form (4.1) with α, β in the ideal generated by: $\bar\varepsilon_2, \bar\varepsilon_3, \nu\varepsilon_4$.

We now proceed to obtain more information about α and β. Recall $\{\bar y\} = 0$ is an orbit of $\bar{\mathcal{X}}_{\bar\lambda}$ when $\bar\varepsilon_2 = 0$. Along this orbit the differential equation reduces to

$$\dot{\bar x} = a\bar x^2 + \bar\varepsilon_1. \tag{4.2}$$

It follows that the time to go from $-\bar X$ to some value $\bar x \in [-\bar X, \bar X]$ is equal to:

$$t(\bar x) = \frac{1}{\sqrt{a\bar\varepsilon_1}}\left(\arctan\left(\sqrt{\frac{a}{\bar\varepsilon_1}}\bar x\right) + \arctan\left(\sqrt{\frac{a}{\bar\varepsilon_1}}\bar X\right)\right)$$

and if $\bar x(t)$ is the inverse function of $t(\bar x)$, $g(t) = (\bar x(t), 0)$ is the trajectory with initial condition $\bar x(0) = -\bar X, \bar y(0) = 0$, when $\bar\varepsilon_2 = 0$. Let us notice that $g(t)$ is in fact a trajectory of $\bar{\mathcal{X}}_{\bar\lambda}$ as soon as $\bar\varepsilon_2 = 0$ (for any value of $\bar\varepsilon_3, \nu, \varepsilon_4$). It follows that α is divisible by $\bar\varepsilon_2 : \alpha(\bar\lambda) = c_2(\bar\lambda)\bar\varepsilon_2$, for an analytic function c_2. We must show that $c_2(\bar\lambda) > 0$. Now, for $\nu = 0$, the $\bar\varepsilon_2$-derivative $\nabla(t)$ of the $\bar y$-component of the flow of (3.6) at the time t, in the neighborhood of the trajectory $g(t)$, verifies the differential equation $\frac{d}{dt}\nabla = (\bar x(t) + \bar\varepsilon_3)\nabla(t) + 1$ with $\nabla(0) = 0$. It follows easily by integration of the differential equation that $\nabla(t) > 0$ for any $t > 0$ and then that $c_2|_{\{\bar\varepsilon_2 = 0\}} = \nabla(t(\bar X)) > 0$. Then $c_2(\bar\lambda) > 0$.

The coefficient $\beta(\bar\lambda)$ is equal to the derivative $\frac{d\bar T_{\bar\lambda}}{d\bar y}(0) - 1$. For $\bar\varepsilon_2 = 0$, one has:

$$\frac{d\bar T_{\bar\lambda}}{d\bar y}(0) = \exp\int_{-\bar X}^{+\bar X} \mathrm{div}(\bar{\mathcal{X}}_{\bar\lambda})dt = \exp\int_{-\bar X}^{+\bar X} \frac{2a\bar x + \bar x + \bar\varepsilon_3}{a\bar x^2 + \bar\varepsilon_1}d\bar x.$$

The integral reduces to

$$\int_{-\bar X}^{+\bar X} \frac{\bar\varepsilon_3}{a\bar x^2 + \bar\varepsilon_1}d\bar x = \frac{\bar\varepsilon_3}{\sqrt{\bar\varepsilon_1}}\int_{-\frac{\bar X}{\sqrt{\bar\varepsilon_1}}}^{\frac{\bar X}{\sqrt{\bar\varepsilon_1}}} \frac{d\xi}{a\xi^2 + 1} = c\frac{\bar\varepsilon_3}{\sqrt{\bar\varepsilon_1}}(1 + O(\sqrt{\bar\varepsilon_1})),$$

for some constant $c > 0$. Finally we have that $\frac{d\bar T_{\bar\lambda}}{d\bar y}(0) = \exp c\frac{\bar\varepsilon_3}{\sqrt{\bar\varepsilon_1}}(1 + O(\sqrt{\bar\varepsilon_1}))$. This implies that $\beta(\bar\lambda)$ is divisible by $\bar\varepsilon_3$ when $\bar\varepsilon_2 = 0$. As a consequence one has that $\beta(\bar\lambda) = c_3(\bar\lambda)\frac{\bar\varepsilon_3}{\sqrt{\bar\varepsilon_1}} + O(\bar\varepsilon_2)$ with a function c_3 as in the statement. It is possible to show that c_3 is bounded and bounded away from 0 in the neighborhood of $\bar\varepsilon_3 = \pm 1$. We do not write the details since the study of the finite cyclicity in the neighborhood of $\bar\varepsilon_3 = \pm 1$ is covered by Remark 3.1. □

4.2 Presentation of the Dulac transitions near the points p_l and p_r

These transitions are commonly called Dulac maps in the literature: see Figure 5 for a Dulac map near p_l.

For $i = l, r$, the singular point p_i is located in a phase-directional chart defined by $\bar x = \pm 1$ (-1 for $i = l$ and $+1$ for $i = r$). In these charts the blown-up field $\bar{\mathcal{X}}$ is a family of vector fields $\bar{\mathcal{X}}_\mu$ defined in a neighborhood of $0 \in \mathbb{R}^3$ with coordinates $(\bar y, \rho, r)$ and parameter defined in (3.8). (The difference with $\bar\lambda$ is that now ν is a variable instead of a parameter. Recall that

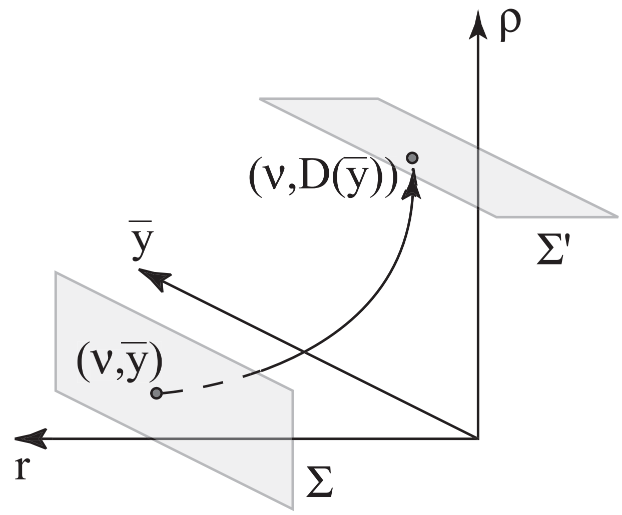


Figure 5: A Dulac map

$E =]-1,+1[$ and that $\bar{\varepsilon}_1 = \sqrt{1-\bar{\varepsilon}_3^2}$.) We have integrability through symmetry precisely when $\bar{\mu} = 0$ with $\bar{\mu}$ given in (3.7). In this subsection we will rely heavily on the paper [7], some results of which we recall. First, we recall that the point p_i is a saddle point with eigenvalues $\{-1, +1, -\sigma(a)\}$, where

$$\sigma(a) = \frac{1-2a}{a} \tag{4.3}$$

and we recall that the graphic occurs for $\sigma(a) > 0$.

In the chart, we consider sections $\bar{\Sigma}_i$ given by $\rho = \rho_0$ and sections $\bar{\Sigma}_i'$ given by $r = r_0$, where ρ_0, r_0 are multiples of the above constants $X, \bar{X}$ (this will be discussed in more details below). Also in the chart, ν appears as a first integral: $\nu = r\rho$. In the neighborhood of p_i the natural coordinates are $(r, \rho, \bar{y})$. When $\bar{x} = \pm 1$, modulo a time scaling the system is given by

$$\begin{array}{rcl} \dot{r} & = & \pm r \\ \dot{\rho} & = & \mp \rho \\ \dot{\bar{y}} & = & \pm \frac{(1-2a)\bar{y} - 2\bar{y}^2 \pm \bar{\varepsilon}_3 \rho \bar{y} - 2\varepsilon_1 \rho^2 \bar{y} \mp 2\varepsilon_4 r \bar{y}^2 + 2r^2 \bar{y}^3 \pm \bar{\varepsilon}_2 \rho^3}{a + \bar{y} - r^2 \bar{y}^2 \pm \varepsilon_4 r \bar{y} + \rho^2 \bar{\varepsilon}_1}. \end{array} \tag{4.4}$$

Each section is naturally parameterized by $(\nu, \bar{y})$. With these coordinates the transition from $\bar{\Sigma}_i$ to $\bar{\Sigma}_i'$ takes the special form $(\nu, \bar{y}) \mapsto (\nu, D_\mu^i(\bar{y}, \nu))$. By abuse of notation, we will forget the first coordinate, so ν becomes a parameter for the second coordinate and for each value ν, we will denote by $D_{\bar{\lambda}}^i(\bar{y})$ the transition from $\bar{\Sigma}_i$ to $\bar{\Sigma}_i'$ (when $i = r$ the transition follows the flow backwards.) The graphic of $\bar{\mathcal{X}}_\mu$ we want to consider cuts Σ_i at the value $\bar{y} = \bar{y}_0^i$. It depends only on ε_4 and ε_0 (since the graphic is located in the plane $r = 0$).

The point p_i corresponds to $(r, \rho, \bar{y}) = (0, 0, 0)$ in (4.4). It is possible to bring (4.4) to normal form in the neighborhood of this point. Because of the form of the system, the normalizing change of coordinates to normal form has the simple form

$$\hat{y}^i = \bar{y} + o(|r, \rho, \theta|).$$

Hence it is remarkable that the sections $\bar{\Sigma}_i$ and $\bar{\Sigma}_i'$ above are parallel to the coordinate planes in the normalizing coordinate system. In the normalizing coordinate system the transition map is given by $\hat{D}_{\bar{\lambda}}(\hat{y}^i)$. Its special form is well described in [7]. In particular it is linear in the case $\sigma(a_0) \notin \mathbb{Q}$:

$$\hat{D}^i_{\bar{\lambda}}(\hat{y}^i) = \Big(\frac{\nu}{\nu_0}\Big)^{\bar{\sigma}^i} \hat{y}^i, \tag{4.5}$$

and all the intuition can be built on this case. This form comes from the fact that a $C^{N(K)}$ normal form is given by

$$\begin{array}{rcl} \dot{r} & = & \pm r \\ \dot{\rho} & = & \mp \rho \\ \dot{\hat{y}} & = & \pm \hat{y}\Big(\frac{1-2a}{a} + \sum_{j=1}^K \gamma_j^i \nu^i\Big), \end{array} \tag{4.6}$$

since all resonances involve powers of $\nu = r\rho$. We have that $N(K) \to +\infty$ when $K \to \infty$ and then $N(K)$ can be chosen arbitrarily large, but the neighborhood on which we can use (4.6) shrinks with K. Then

$$\bar{\sigma}^i = \frac{1-2a}{a} + \sum_{j=1}^K \gamma_j^i \nu^i.$$

Additional resonant terms have to be added in the case $\sigma(a_0) = \frac{p}{q}$. The computation of the Dulac map involves a compensator with parameter of the form

$$\alpha_1^i = p - \bar{\sigma}^i q$$

which is a smooth function of (μ, ν) (details in [7]). Also in the particular case $a_0 \in \mathbb{N}$ there is in general an additional constant term coming from the existence of a resonant term between r and $\hat{y}$. However this term does not exist in our family (4.4). This comes from the fact that we have been able to remove all pure terms in x in the $\dot{y}$ term of (2.3).

Notation 4.2

1. The symbol $O_P(\bar{\mu})$ where $\bar{\mu}$ has been defined in (3.7) refers to a function in the parameter $\bar{\lambda}$, belonging to the ideal generated by $\bar{\varepsilon}_2, \bar{\varepsilon}_3, \varepsilon_4$. Such a function is expressed as a combination: $\bar{\varepsilon}_2\Phi_2(\mu,\nu) + \bar{\varepsilon}_3\Phi_3(\mu,\nu) + \varepsilon_4\Phi_4(\mu,\nu)$.

2. The symbol $O_G(\bar{\mu})$ will denote a function of $(z, \bar{\lambda})$ also in the ideal generated by $\bar{\varepsilon}_2, \bar{\varepsilon}_3, \varepsilon_4$, but in the space of functions of $(z, \bar{\lambda})$. Such a function is expressed as a combination: $\bar{\varepsilon}_2\Phi_2(z,\mu,\nu) + \bar{\varepsilon}_3\Phi_3(z,\mu,\nu) + \varepsilon_4\Phi_4(z,\mu,\nu)$.

3. ω is the compensator defined for $z > 0$ by:

$$\omega(z,\alpha) = \begin{cases} \frac{z^{\alpha}-1}{\alpha} & \alpha \neq 0 \\ \ln z & \alpha = 0. \end{cases}$$

Summing all the results for D, introducing the variable $z = \hat{y}^i - \hat{y}_0^i$ and keeping the same letter $\hat{D}_{\mu,\nu}$ for the transition written in the z-coordinate we get:

Proposition 4.3 *For any $a_0 \in (0, \frac{1}{2})$ (rational or not) the map $\hat{D}^i_{\bar{\lambda}}$ has the form*

$$\hat{D}^i_{\bar{\lambda}}(z) = \hat{\eta}^i\Big(\bar{\lambda}\Big) + \Big(\frac{\nu}{\nu_0}\Big)^{\bar{\sigma}^i}\Big[z + \hat{\phi}^i\Big(z, \mu, \nu, \omega\Big(\frac{\nu}{\nu_0}, -\alpha_1^i\Big)\Big)\Big]. \tag{4.7}$$

In (4.7) $\bar{\sigma}^i$ *is a function of the parameter* $\bar{\lambda} = (\mu, \nu)$ *of the form*

$$\bar{\sigma}^i(\bar{\lambda}) = \sigma(a) + O_P(\bar{\mu}). \tag{4.8}$$

The function $\hat{\phi}$ *is* C^∞*. Letting* $\nu_0 = r_0\rho_0$*, one has that*

$$\hat{\eta}^i(\bar{\lambda}) = \hat{y}_0^i\Big(\frac{\nu}{\nu_0}\Big)^{\bar{\sigma}^i} \tag{4.9}$$

When $\sigma(a_0) \notin \mathbb{Q}$*, we have* $\hat{\phi}^i \equiv 0$ *yielding the particular case* (4.5)*.*

Proof. The formula (4.7) comes from [7]. (4.8) comes from the integrability of p_i when $\bar{\mu} = 0$.

The simple form of $\hat{\eta}$ in (4.9) comes from the special form of (4.6). In general, when $\sigma(a_0) = p \in \mathbb{N}$, then we could have one resonant term of the form r^p, which yields to the more complicated form of $\hat{\eta}$ described in [7]. This term does not appear in (4.6) which comes from the blow-up of a family of quadratic systems. □

Let us define the function

$$\hat{\varphi}^i(z, \mu, \nu) = \hat{\phi}^i\Big(z, \mu, \nu, \omega\Big(\frac{\nu}{\nu_0}, -\alpha_1^i\Big)\Big). \tag{4.10}$$

There exists $\tau > 0$ such that each partial derivative of $\hat{\varphi}^i$ in the variables $z, \bar{\varepsilon}_3, \bar{\varepsilon}_2, \varepsilon_4, \varepsilon_0$ is $O(\nu^\tau)$. The function $\hat{\varphi}^i$ is ν-regularly smooth in (z, μ) in the following sense:

Definition 4.4 A function $f(u, \nu)$ defined for $(u, \nu) \in \mathbb{R}^k \times \mathbb{R}^+$ is ν-regularly smooth in u if all the partial derivatives of f in u exist and are continuous in ν (including at the value $\nu = 0$).

Remark 4.5 (i) Let $f(u, \nu)$ be a function, ν-regularly smooth in u. If there exists some $\tau > 0$ such that each partial derivative of f in u is divisible by ν^τ, it is clear that we can write $f(u, \nu) = \nu^\tau \tilde{f}(u, \nu)$ with a function $\tilde{f}$ which is also ν-regularly smooth in u.

(ii) If the function $f(u, \nu)$ is ν-regularly smooth in u and $f(u, \nu) = 0$ when $u_1 = \cdots = u_j = 0$, for some j: $1 \le j \le k$, then we can write $f(u, \nu) = \sum_{i=1}^j u_i f_i(u, \nu)$, with factor functions f_i which are ν-regularly smooth in u. This can be proved by using a Taylor formula with an integral remainder.

Lemma 4.6 *In the family* (2.3) *there exists* $\tau > 0$ *such that* $\hat{\varphi}^i(z, \mu, \nu)$ *defined in* (4.10) *always has the form*

$$\hat{\varphi}^i(z, \mu, \nu) = \nu^\tau O_G(\bar{\mu}), \tag{4.11}$$

and the factors in the ideal decomposition are ν*-regularly smooth in* (z, μ)*.*

Proof. The singular point p_i is integrable for $\bar{\mu} = 0$: this is a direct consequence of the form of (2.6). So $\varphi|_{\bar{\mu}=0} \equiv 0$. It is proved in Theorem 4.10 of [7] that all partial derivatives of the functions $\hat{\varphi}^i$, in terms of z and the parameter μ are $O(\nu^\tau)$ for some $\tau > 0$. Then, using the Remark 4.5 for the division by ν^τ and next for the ideal decomposition in the $\bar{\mu}$-coordinates, we obtain (4.11). □

Now, an important remark is the symmetry that we have already noticed in the parameter-directional chart and which extends at the boundary to give the following symmetry property between the two phase-directional charts: for $\bar{\mu} = 0$, one has that $(\bar{\mathcal{X}}_0, p_r) = -(\bar{\mathcal{X}}_0, p_l)$. In fact, it was already established in [6], equation (4.10), page 349 that:

$$\bar{\sigma}^r\left(\bar{\lambda}\right) = \bar{\sigma}^l\left(\bar{\lambda}\right) + \nu O_P(\bar{\mu}), \quad \alpha_1^r\left(\bar{\lambda}\right) = \alpha_1^l\left(\bar{\lambda}\right) + \nu O_P(\bar{\mu}). \tag{4.12}$$

We use these equations (4.12) to obtain further division properties by a power of ν:

Proposition 4.7 *There exists $\tau > 0$ such that:*

$$\hat{\eta}^r\left(\bar{\lambda}\right) - \hat{\eta}^l\left(\bar{\lambda}\right) = \nu^\tau \hat{\Xi}\left(\bar{\lambda}\right) = \nu^\tau O_P(\bar{\mu}), \tag{4.13}$$

and $\hat{\Xi}\left(\bar{\lambda}\right)$ is ν-regularly smooth in μ and belongs to the ideal generated by $\bar{\varepsilon}_2, \bar{\varepsilon}_3, \varepsilon_4$.

Proof. The divisibility by a power of ν^τ with τ any positive real number strictly less than $\sigma(a)$ is obvious yielding $(\hat{\eta}^r - \hat{\eta}^l)(\bar{\lambda}) = \nu^\tau \hat{\Xi}(\bar{\lambda})$. We can decompose

$$\hat{\eta}^r\left(\bar{\lambda}\right) - \hat{\eta}^l\left(\bar{\lambda}\right) = \hat{y}_0^l\left(\left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}^r} - \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}^l}\right) + (\hat{y}_0^r - \hat{y}_0^l)\left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}^r}.$$

Now, we have:

$$\left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}^r} - \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}^l} = \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}^r}\left(\left(\frac{\nu}{\nu_0}\right)^{\nu O_P(\bar{\mu})} - 1\right) = \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}^r}\left(1 + \nu \log\left(\frac{\nu}{\nu_0}\right) O_P(\bar{\mu})\right). \tag{4.14}$$

The fact that $\hat{\Xi}$ is ν-regularly smooth in μ follows directly from (4.14). Now, as a consequence of $\hat{y}_0^r - \hat{y}_0^l = O_P(\bar{\mu})$, we have that $\hat{\Xi}(\mu, \nu) = O_P(\bar{\mu})$. □

Passing form $\hat{D}^i_{\bar{\lambda}}$ to $D^i_{\bar{\lambda}}$.

The first remark we can make is that we work in the charts: $\bar{x} = \pm 1$. Hence we can always take $r_0 = \pm X$, i.e. $\Sigma_i = \bar{\Sigma}_i$. Similarly we can take $\rho_0 = \pm \bar{X}_0$, i.e. $\Sigma'_i = \bar{\Sigma}'_i$. So the only change is the parametrization of the sections. The sections Σ_i and Σ'_i are parameterized by $\bar{y}$, while the sections $\bar{\Sigma}_1$ and $\bar{\Sigma}'_i$ are parameterized by $\hat{y}^i$, and the changes $\bar{y} \mapsto \hat{y}^i = h^i_\mu(\bar{y}, r, \rho)$ are $C^{N(K)}$ diffeomorphisms for some $N(K)$ larger than the order of the graphic to be defined below in Definition 4.11.

Because of the symmetry we also have that

$$h^r_\mu(\bar{y}, r, \rho) - h^l_\mu(\bar{y}, r, \rho) = O_G(\bar{\mu}).$$

This gives us for the maps $D^i_{\bar{\lambda}}(z)$

Proposition 4.8 *The maps $D^i_{\bar{\lambda}}(z)$ have the form*

$$D^i_{\bar{\lambda}}(z) = \xi^i\left(\bar{\lambda}\right) + \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}^i}\left[z + \psi^i\left(z, \bar{\lambda}\right)\right)\Big], \tag{4.15}$$

where

$$\begin{cases} \xi^l - \xi^r = \nu^\tau O_P(\bar{\mu}) \\ \psi^i = \nu^\tau O_G(\bar{\mu}), \end{cases}$$

and the functions ξ^i (resp. ψ^i) is ν-regularly smooth in μ (resp. (z, μ)).

We will go further and expand ψ^i as

$$\psi^i(z,\bar\lambda) = b_0^i(\bar\lambda) + b_1^i(\bar\lambda)z + z^2 g^i(z,\bar\lambda).$$

Then necessarily $b_0^i(\bar\lambda) = \nu^\tau O_P(\bar\mu)$, $b_1^i(\bar\lambda) = \nu^\tau O_P(\bar\mu)$ and $g^i(z,\bar\lambda) = \nu^\tau O_G(\bar\mu)$.

This gives us the final form for the maps $D^i_{\bar\lambda}(z)$

Proposition 4.9 *The maps $D^i_{\bar\lambda}(z)$ have the form*

$$D^i_{\bar\lambda}(z) = \eta^i\Big(\bar\lambda\Big) + \Big(\frac{\nu}{\nu_0}\Big)^{\bar\sigma^i}\Big[c^i(\bar\lambda)z + \varphi^i\Big(z,\bar\lambda\Big)\Big)\Big], \tag{4.16}$$

where

$$\begin{cases} \eta^l - \eta^r = \nu^\tau O_P(\bar\mu) \\ \varphi^i = \nu^\tau O_G(\bar\mu) O(z^2) \\ c^i(\bar\lambda) = 1 + O_P(\bar\mu), \end{cases}$$

and the functions η^i, c^i (resp. φ^i) are ν-regularly smooth in μ (resp. (z,μ)).

4.3 Presentation of the regular transition R_ε

Let us recall that the regular transition goes from Σ_l to Σ_r following the flow backwards. As the vector field $\mathcal{X}_\varepsilon$ has a symmetric phase diagram for $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0$ one has that:

$$R_\varepsilon(y) = R(y,\varepsilon) = y + \varepsilon_2 R_2(y,\varepsilon) + \varepsilon_3 R_3(y,\varepsilon) + \varepsilon_4 R_4(y,\varepsilon)$$

where the R_i are analytic functions of (y,ε). Moreover we can prove:

Proposition 4.10 *The function $y \to R_4(y,0)$ is not affine, or in other words $\frac{\partial^2 R_4}{\partial y^2}(y,0) \not\equiv 0$. As a consequence, for each y_0, there exists a minimum integer $k = k(y_0)$, $2 \le k < +\infty$ such that $\frac{\partial^k R_4}{\partial y^k}(y_0,0) \neq 0$. Moreover for all y_0 except a discrete subset we have $k(y_0) = 2$.*

PROOF. From the definition of Σ_i, $i \in \{r,l\}$ it is clear that R is analytic in (y,ε). It is shown in [6] (which studies the generic case) that $\frac{\partial^2 R}{\partial y^2}(y_0,\varepsilon) \neq 0$ for $\varepsilon_2 = \varepsilon_3 = 0$ and $\varepsilon_4 \neq 0$ and y_0 small. Indeed $y_0 = 0$ corresponds to the hemicycle. So it is possible to consider the expansion of R along the hemicycle. The calculation of this expansion makes use of the fact that the hyperbolicity ratio of the right (resp. left) saddle at infinity on the hemicycle is $s_r \in (0,1)$ (resp. $s_l = \frac{1}{s_r} > 1$) and that the regular transition along the upper half of the equator has a nonzero second derivative (see also the discussion of the hemicycle in Section 4.5 below). Consequently $R_4(y,0) = \frac{\partial R}{\partial \varepsilon_4}(y,0)$ is a nonlinear analytic map in y. In particular for each y_0 there exists $k \ge 2$ such that $\frac{\partial^k R_4}{\partial y^k}(y_0,0) \neq 0$. Moreover, except on a discrete subset we have $k(y_0) = 2$. □

Definition 4.11 If Γ is the graphic passing through $y_0 \in \Sigma^l$, we call order of Γ and denote by $\mathrm{ord}(\Gamma)$ the integer $k(y_0)$ given by Proposition 4.10.

Now, when we consider the displacement map from Σ_l to Σ_r we will consider it as a function of $\overline{y} = \frac{y}{r_0^2}$. As this change is linear it preserves the nonlinearity properties of R. We introduce the local coordinate $z = \bar{y} - \bar{y}_0$.

Notation. Using the blowing up formulas (3.2) we can use the parameter $\bar{\lambda} = (\mu, \nu)$ as the parameter of R, as well as the variable z. By abuse of notation we will use the same letters R (or $R_{\bar{\lambda}}$) and the R_i for the expression of R in the z coordinate and parameter $\bar{\lambda}$.

We have:

Corollary 4.12 *The expression of the regular transition from Σ_l to Σ_r following the flow backwards is given by*

$$R_{\bar{\lambda}}(z) = R(z, \bar{\lambda}) = z + \nu^3 \bar{\varepsilon}_2 R_2(z, \bar{\lambda}) + \nu \bar{\varepsilon}_3 R_3(z, \bar{\lambda}) + \varepsilon_4 R_4(z, \bar{\lambda}). \tag{4.17}$$

The functions R_i are analytic and the functions $R_i(z, (\mu, 0))$ are independent of $\bar{\mu}$. Moreover $\frac{\partial^k R_4}{\partial y^k}(0,0) \neq 0$, with $k = \mathrm{ord}(\Gamma)$, is the first non zero derivative of R_4 of order $k \geq 2$.

4.4 Change of parametrization on Σ_i'

We used the same letter $\bar{y}$ for the $\bar{y}$-coordinate when we calculated $D^i_{\bar{\lambda}}$ and $\bar{T}_{\bar{\lambda}}$. This is an abuse of notation as we calculate $D^i_{\bar{\lambda}}$ in the chart $\bar{x} = \pm 1$ and we compute $\bar{T}_{\bar{\lambda}}$ in the chart $\rho = 1$. The change from one to the other is a positive constant bounded and bounded away from zero. Moreover the constant depends only on $\bar{X}$, so it is the same on the left and on the right. This justifies a posteriori the abuse of notation.

4.5 Presentation of the transition $R_{\bar{\lambda}}$ for the graphic (H_7^1)

In the case of the graphic (H_7^1) there are two additional saddles P_l and P_r on the graphic, so the map $R_{\bar{\lambda}}$ is no longer regular at $\bar{y}_0 = 0$. Fortunately the connection between the two saddles is fixed allowing a nice expansion for $R_{\bar{\lambda}}$. All necessary calculations on the form of $R_{\bar{\lambda}}$ have been done in [6] in the generic case $\varepsilon_4 \neq 0$. They are still valid here with ε_4 replacing the nonzero coefficient of the xy-term in the first equation of (2.3). Let us discuss them.

As in the previous case the center case corresponds precisely to the case where there is symmetry with respect to the y-axis.

To calculate $R_{\bar{\lambda}}$ let us look at Figure 6.

It is necessary to take two additional sections Π_i and Π_i' in the neighborhood of each singular point at infinity P_i, $i = l, r$. These sections should be taken parallel to the coordinate axes in the normalizing coordinates near P_l and P_r. The hyperbolicity ratios at P_r and P_l are given by s_r and $s_l = \frac{1}{s_r}$ with $s_r < 1$. The transition map $R_{\bar{\lambda}}$ (following the flow backwards) is given by (to simplify, we do not write the dependence of all intermediate maps on $\bar{\lambda}$)

$$R_{\bar{\lambda}} = (S_r)^{-1} \circ (\Delta_r)^{-1} \circ (S_{inf})^{-1} \circ (\Delta_l)^{-1} \circ (S_l)^{-1}.$$

The functions S_l, S_r and S_{inf} are $C^{N(K)}$ regular transitions. The Dulac maps Δ_i are given by

$$\begin{cases} (\Delta_l)^{-1}(u) = u^{\frac{1}{s_l}}(1 + \Psi_l(\varepsilon, u)) \\ (\Delta_r)^{-1}(u) = u^{\frac{1}{s_r}}(1 + \Psi_r(\varepsilon, u)) \end{cases} \tag{4.18}$$

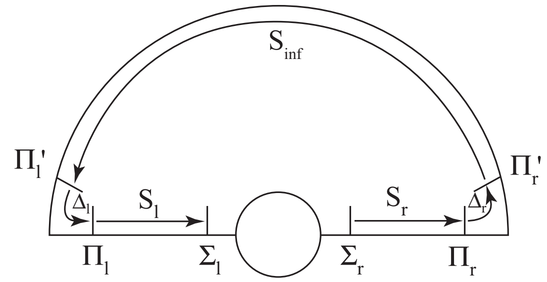


Figure 6: The sections for the graphic (H_7^1)

where the functions Ψ_i have the property (I) of Mourtada defined below in Definition 4.14, which implies

$$\frac{\partial^j(\Delta_l)^{-1}}{\partial u^j} = *u^{\frac{1}{s_l}-j}(1+\hat{\Psi}_i(\bar{\lambda},u)) \tag{4.19}$$

for $*$ a nonzero constant and $\hat{\Psi}_i(\varepsilon,u) = O(u)$, all this holding uniformly in ε for ε in a neighborhood of the origin. We also have

$$\Delta_r(u) = u^{\frac{1}{s_l}}(1+\Phi_r(\varepsilon,u))$$

with Φ_r of class (I) and moreover $\Psi_l(\varepsilon,u) - \Phi_r(\varepsilon,u) = O_G(\bar{\mu})$.

We will need the expression of $R_{\bar{\lambda}}$ in the variable $\bar{y} = \frac{y}{r_0^2}$, By abuse of notation we keep the same letter $R_{\bar{\lambda}}$ and we switch to the parameter $\bar{\lambda}$.

We have $S_l(y) = \varepsilon_2\xi_1(\bar{\lambda}) + S(y)$ with $S(y) = O(y)$ a $C^{N(K)}$-diffeomorphism, so that

$$\begin{array}{rcl} (S_l)^{-1}(y) & = & S^{-1}(y-\varepsilon_2\xi_1(\bar{\lambda})) \\ & = & S^{-1}\Big(r_0^2\big(\bar{y}-\varepsilon_2\frac{\xi_1(\bar{\lambda})}{r_0^2}\big)\Big) \\ & = & S^{-1}\Big(r_0^2\big(\bar{y}-\varepsilon_2\xi(\bar{\lambda})\big)\Big), \end{array}$$

with $\xi(\bar{\lambda}) = \frac{\xi_1(\bar{\lambda})}{r_0^2}$.

The form of the transition map $R_{\bar{\lambda}}$ is best given in the variable $\check{y} = \bar{y} - \varepsilon_2\xi(\bar{\lambda}) = (\zeta_{\bar{\lambda}})^{-1}(\bar{y})$ with image in the variable $\bar{y}$. It is given by

$$\check{R}(\check{y}) = \check{y} + \nu C_1(\bar{\lambda})\check{y} + \check{y}^{1+s_r}\Big(\nu^3\bar{\varepsilon}_2\psi_2(\bar{\lambda},\check{y}) + \nu\bar{\varepsilon}_3\psi_3(\bar{\lambda},\check{y}) + \varepsilon_4\psi_4(\bar{\lambda},\check{y})\Big) \tag{4.20}$$

where $\check{R}(\tilde{y}) - \check{y} = O_G(\bar{\mu})$ has the property (I) of Mourtada, (implying $C_1(\bar{\lambda}) = O_P(\bar{\mu})$) and moreover $\psi_4(0,0) \neq 0$. This last property comes from the fact that $S_{inf} - id = O_G(\mu)$ and the direct calculation $\frac{\partial^3 S_{inf}}{\partial u^2 \partial\varepsilon_4} \neq 0$ following from [6].

We have proved

Proposition 4.13 *There exists $\tau > 0$ such that the transition map $R_{\bar{\lambda}}$ in $\bar{y}$-coordinate, composed with the translation*

$$\bar{y} = \zeta_{\bar{\lambda}}(\check{y}) = \check{y} + \varepsilon_2\xi(\bar{\lambda}) = \check{y} + \nu^\tau O_P(\bar{\mu}) \tag{4.21}$$

has the form

$$\check{R}_{\bar{\lambda}}(\check{y}) = \check{y} + \nu^{\tau} O_G(\bar{\mu}) + \varepsilon_4 \Psi_4(\bar{\lambda}, \check{y}) \tag{4.22}$$

where

$$\Psi_4(\check{y}) = c_1(\bar{\lambda})\check{y} + c_2(\bar{\lambda})\check{y}^{1+s_r}(1 + \psi(\bar{\lambda}, \check{y})),$$

$c_2(0) \neq 0$ *and* ψ *has the property* (I) *defined in Definition* 4.14, *for some* N. *(*N *can be chosen arbitrarily large provided the sections* Π_i *and* Π_i' *are chosen sufficiently close to the singular points* P_i.*)*

Definition 4.14 A function $\Psi(\varepsilon, u)$ has the property (I) of Mourtada if Ψ is C^N for some N on $W \times]0, u_0[$, where W is some neighborhood of the origin in ε-space and if there exists some neighborhood W' of the origin in ε-space such that, for all $0 \leq j \leq N$,

$$\lim_{u \to 0} u^j \frac{\partial^j \Psi}{\partial u^j} = 0$$

uniformly for $\varepsilon \in W'$.

5 Equation for the limit cycles for the graphic (F_{7a}^1)

Let us recall that $\bar{\mu} = (\bar{\varepsilon}_2, \bar{\varepsilon}_3, \varepsilon_4)$ and that $O(\bar{\mu})$ denotes a function divisible in the ideal generated by $\bar{\varepsilon}_2, \bar{\varepsilon}_3, \varepsilon_4$. Recall that we have also introduced more precise notations in Notation 4.2: $O_P(\bar{\mu})$ and $O_G(\bar{\mu})$. For a local positive numerical function g we will also use the usual Landau notation $O(g)$.

5.1 The displacement map

It is convenient to interpret the formula (4.16) for $D_{\bar{\lambda}}^i$ as a composition of three maps: $D_{\bar{\lambda}}^i = \mathcal{T}_{\bar{\lambda}}^i \circ H_{\bar{\lambda}}^i \circ \tilde{D}_{\bar{\lambda}}^i$ where

$$\mathcal{T}_{\bar{\lambda}}^i(u) = u + \eta^i(\bar{\lambda}), \quad H_{\bar{\lambda}}^i(u) = \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}^i(\bar{\lambda})} u, \quad \tilde{D}_{\bar{\lambda}}^i(u) = c^i(\bar{\lambda})u + \varphi^i(u, \bar{\lambda}) \tag{5.1}$$

and $\varphi = O(u^2)$. Indeed as $\bar{T}_{\bar{\lambda}}(0)$ is usually nonzero, it could become large after composition by $(H_{\bar{\lambda}}^r)^{-1}$ and then cause problems in the composition with the nonlinear map $(\tilde{D}_{\bar{\lambda}}^r)^{-1}$. So we must avoid performing composition with $(\tilde{D}_{\bar{\lambda}}^r)^{-1}$ when $\bar{T}_{\bar{\lambda}}(0) \neq 0$.

The trick to avoid this difficulty is to use the diffeomorphisms $\tilde{D}_{\bar{\lambda}}^i$ to reparameterize the sections Σ_i. In these new parameterizations of Σ_i (that we continue to call z), the transition $T_{\bar{\lambda}}$ becomes:

$$\tilde{T}_{\bar{\lambda}} = (H_{\bar{\lambda}}^r)^{-1} \circ (\mathcal{T}_{\bar{\lambda}}^r)^{-1} \circ \bar{T}_{\bar{\lambda}} \circ \mathcal{T}_{\bar{\lambda}}^l \circ H_{\bar{\lambda}}^l$$

and the regular transition R becomes:

$$\tilde{R}_{\bar{\lambda}} = \tilde{D}_{\bar{\lambda}}^r \circ R_{\bar{\lambda}} \circ (\tilde{D}_{\bar{\lambda}}^l)^{-1}$$

Then in this new coordinate on the sections Σ_i, the displacement map is equal to $\delta_{\bar{\lambda}} = \tilde{R}_{\bar{\lambda}} - \tilde{T}_{\bar{\lambda}}$ and the equation for limit cycles is given by

$$\delta_{\bar{\lambda}}(z) = \tilde{R}_{\bar{\lambda}}(z) - \tilde{T}_{\bar{\lambda}}(z) = 0.$$

We now want to make precise $\tilde{R}_{\bar{\lambda}}, \tilde{T}_{\bar{\lambda}}$.

5.2 The transition $\tilde{T}_{\bar{\lambda}}$

We begin by looking to the composition $(\mathcal{T}^r_{\bar{\lambda}})^{-1} \circ \bar{T}_{\bar{\lambda}} \circ \mathcal{T}^l_{\bar{\lambda}}$.

Proposition 5.1

$$(\mathcal{T}^r_{\bar{\lambda}})^{-1} \circ \bar{T}_{\bar{\lambda}} \circ \mathcal{T}^l_{\bar{\lambda}}(u) = u + \tilde{\alpha} + \tilde{\beta}u + u^2(\bar{\varepsilon}_2\tilde{\Phi}_2 + \bar{\varepsilon}_3\tilde{\Phi}_3 + \nu\varepsilon_4\tilde{\Phi}_4) \tag{5.2}$$

where there exists a $\tau > 0$ such that

$$\tilde{\alpha} = \tilde{\alpha}(\bar{\lambda}) = \alpha(\bar{\lambda}) + \nu^\tau O_P(\bar{\mu}),$$
$$\tilde{\beta} = \tilde{\beta}(\bar{\lambda}) = \beta(\bar{\lambda}) + \nu^\tau O_P(\bar{\mu})$$

and α and β are the functions defined in Proposition 4.1. The functions $\tilde{\alpha}$ and $\tilde{\beta}$ are ν-regularly smooth in μ. The $\tilde{\Phi}_i = \tilde{\Phi}_i(u,\bar{\lambda})$ are functions of (u,μ,ν) which are ν-regularly smooth in (u,μ). Moreover $\tilde{\alpha} = O_P(\bar{\mu})$ and $\tilde{\beta} = O_P(\bar{\mu})$.

Proof. Introducing $\Psi_j(\bar{y},\bar{\lambda}) = \bar{y}^2\Phi_j(\bar{y},\bar{\lambda})$, $j = 2,3,4$, the formula (4.1) can be written:

$$\bar{T}_{\bar{\lambda}}(\bar{y}) = \bar{y} + \alpha(\bar{\lambda}) + \beta(\bar{\lambda})\bar{y} + \bar{\varepsilon}_2\Psi_2(\bar{y},\bar{\lambda}) + \bar{\varepsilon}_3\Psi_3(\bar{y},\bar{\lambda}) + \nu\varepsilon_4\Psi_4(\bar{y},\bar{\lambda}).$$

Let us consider $(\mathcal{T}^r_{\bar{\lambda}})^{-1} \circ \bar{T}_{\bar{\lambda}} \circ \mathcal{T}^l_{\bar{\lambda}}(u) = \bar{T}_{\bar{\lambda}}(u+\eta^l) - \eta^r$. It is equal to

$$u + \eta^l - \eta^r + \beta\eta^l + \alpha + \beta u + \bar{\varepsilon}_2\Psi_2(u+\eta^l,\bar{\lambda}) + \bar{\varepsilon}_3\Psi_3(u+\eta^l,\bar{\lambda}) + \nu\varepsilon_4\Psi_4(u+\eta^l,\bar{\lambda})$$

with

$$\alpha = \alpha(\bar{\lambda}), \quad \beta = \beta(\bar{\lambda}), \quad \eta^i = \eta^i(\bar{\lambda}).$$

We can expand:

$$\Psi_i(u+\eta^l,\bar{\lambda}) = \Psi_i(\eta^l,\bar{\lambda}) + u\frac{\partial\Psi_i}{\partial u}(\eta^l,\bar{\lambda}) + u^2\tilde{\Phi}_i(u,\bar{\lambda})$$

where $\tilde{\Phi}_i(u,\bar{\lambda})$ is ν-regularly smooth in (u,μ). One has that

$$\Psi_i(\eta^l,\bar{\lambda}) = O((\eta^l)^2) \text{ and } \frac{\partial\Psi_i}{\partial u}(\eta^l,\bar{\lambda}) = O(\eta^l).$$

Taking any strictly positive $\tau < \sigma(a)$ we have that $\eta^l = O(\nu^\tau)$ and then:

$$\tilde{\Phi}_i(u+\eta^l,\bar{\lambda}) = O(\nu^{2\tau}) + uO(\nu^\tau) + u^2\Psi_i(u,\bar{\lambda}) \tag{5.3}$$

It suffices now to bring these expansions to obtain the expansion (5.2) with:

$$\tilde{\alpha}(\bar{\lambda}) = \alpha(\bar{\lambda}) + \eta^l(\bar{\lambda}) - \eta^r(\bar{\lambda}) + \beta(\bar{\lambda})\eta^l(\bar{\lambda}) + \nu^{2\tau}O_P(\bar{\mu}) = \alpha(\bar{\lambda}) + \nu^\tau O_P(\bar{\mu})$$

as $\eta^l(\bar{\lambda}) - \eta^r(\bar{\lambda}) = \nu^\tau O_P(\bar{\mu})$ by Proposition 4.7. We have also that $\tilde{\beta}(\bar{\lambda}) = \beta(\bar{\lambda}) + \nu^\tau O_P(\bar{\mu})$.
□

We can now compute $\tilde{T}_{\bar{\lambda}} = (H^r_{\bar{\lambda}})^{-1} \circ \left((\mathcal{T}^r_{\bar{\lambda}})^{-1} \circ \bar{T}_{\bar{\lambda}} \circ \mathcal{T}^l_{\bar{\lambda}}\right) \circ H^l_{\bar{\lambda}}$, using the formula (5.2):

$$\tilde{T}_{\bar{\lambda}}(z) = \tilde{\alpha}(\bar{\lambda})\Big(\frac{\nu}{\nu_0}\Big)^{-\bar{\sigma}^r} + (1+\tilde{\beta}(\bar{\lambda}))z\Big(\frac{\nu}{\nu_0}\Big)^{\bar{\sigma}^l-\bar{\sigma}^r} + z^2\Big(\frac{\nu}{\nu_0}\Big)^{2\bar{\sigma}^l-\bar{\sigma}^r}(\bar{\varepsilon}_2\hat{\Phi}_2 + \bar{\varepsilon}_3\hat{\Phi}_3 + \nu\varepsilon_4\hat{\Phi}_4)$$

where $\bar\sigma^i = \bar\sigma^i(\bar\lambda)$ and $\hat\Phi_i = \hat\Phi_i(z,\bar\lambda) = \tilde\Phi_i\Big(z\big(\frac{\nu}{\nu_0}\big)^{\bar\sigma^l}, \bar\lambda\Big)$ is ν-regularly smooth in (z,μ). We have that

$$\Big(\frac{\nu}{\nu_0}\Big)^{\bar\sigma^l-\bar\sigma^r} = \Big(\frac{\nu}{\nu_0}\Big)^{\nu O_P(\bar\mu)} = 1 + \nu\log\Big(\frac{\nu}{\nu_0}\Big)O_P(\bar\mu)$$

and then:

$$(1+\tilde\beta(\bar\lambda))\Big(\frac{\nu}{\nu_0}\Big)^{\bar\sigma^l-\bar\sigma^r} = 1+\beta(\bar\lambda)+\nu^\tau O_P(\bar\mu) \tag{5.4}$$

for some new value of $\tau > 0$. Finally, we have obtained the following representation of $\tilde T_{\bar\lambda}$:

Proposition 5.2 *There exists some $\tau > 0$ such that:*

$$\tilde T_{\bar\lambda}(z) = z + \big(\alpha(\bar\lambda)+\nu^\tau O_P(\bar\mu)\big)\Big(\frac{\nu}{\nu_0}\Big)^{-\bar\sigma^r} + \big(\beta(\bar\lambda)+\nu^\tau O_P(\bar\mu)\big)z + \nu^\tau O_G(\bar\mu)z^2 \tag{5.5}$$

where $\alpha(\bar\lambda)$ and $\beta(\bar\lambda)$ are the parameter functions defined in Proposition 4.1 *and the functions represented by $O_P(\bar\mu)$ (resp. $O_G(\bar\mu)$) are ν-regularly smooth in μ (resp. (z,μ)).*

Remark 5.3 In order to apply the formula (5.5) we have to assume that the coefficient $\big(\alpha+\nu^\tau O_P(\bar\mu)\big)\big(\frac{\nu}{\nu_0}\big)^{-\bar\sigma^r}$ remains bounded. This implies that the parameter $\overline{\varepsilon}_2$ must be chosen in an interval of order $\big(\frac{\nu}{\nu_0}\big)^{\bar\sigma^r} \approx \nu^{\sigma(a)}$. In fact, outside this interval, the possible limit cycles have already escaped the neighborhood of Γ that is chosen for the study.

5.3 The regular transition $\tilde R_{\bar\lambda}$

Below, $\tau > 0$ is an arbitrarily small constant which may be adapted at each step. The map $R_{\bar\lambda}$ can be written:

$$R_{\bar\lambda}(z) = z + \nu O_G(\bar\mu) + \varepsilon_4 R_4(z,\bar\lambda)$$

using (4.17). We first compute:

$$R_{\bar\lambda}\circ(\tilde D^l_{\bar\lambda})^{-1}(z) = (\tilde D^l_{\bar\lambda})^{-1}(z) + \nu O_G(\bar\mu) + \varepsilon_4 R_4((\tilde D^l_{\bar\lambda})^{-1}(z),\bar\lambda)$$

As $(\tilde D^l_{\bar\lambda})^{-1}(z) = z + O(\nu^\tau)$ by inversion of a similar formula for $\tilde D^l_{\bar\lambda}(z)$, we have that

$$R_4((\tilde D^l_{\bar\lambda})^{-1}(z),\bar\lambda) = R_4(z,\bar\lambda) + O(\nu^\tau)$$

and then, writting $O(\nu^\tau)\varepsilon_4 + \nu O_G(\bar\mu)$ as $\nu^\tau O_G(\bar\mu)$ (for a sufficiently small new τ), one has

$$R_{\bar\lambda}\circ(\tilde D^l_{\bar\lambda})^{-1}(z) = (\tilde D^l_{\bar\lambda})^{-1}(z) + \nu^\tau O_G(\bar\mu) + \varepsilon_4 R_4(z,\bar\lambda)$$

We can now compute

$$\tilde R_{\bar\lambda}(z) = \tilde D^r_{\bar\lambda}\Big((\tilde D^l_{\bar\lambda})^{-1}(z) + \nu^\tau O_G(\bar\mu) + \varepsilon_4 R_4(z,\bar\lambda)\Big)$$

Expanding this expression at order $k = ord(\Gamma)$ we obtain

$$\tilde R_{\bar\lambda}(z) = \tilde D^r_{\bar\lambda}\circ(\tilde D^l_{\bar\lambda})^{-1}(z) + \sum_{j=1}^{k-1}\frac{1}{j!}\frac{\partial^j \tilde D^r_{\bar\lambda}}{\partial z^j}((\tilde D^l_{\bar\lambda})^{-1}(z))\Big(\nu^\tau O_G(\bar\mu) + \varepsilon_4 R_4(z,\bar\lambda)\Big)^j$$

$$+\frac{1}{k!}\int_0^1(1-s)^k\frac{\partial^k\tilde{D}^r_{\bar\lambda}}{\partial z^k}\Big((\tilde{D}^l_{\bar\lambda})^{-1}(z)+s\big(\nu^\tau O_G(\bar\mu)+\varepsilon_4R_4(z,\bar\lambda)\big)\Big)\big(\nu^\tau O_G(\bar\mu)+\varepsilon_4R_4(z,\bar\lambda)\big)^k ds$$

Now, as $\tilde{D}^r_{\bar\lambda}=\tilde{D}^l_{\bar\lambda}+\nu^\tau O_G(\bar\mu)$, we have that

$$\tilde{D}^r_{\bar\lambda}\circ(\tilde{D}^l_{\bar\lambda})^{-1}(z)=z+\nu^\tau O_G(\bar\mu).$$

From $\tilde{D}^i_{\bar\lambda}(z)=z+O(\nu^\tau)$ we deduce that

$$\frac{\partial\tilde{D}^r_{\bar\lambda}}{\partial z}\big((\tilde{D}^l_{\bar\lambda})^{-1}(z)\big)=1+O(\nu^\tau)$$

and also that

$$\frac{\partial^j\tilde{D}^r_{\bar\lambda}}{\partial z^j}\Big((\tilde{D}^l_{\bar\lambda})^{-1}(z)+s\big(O(\nu^\tau)O(\bar\mu)+\varepsilon_4R_4(z,\bar\lambda)\big)\Big)=O(\nu^\tau)$$

Bringing these estimates in the above expression of $\tilde{R}_{\bar\lambda}$ we obtain

$$\tilde{R}_{\bar\lambda}(z)=z+\nu^\tau O_G(\bar\mu)+(1+O(\nu^\tau))\Big(\nu^\tau O_G(\bar\mu)+\varepsilon_4R_4(z,\bar\lambda)\Big)+\nu^\tau O_G(\bar\mu)$$

Expanding this expression, one obtains the following result:

Proposition 5.4

$$\tilde{R}_{\bar\lambda}(z)=z+\nu^\tau O_G(\bar\mu)+\varepsilon_4R_4(z,\bar\lambda), \tag{5.6}$$

for some $\tau>0$. The term $O_G(\bar\mu)$ and the function R_4 are functions of (z,μ,ν) and are ν-regularly smooth in (z,μ). In agreement with Proposition 4.10, one has that $\frac{\partial^k R_4}{\partial z^k}(0,0,0)\neq 0$, where $k=\mathrm{ord}(\Gamma)<\infty$ is the order of the graphic Γ defined in Definition 4.11.

Then, we have proved that $\tilde{R}_{\bar\lambda}$ has an expression similar to the one for $R_{\bar\lambda}$.

5.4 Presentation of the displacement map

We now use the expressions (5.5) and (5.6) to obtain a good presentation of $\delta_{\bar\lambda}=\tilde{R}_{\bar\lambda}-\tilde{T}_{\bar\lambda}$. We have, using the expressions of α,β, that

$$\begin{aligned}\delta_{\bar\lambda}(z)=\ &\nu^\tau O_G(\bar\mu)+\varepsilon_4R_4(z,\bar\lambda)\\ &-\Big(\big(c_2\bar\varepsilon_2+\nu^\tau O_P(\bar\mu)\big)\Big(\tfrac{\nu}{\nu_0}\Big)^{-\bar\sigma^r}+\big(c_3\tfrac{\bar\varepsilon_3}{\sqrt{\bar\varepsilon_1}}+O(\bar\varepsilon_2)+\nu^\tau O_P(\bar\mu)\big)z+\nu^\tau O_G(\bar\mu)z^2\Big).\end{aligned}$$

where $c_2=c_2(\bar\lambda)$ and $c_3=c_3(\bar\lambda)$ are the strictly positive analytic functions defined in Proposition 4.1 (in particular, $c_2(\bar\lambda),c_3(\bar\lambda)\geq C$ for some constant $C>0$).

We expand the first term $\nu^\tau O_G(\bar\mu)$ of $\tilde{R}_{\bar\lambda}$ at order 2 in z :

$$\nu^\tau O_G(\bar\mu)=\nu^\tau O_P(\bar\mu)+\nu^\tau O_P(\bar\mu)z+\nu^\tau O_G(\bar\mu)z^2$$

All the terms in this sum are ν-regularly smooth (in their other variables). We now regroup these terms in δ with the corresponding terms in $\tilde{T}_{\bar\lambda}$. (Let us remark for instance that the term $\nu^\tau O_P(\bar\mu)$ of $\tilde R_{\bar\lambda}$ is of course absorbed by the term $\nu^\tau O_P(\bar\mu)\left(\frac{\nu}{\nu_0}\right)^{-\bar\sigma^r}$ of $\tilde T_{\bar\lambda}$). We obtain the following expansion:

$$\begin{aligned}\delta_{\bar\lambda}(z) = \quad & \varepsilon_4 R_4(z,\bar\lambda) - \left(c_2\bar\varepsilon_2 + \nu^\tau O_P(\bar\mu)\right)\left(\frac{\nu}{\nu_0}\right)^{-\bar\sigma^r} \\ & -\left(c_3\frac{\bar\varepsilon_3}{\sqrt{\bar\varepsilon_1}} + O(\bar\varepsilon_2) + \nu^\tau O_P(\bar\mu)\right)z + \nu^\tau O_G(\bar\mu)z^2,\end{aligned} \tag{5.7}$$

for some new terms $O_P(\bar\mu), O_G(\bar\mu)$.

We introduce now the coefficients of order 0 and 1 in (5.7) as new parameters, namely:

$$\begin{cases} -\tilde\varepsilon_2(\bar\lambda) = c_2(\bar\lambda)\bar\varepsilon_2 + \nu^\tau O_P(\bar\mu) = \delta_{\bar\lambda}(0) \\ -\tilde\varepsilon_3(\bar\lambda) = c_3(\bar\lambda)\frac{\bar\varepsilon_3}{\sqrt{\bar\varepsilon_1}} + O(\bar\varepsilon_2) + \nu^\tau O_P(\bar\mu) = \delta'_{\bar\lambda}(0). \end{cases} \tag{5.8}$$

(This change of parameters is invertible). In fact, writing

$$\tilde\mu = (\tilde\varepsilon_2, \tilde\varepsilon_3, \varepsilon_4),$$

the map $P_\nu : (\bar\mu, \varepsilon_0) \to (\tilde\mu, \varepsilon_0)$ is a ν-family of smooth diffeomorphisms which are ν-regularly smooth in $(\bar\mu, \varepsilon_0)$. Of course, a function in the parameter $\bar\lambda = (\bar\mu, \varepsilon_0, \nu)$ is $O_P(\bar\mu)$ if and only it is $O_P(\tilde\mu)$ and it is ν-regularly smooth in $\mu = (\bar\mu, \varepsilon_0)$ if it is ν-regularly smooth in $(\tilde\mu, \varepsilon_0)$. Similar remarks can be made about the symbols $O_G(\bar\mu), O_G(\tilde\mu)$ and the ν-regularly smoothness.

We can now expand the term $\nu^\tau O_G(\bar\mu)z^2$ of (5.7) in the ideal generated by $\tilde\varepsilon_2, \tilde\varepsilon_3, \varepsilon_4$:

$$\nu^\tau O_G(\bar\mu)z^2 = \nu^\tau\left(\tilde\varepsilon_2 z^2\Omega_2(z,\tilde\mu,\varepsilon_0,\nu) + \tilde\varepsilon_3 z^2\Omega_3(z,\tilde\mu,\varepsilon_0,\nu) + \varepsilon_4 z^2\Omega_4(z,\tilde\mu,\varepsilon_0,\nu)\right)$$

where the functions Ω_i are ν-regularly smooth in $(z, \tilde\mu, \varepsilon_0)$. Putting this expression in (5.7), we obtain:

Theorem 5.5

$$\delta(z,\tilde\mu,\varepsilon_0,\nu) = \tilde\varepsilon_2\left(\frac{\nu}{\nu_0}\right)^{-\bar\sigma^r}\left(1+\nu^\tau\left(\frac{\nu}{\nu_0}\right)^{\bar\sigma^r}z^2\Omega_2\right) + \tilde\varepsilon_3 z\left(1+\nu^\tau z\Omega_3\right) + \varepsilon_4\tilde R_4 \tag{5.9}$$

where $\tilde R_4 = R_4 + \nu^\tau z^2\Omega_4$. *The parameters* $\tilde\varepsilon_2, \tilde\varepsilon_3$ *are defined by* (5.8) *and the functions* $\tilde R_4 = \tilde R_4(z,\tilde\mu,\varepsilon_0,\nu)$, $\Omega_i = \Omega_i(z,\tilde\mu,\varepsilon_0,\nu)$ *are* ν*-regularly smooth in* $(z,\tilde\mu,\varepsilon_0)$. *Moreover*

$$\frac{\partial^k\tilde R_4}{\partial z^k}(0,\tilde\mu,\varepsilon_0,\nu) \neq 0,$$

where $k = \mathrm{ord}(\Gamma) < \infty$.

6 Finite cyclicity of (F_{7a}^1) and (H_7^1)

6.1 The finite cyclicity of (F_{7a}^1)

We will write

$$\tilde\lambda = (\tilde\mu, \varepsilon_0, \nu)$$

and we recall that $\varepsilon_4 = \tilde{\varepsilon}_4$. The parameter $\tilde{\lambda}$ is a parameter locally diffeomorphic to the parameter $\bar{\lambda}$ and it is of course equivalent to work with it to study the cyclicity.

Let Γ be any graphic of type $\left(F_{7a}^1\right)$ for the unfolding $\mathcal{X}_\varepsilon$, and $\delta(z, \tilde{\lambda}, \nu)$ its local displacement map. We will use the equation (5.9) for $\delta(z, \tilde{\lambda}, \nu)$ to compute the cyclicity of the graphics $\Gamma_{\bar{\varepsilon}_3}$. These graphics of the blown-up unfolding $\bar{\mathcal{X}}_{\tilde{\lambda}}$, are the graphics associated to the graphic Γ of $\mathcal{X}_\varepsilon$, and we will prove that Γ has a finite cyclicity using the formula (3.9).

6.1.1 The cyclicity of $\Gamma_{\bar{\varepsilon}_3}$, for $\bar{\varepsilon}_3 \neq 0$

If $\bar{\varepsilon}_3 \neq \pm 1, 0$ (i.e. if $\bar{\varepsilon}_1 \neq 0, 1$), the derivative $\frac{\partial \delta}{\partial z}(0, 0, \bar{\varepsilon}_3, 0, 0) = c_3 \frac{\bar{\varepsilon}_3}{\sqrt{\bar{\varepsilon}_1}} \neq 0$ and the unfolding is generic in the direction of the parameter $\bar{\varepsilon}_2$. Then $\mathrm{Cycl}(\bar{\mathcal{X}}_{\tilde{\lambda}}, \Gamma_{\bar{\varepsilon}_3}) = 1$, for $\bar{\varepsilon}_3 \neq \pm 1, 0$.

It was proved in [6] that $\mathrm{Cycl}(\bar{\mathcal{X}}_{\tilde{\lambda}}, \Gamma_{\pm 1}) = 1$. It is also possible to deduce this from (5.9). In fact, as the function $c_3(\bar{\lambda}) > C > 0$, we have that $|\tilde{\varepsilon}_3| \to +\infty$ when $\bar{\varepsilon}_3 \to \pm 1$ in (5.8) (recall that $\bar{\varepsilon}_1 = \sqrt{1 - \bar{\varepsilon}_3^2}$). Then, we also have $\mathrm{Cycl}(\bar{\mathcal{X}}_{\tilde{\lambda}}, \Gamma_{\pm 1}) = 1$.

6.1.2 The cyclicity of Γ_0

Theorem 6.1 *Let the order of the graphic* Γ, $\mathrm{ord}(\Gamma)$*, be defined through the regular transition* $R_{\bar{\lambda}}$ *in Definition* 4.11*. Then*

$$\mathrm{Cycl}(\bar{\mathcal{X}}_{\tilde{\lambda}}, \Gamma_0) \leq \mathrm{ord}(\Gamma) < \infty.$$

Proof. The proof uses a procedure of derivation-division applied to (5.9), very similar to the one first introduced in [8]. To make the text more readable, we will give a short sketch of the proof. To simplify, we do not write everywhere the variables.

Let $\mathcal{W} = \mathcal{U} \setminus \{0\}$, where $\mathcal{U}$ is some compact neighborhood of $\tilde{\lambda} = 0$ in the parameter space, in which the formula (5.9) makes sense (for z small enough). We can write

$$\mathcal{W} = \mathcal{W}_2 \cup \mathcal{W}_3 \cup \mathcal{W}_4$$

where

$$\mathcal{W}_i = \{\tilde{\lambda} \mid |\tilde{\varepsilon}_i| \geq |\tilde{\varepsilon}_j|, \text{ for } j \in \{2, 3, 4\} \setminus \{i\} \},$$

where we have written $\varepsilon_4 = \tilde{\varepsilon}_4$ for convenience. Let us notice that $\tilde{\varepsilon}_i \neq 0$ when $\tilde{\lambda} \in \mathcal{W}_i$.

1. On $\mathcal{W}_2$ one can consider:

$$\begin{aligned} \frac{\delta(z, \tilde{\mu}, \varepsilon_0, \nu)}{\tilde{\varepsilon}_2} &= \left(\frac{\nu}{\nu_0}\right)^{-\bar{\sigma}^r} \left(1 + \nu^\tau \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}^r} z^2 \Omega_2\right) + \frac{\tilde{\varepsilon}_3}{\tilde{\varepsilon}_2} z \left(1 + \nu^\tau z \Omega_3\right) + \frac{\tilde{\varepsilon}_4}{\tilde{\varepsilon}_2} \tilde{R}_4 \\ &= \left(\frac{\nu}{\nu_0}\right)^{-\bar{\sigma}^r} \left(1 + \nu^\tau \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}^r} z^2 \Omega_2\right) + O(z). \end{aligned}$$

   As $\mathcal{U}$ is compact, this function is everywhere non-zero for $\tilde{\lambda} \in \mathcal{W}_2$, and z small enough. Then, the displacement function does not vanish for $\tilde{\lambda} \in \mathcal{W}_2$, and z sufficiently small.

2. If z is small enough, one can consider on $\mathcal{W}_3$ the function

$$\tilde{\delta} = \frac{\delta(z, \tilde{\mu}, \varepsilon_0, \nu)}{1 + \nu^\tau \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}^r} z^2 \Omega_2} = \tilde{\varepsilon}_2 \left(\frac{\nu}{\nu_0}\right)^{-\bar{\sigma}^r} + \tilde{\varepsilon}_3 z \frac{1 + \nu^\tau z \Omega_3}{1 + \nu^\tau \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}^r} z^2 \Omega_2} + \varepsilon_4 \overline{R}_4.$$

where $\overline{R}_4 = \tilde{R}_4/\left(1+\nu^\tau\left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}^r} z^2\Omega_2\right)$ is nonlinear of order $k \geq 2$ in z at $z=0, \tilde{\lambda}=0$, as $\tilde{R}_4$. Let us consider $\delta_1 = \frac{\partial\tilde{\delta}}{\partial z}$. One has

$$\delta_1 = \tilde{\varepsilon}_3(1+\nu^\tau z\hat{\Omega}_3) + \tilde{\varepsilon}_4\hat{R}_4,$$

for functions $\hat{\Omega}_3, \hat{R}_4$, which are ν-regularly smooth in $(z,\tilde{\mu},\varepsilon_0)$. The function $\hat{R}_4$ is of order $k-1$ in z at $z=0, \tilde{\lambda}=0$ and then $\hat{R}_4 = O(|z|+||\tilde{\lambda}||)$. Then the function $\frac{\delta_1}{\tilde{\varepsilon}_3} = 1+O(|z|+||\tilde{\lambda}||)$, does not vanish on $\mathcal{W}_3$ as soon as we choose the size of $\mathcal{U}$ small enough. Using Rolle's Theorem, one obtains that the displacement function has at most one zero for $\tilde{\lambda} \in \mathcal{W}_2$, and z sufficiently small.

3. If z is small enough, one can consider on $\mathcal{W}_4$ the function

$$\tilde{\delta}_1 = \frac{\delta_1}{1+\nu^\tau z\hat{\Omega}_3} = \tilde{\varepsilon}_3 + \tilde{\varepsilon}_4\frac{\hat{R}_4}{1+\nu^\tau z\hat{\Omega}_3},$$

where the function $\hat{R}_4/(1+\nu^\tau z\hat{\Omega}_3)$ is of order $k-1$ in z at $z=0, \tilde{\lambda}=0$. Then, if $\delta^k = \frac{\partial^{k-1}}{\partial z^{k-1}}\tilde{\delta}_1$, one has

$$\delta^k(z,\tilde{\lambda}) = \tilde{\varepsilon}_4 U(z,\tilde{\mu},\varepsilon_0,\nu)$$

where U is a function ν-regularly smooth in $(z,\tilde{\mu},\varepsilon_0)$ such that $U(0,0,0)\neq 0$. Applying again k times Rolle's Theorem, one obtains that the displacement function has at most k zeros for $\tilde{\lambda} \in \mathcal{W}_2$, with z and the size of $\mathcal{U}$ sufficently small.

The result follows now from the three above points. □

6.1.3 The cyclicity of Γ

By definition, $\mathrm{ord}(\Gamma) \geq 2$. Then it follows from the above results and the formula (3.9) that

Theorem 6.2

$$\mathrm{Cycl}(\mathcal{X}_\varepsilon, \Gamma) \leq \mathrm{ord}(\Gamma) < \infty.$$

Moreover for all graphics (F_7^1) *except a discrete set we have* $\mathrm{Cycl}(\mathcal{X}_\varepsilon, (F_7^1)) \leq 2$.

6.2 The finite cyclicity of (H_7^1)

The transition $R_{\bar{\lambda}}$ (which is no more regular in that case) has been calculated in Proposition 4.13 in the variable $\check{y} = (\zeta_{\bar{\lambda}})^{-1}(\bar{y})$. This allows to get the exact cyclicity theorem:

Theorem 6.3

$$\mathrm{Cycl}(\mathcal{X}_\varepsilon, H_7^1) \leq 2.$$

PROOF. We consider the displacement map

$$\check{\delta}_{\bar{\lambda}}(\check{y}) = R_{\bar{\lambda}} \circ \zeta_{\bar{\lambda}} - T_{\bar{\lambda}} \circ \zeta_{\bar{\lambda}}.$$

We have that $T_{\bar\lambda} = (D^r_{\bar\lambda})^{-1} \circ \bar T_{\bar\lambda} \circ D^l_{\bar\lambda}$. We consider the composition $D^l_{\bar\lambda} \circ \zeta_{\bar\lambda} = \bar D^l_{\bar\lambda}$. It is a map which has the same form as $D^l_{\bar\lambda}$ as in (4.16). So it an also be written as a composition of three maps: $\bar D^l_{\bar\lambda} = \check{\mathcal{T}}^l_{\bar\lambda} \circ \check H^l_{\bar\lambda} \circ \check D^l_{\bar\lambda}$ where

$$\check{\mathcal{T}}^l_{\bar\lambda}(u) = u + \check\eta^l(\mu,\nu), \quad \check H^l_{\bar\lambda}(u) = \Big(\frac{\nu}{\nu_0}\Big)^{\bar\sigma^l} u, \quad \check D^l_{\bar\lambda}(u) = u + \check\varphi^i(u,\mu,\nu) \tag{6.1}$$

and

$$\begin{cases} \check\eta^l(\mu,\nu) = \eta^l + \nu^\tau O_P(\bar\mu) \\ \check\varphi(u,\mu,\nu) = u^2 O_G(\bar\mu). \end{cases}$$

The function $\check\eta^l$ (resp. $\check D_{\bar\lambda}$) is ν-regularly smooth in μ (resp. (u,μ)).

Rather than working with $\check\delta_{\bar\lambda}$ we will work with

$$\begin{aligned} \delta_{\bar\lambda} &= \tilde D^r_{\bar\lambda} \circ \check\delta_{\bar\lambda} \circ \check D^l_{\bar\lambda} \\ &= \tilde D^r_{\bar\lambda} \circ R_{\bar\lambda} \circ \check D^l_{\bar\lambda} - (H^r_{\bar\lambda})^{-1} \circ (\mathcal{T}^r_{\bar\lambda})^{-1} \circ \bar T_{\bar\lambda} \circ \check{\mathcal{T}}^l_{\bar\lambda} \circ \check H^l_{\bar\lambda} \\ &= \tilde R_{\bar\lambda} - \tilde T_{\bar\lambda}, \end{aligned}$$

where

$$\begin{cases} \tilde R_{\bar\lambda} = \tilde D^r_{\bar\lambda} \circ R_{\bar\lambda} \circ \check D^l_{\bar\lambda} \\ \tilde T_{\bar\lambda} = (H^r_{\bar\lambda})^{-1} \circ (\mathcal{T}^r_{\bar\lambda})^{-1} \circ \bar T_{\bar\lambda} \circ \check{\mathcal{T}}^l_{\bar\lambda} \circ \check H^l_{\bar\lambda}. \end{cases}$$

The form of $\tilde T_{\bar\lambda}$ is exactly the one given by Proposition 5.2. Also the composition $\tilde R_{\bar\lambda}$ has the same form as $\tilde R_{\bar\lambda}$ given in Proposition 4.13.

This allows to give a nice decomposition for δ which has exactly the form of (5.7), the only difference being the form of R_4. Finally we can reparameterize as in the case of (F^1_{7a}) and get

$$\delta(z,\tilde\mu,\nu) = \tilde\varepsilon_2\Big(\frac{\nu}{\nu_0}\Big)^{-\bar\sigma^r}\Big(1+\nu^\tau z\Omega_2\Big) + \tilde\varepsilon_3 z\Big(1+\nu^\tau z\Omega_3\Big) + \tilde\varepsilon_4 z^{1+s_r}\Big(1+\Omega_4\Big) \tag{6.2}$$

where $\Omega_4(z,\tilde\mu,\varepsilon_0,\nu) = O(z)$ and the functions $\Omega_i = \Omega_i(z,\tilde\mu,\varepsilon_0,\nu)$ have the property (I) of Mourtada (see Definition 4.14).

As $\{1, z, z^{1+s_r}\}$ is a Tchebychev system, a standard division of the parameter space in cones and derivation-division yields that δ has at most two positive zeroes in z in a neighborhood of $(z,\tilde\mu,\varepsilon_0,\nu) = 0$. □

7 The graphics (I^1_{6a}) and (H^3_{11})

Here most of the development is identical to the case of the graphics (F^1_{7a}) and (H^1_7), so we will only insist on the differences. We drop the dependence on $\bar\lambda$ in all expressions.

The main difference comes from the fact that the connection on the blow-up sphere is fixed (see Figure 7). So the map $\bar T$ has the property that $\bar T(0) = 0$. One consequence is that the reparametrization from R, T to $\tilde R$, $\tilde T$ is no more necessary.

The formulas for the blow-up of (2.19) have been given in Section 3.2. The sections used for defining the displacement map are now given by $\{v = \pm V_0\}$ and they are parameterized by $\bar w$. Consider the cyclicity of a graphic Γ cutting the sections at w_0 in w-variable ($\bar w_0$ in $\bar w$-variable). We will let $z = \bar w - \bar w_0$, so the graphic occurs at $z = 0$. Moreover $w_0 = 0$ for the graphic (H^3_{11}).

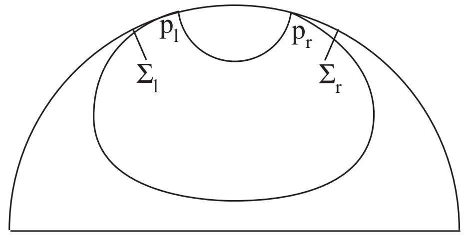


Figure 7: The sections Σ_r and Σ_l for a graphic (I_{6a})

7.1 Finite cyclicity of (I^1_{6a})

In the previous section, when we were considering the equation for limit cycles we were having terms of different orders controlled by independent parameters. Let us review this and compare with what we now have:

- The dominant nonlinear term was coming from R, since the nonlinear terms of $\bar{T}$ were multiplied by a factor ν^τ. It was controlled by the parameter ε_4. This property will remain the same for the displacement maps associated to (I_{6a}) and (H^3_{11}).
- The dominant linear term was coming from $\bar{T}$: this property will remain the same. In the previous section the analysis has allowed us to reparameterize by $\tilde{\varepsilon}_3$ which was the coefficient of the linear term minus 1. Moreover the form of $\tilde{\varepsilon}_3$ was a nonzero multiple of $\bar{\varepsilon}_3$ plus a term of the form $O(\bar{\varepsilon}_2)$. Here is is even simpler as $\bar{T}'(0)$ is exactly a nonzero multiple of $\bar{\varepsilon}_3$.
- The dominant constant term was coming from $\bar{T}$. It was dominant over the constant term coming from R because of the factor ν^τ in the constant term of R. It was controlled by a parameter $\tilde{\varepsilon}_2$ which was "essentially" a large multiple of ε_2.

  This is no more valid as $\bar{T}$ and D^i, $i = l, r$, have no constant terms. So the constant term is exactly that of R. Fortunately there is an easy way to handle this. Indeed only the parameters ε_1 and ε_3 have been blown-up in Section 3.2. Moreover the ε_2-term is without contact in (2.13). This yields (keeping z for the name of the variable in R so that the graphic occurs at $z = 0$):

$$\frac{\partial R(0)}{\partial \varepsilon_2} = c_2 \neq 0. \tag{7.1}$$

  Since R is analytic in ε and $R \equiv id$ when $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0$ we have moreover

$$\frac{\partial R(0)}{\partial \bar{\varepsilon}_3} = O(\nu). \tag{7.2}$$

In the previous section it was obvious without calculations that $(\varepsilon_2, \bar{\varepsilon}_3, \varepsilon_4) \mapsto (\tilde{\varepsilon}_2, \tilde{\varepsilon}_3, \varepsilon_4)$ was an invertible change of parameters because of the presence of the factors ν^τ. In the context of (I_{16a}) and (H^3_{11}) we need to be more careful. A natural reparametrization is given by

$$(\varepsilon_2, \bar{\varepsilon}_3, \varepsilon_4) \mapsto (\tilde{\varepsilon}_2, \tilde{\varepsilon}_3, \tilde{\varepsilon}_4) = (R(0), R'(0) - T'(0), \varepsilon_4). \tag{7.3}$$

Lemma 7.1 *The change of parameters* (7.3) *is invertible.*

The first thing to remark is that $R'(0) - T'(0) = c_3\bar{\varepsilon}_3 + O(\nu)$ with $c_3 \neq 0$. So it suffices to show that for $\varepsilon_3 = 0$ the change of parametrization $(\varepsilon_2, \varepsilon_4) \mapsto (\tilde{\varepsilon}_2, \tilde{\varepsilon}_4) = (R(0), \varepsilon_4)$ is invertible. This follows from (7.1). $\square$

The transition map R from $\Sigma_r = \{v = V_0\}$ to $\Sigma_l = \{v = -V_0\}$ following the flow backwards is analytic. Moreover we have that $R \equiv id$ for $\tilde{\varepsilon}_2 = \tilde{\varepsilon}_3 = \tilde{\varepsilon}_4 = 0$. We will stress that the map δ smoothly depends on $\bar{w}_0$ and we will make this dependence explicit. This yields a decomposition

$$R(z, w_0) - z = \tilde{\varepsilon}_2 h_2(z, w_0) + \tilde{\varepsilon}_3 z h_3(z, w_0) + \tilde{\varepsilon}_4 z^2 h_4(z, w_0), \tag{7.4}$$

where $h_2(0, w_0) \neq 0$.

Proposition 7.2 *For each $\bar{w}_0$ there exists $k = k(\bar{w}_0) \geq 0$ such that $\frac{\partial^k h_4}{\partial z^k} \neq 0$.*

Proof. We want to use an analyticity argument as in the proof of Proposition 4.10. Indeed for $w_0 \neq 0$ the maps $h_j(z, w_0)$ depend analytically on z, w_0 and the parameters. Moreover we can show that $h_4 \neq 0$ for w_0 small (see Lemma 7.4 below). This follows by analyzing R in the neighborhood of the hemicycle. The conclusion follows from the analyticity of h_4. $\square$

Definition 7.3 Let $k(w_0)$ be the minimum k with this property, then the order of Γ is defined as $\mathrm{ord}(\Gamma) = k(w_0) + 2$.

Lemma 7.4 *The function h_4 defined in Proposition 7.2 does not vanish in a neighborhood of 0 for w_0 small.*

Proof. An asymptotic expansion of the transition R along the hemicycle can only be explicitly calculated when $\varepsilon_2 = 0$, i.e. the invariant line remains unbroken (more details in [6] and in Section 7.2 below). Let us look at Figure 8. Then we have $R = \left(S_r \circ \Delta_r \circ S_{fin} \circ \Delta_l \circ S_l\right)^{-1}$. It has the form

$$R(z) = c_1 z + c_2 z^{1+s_l} + o(z^{1+s_l})$$

where $s_l \in (0,1)$ is the hyperbolicity ratio of P_l. (This is the case since $1/s_r = s_l$.) The coefficient c_2 is a nonzero multiple of $S''_{fin}(0)$ which was calculated in [6] for $\varepsilon_2 = 0$. The calculation yielded $S''_{fin}(0) = C\varepsilon_4$ for some nonzero constant C. Remark that the function $O(z^{1+s_l})$ has property (I). $\square$

Theorem 7.5 *Let Γ be a graphic of type (I^1_{6a}). Then*

$$\mathrm{Cycl}(\mathcal{X}_\varepsilon, \Gamma) \leq \mathrm{ord}(\Gamma).$$

Proof. Let $k + 2 = \mathrm{ord}(\Gamma)$. As before we consider the displacement function $\delta(z) = R(z) - T(z)$. It can be decomposed as

$$\delta(z, w_0) = \tilde{\varepsilon}_2 \Psi_2(z, w_0) + \tilde{\varepsilon}_3 z \Psi_3(z, w_0) + z^2 \tilde{\varepsilon}_4 \Psi_4(z, w_0), \tag{7.5}$$

where $\Psi_2, \Psi_3 \neq 0$ and $\frac{\partial^k \Psi_4}{\partial z^k} \neq 0$. A separation of the discussion in three cones and an algorithm of derivation-division allows to conclude as in Section 6.1. $\square$

7.2 Finite cyclicity of (H_{11}^3)

Theorem 7.6

$$\mathrm{Cycl}(\mathcal{X}_\varepsilon, H_{11}^3) \leq 2.$$

PROOF. We introduce sections as in Figure 8. All connections along the equator are fixed,

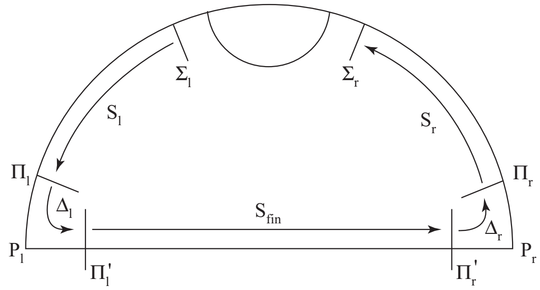


Figure 8: The sections for the graphic (H_{11}^3)

so it is natural to consider a displacement map $\delta : \Sigma_r \rightarrow \Pi_r'$ given by

$$\delta = \Delta_l \circ S_l \circ T - S_{fin}^{-1} \circ \Delta_r^{-1} \circ (S_r)^{-1}.$$

We have that $S_r - S_l = O_G(\bar{\mu})$. Also the Δ_i are given by

$$\begin{cases} \Delta_l(u) = u^{s_l}(1 + \Psi_l(u, \varepsilon)) \\ (\Delta_r)^{-1}(u) = u^{s_l}(1 + \Psi_r(u, \varepsilon)) \end{cases}$$

where the $\Psi_i(u, \varepsilon)$ are of class (I) and $\Psi_l(u, \varepsilon) - \Psi_r(u, \varepsilon) = O_G(\bar{\mu})$. Finally

$$S_{fin}(u) - u = O_G(u) = \varepsilon_2 S_2(u) + \bar{\varepsilon}_3 S_3(u) + \varepsilon_4 S_4(u)$$

with $S_2(0) \neq 0$, $S_3(0) = 0$, $S_3'(0) \neq 0$, $S_4(0) = 0$ and $\frac{\partial^2 S_4}{\partial u^2} \neq 0$ and $s_l \in (0, 1)$. This gives for δ a decomposition

$$\delta(z) = \varepsilon_2 h_2(z, \mu, \nu) + (\bar{\varepsilon}_3 + \nu^\tau O_P(\mu)) z h_3(z, \mu, \nu) + (\varepsilon_4 + \nu^\tau O_P(\bar{\mu})) z^{1+s_l} h_4(z, \mu, \nu)$$

with $h_2, h_3, h_4 \neq 0$ from which cyclicity 2 follows. □

8 Perspectives for proving the finite cyclicity of center graphics

The present paper presents a strategy to study the finite cyclicity of several center graphics occurring in finite-parameter families of analytic vector fields $\mathcal{X}_\varepsilon$. With the following ingredients we can hope proving the finite cyclicity of a center graphic Γ_0:

(i) The Bautin ideal is radical. We recall that the set of zeros of this ideal is the set of parameter values for which there exists an annulus of periodic solutions.

(ii) There exists a regular submanifold M in parameter space such that for $\varepsilon \in M$ then $\mathcal{X}_\varepsilon$ has a graphic Γ_ε. The family $\{\Gamma_\varepsilon\}_{\varepsilon \in M}$ is a continuous family of graphics. All graphics Γ_ε have the "same type" (same kind of singular points and orbits connecting them).

(iii) Any graphic Γ_ε is either a center graphic for ε in the set of zeros of the Bautin ideal or satisfies a genericity condition.

(iv) The finite cyclicity of a generic graphic Γ_{ε_0} is obtained by means of a generalized derivation-division algorithm on a "well-behaved" system of equations whose solutions yield the periodic solutions of $\mathcal{X}_\varepsilon$ for ε close to ε_0.

(v) The finite cyclicity of the generic Γ_ε's in the neighborhood of a center graphic is uniformly bounded in the neighborhood of a center graphic.

These conditions are very general. Conditions (i) and (ii) are satisfied for all center graphics from the DRR program. In [2] and [5], it was pointed that there remains a unique elementary graphic (i.e. with only hyperbolic and semi-hyperbolic singular points), namely (I_{16a}), whose finite cyclicity is not yet proved. This graphic satisfies (i)-(v).

Let us discuss in more detail why the method presented here is powerful for studying the center graphics of the DRR program. The usual problem with studying the exact cyclicity of center graphics is the computation of complicated Abelian integrals. Indeed we need to show that a change of parameters as in Lemma 7.1 is invertible. A direct calculation could be extremely tricky. Our argument is indirect and makes an essential use of analyticity: it applies for graphics occurring in families of graphics. The genericity condition for the generic graphics is defined on analytic transitions between analytic sections and has the property that, once it is satisfied for one graphic, it is satisfied for all graphics in the family. The genericity condition is easy to check (by direct calculation) for a graphic near the boundary of the family of graphics. Indeed the boundary graphic usually involves invariant lines. Passing to the proof that the change of parameters is invertible, we use the analyticity in the parameters: it is easy to show that the change of parameters is invertible near the boundary graphic. Analyticity in the parameters allows to push this property for all graphics. This last property has not yet been noticed at the time of [2], explaining why the finite cyclicity of $(I_{16a)}$ was not proved at the time.

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