What can we learn from singularities?

Structure of the talk

We will illustrate the ideas through examples:

- Dynamical systems as models
- The Hopf bifurcation
- The Lorenz system
- Normal form of an ODE at a singular point
- The center manifold of a 2-dimensional saddle-node

Dynamical systems as models

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Typical examples:

- A difference equation

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Dynamical systems as models

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Typical examples:

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An equilibrium is a fixed point $f(x_0) = x_0$.

- An autonomous ordinary differential equation

$$\dot{X} = v(X)$$

A singular point is a point X_0 such that $v(X_0) = 0$.

Dynamical systems as models

More typical examples:

- A linear differential system

$$\dot{Y} = A(t)Y$$

A singularity is a point t_0 where $A(t_0)$ has a pole.

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- One fruitful approach in the study of PDEs is to look at them as dynamical systems over spaces of infinite dimensions.

Dynamical systems naturally depend on parameters

A singular value of the parameters, called *bifurcation value*, is a value where a qualitative change occurs in the system.

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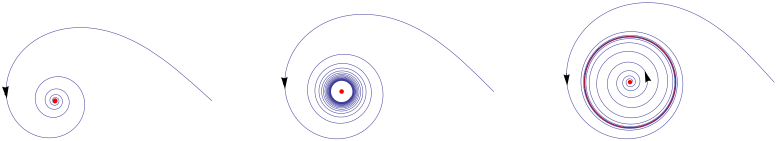
Example: the loss of stability of an equilibrium through the *Hopf bifurcation*

$$\dot{x} = \epsilon x - y - x(x^2 + y^2) \qquad \dot{r} = \epsilon r - r^3$$

$$\dot{y} = x + \epsilon y - y(x^2 + y^2) \qquad \dot{\theta} = 1$$



(a) $\epsilon < 0$ (b) $\epsilon = 0$ (c) $\epsilon > 0$

What do we learn from that?

$$\dot{x} = \epsilon x - y + x(x^2 + y^2)$$
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$$\dot{r} = \epsilon r + r^3$$
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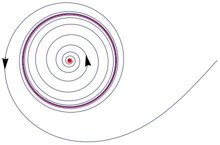

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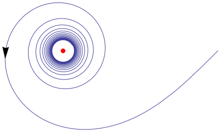
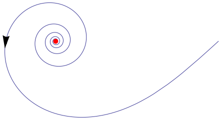


(b) $\epsilon = 0$



(c) $\epsilon > 0$

- The danger on relying on linear analysis

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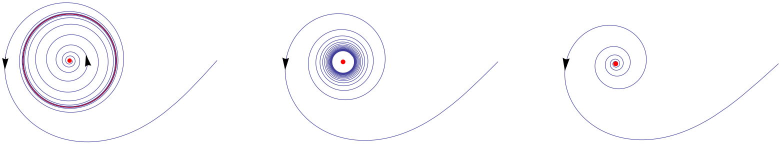


(a) $\epsilon < 0$ (b) $\epsilon = 0$ (c) $\epsilon > 0$

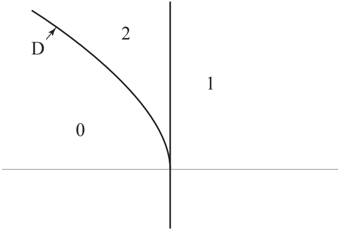
- The danger on relying on linear analysis
- Looking at what happens at the singular point for $\epsilon = 0$ gives us a *hand* on the periodic orbit and on the global behavior. The value $\epsilon = 0$ is an *organizing center*.

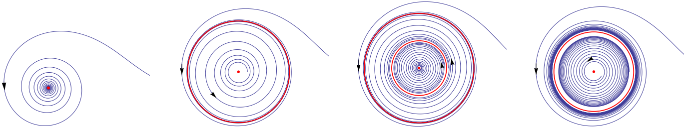
The Hopf bifurcation of order 2

$$\dot{x} = \epsilon_1 x - y - \epsilon_2 x(x^2 + y^2) - x(x^2 + y^2)^2$$
$$\dot{y} = x + \epsilon_1 y - \epsilon_2 y(x^2 + y^2) - y(x^2 + y^2)^2$$

$$\dot{r} = \epsilon_1 r - \epsilon_2 r^3 - r^5$$
$$\dot{\theta} = 1$$

The bifurcation diagram





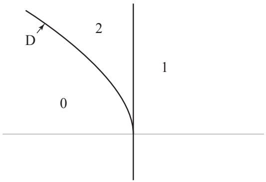
(d) 0

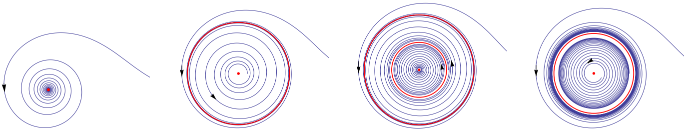
(e) 1

(f) 2

(g) D

The Hopf bifurcation of order 2 is an organizing center of the bifurcation diagram





(h) 0 (i) 1 (j) 2 (k) D

This is a very general principle: The bifurcations of highest order organize the bifurcation diagram.

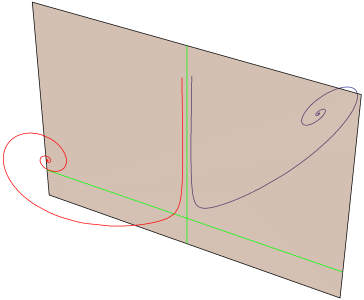
The Lorenz system

Edward Lorenz was a meteorologist. In his 1963 paper, he introduced the Lorenz system, a chaotic system, and the “butterfly effect” in meteorology.

Lorenz system:

$$\dot{x} = 10(y - x)$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = -\frac{8}{3}z + xy$$

The Lorenz system: regular behaviour



(a) $\rho = 4$

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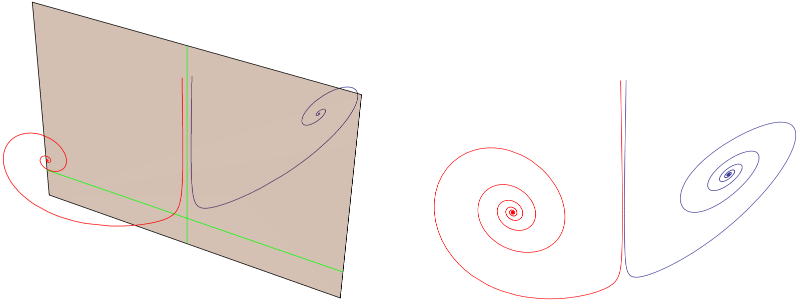
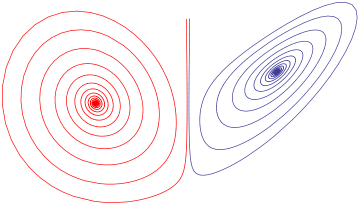


(a) $\rho = 4$ (b) $\rho = 8$

The Lorenz system: homoclinic bifurcation



(a) $\rho = 13.9$

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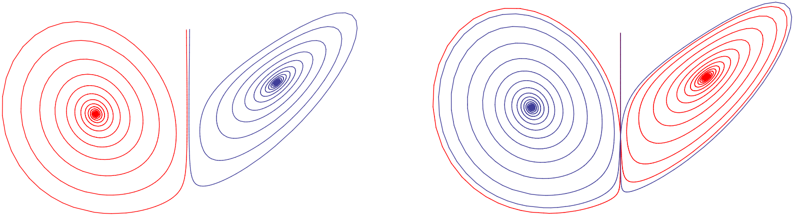
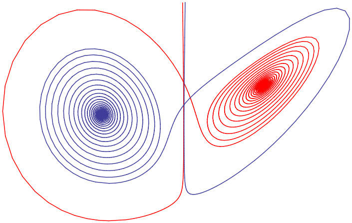


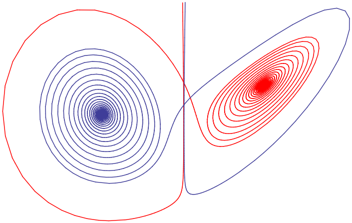
(a) $\rho = 13.9$ (b) $\rho = 14.5$

The Lorenz system: approaching the Hopf bifurcation



(a) $\rho = 18$

The Lorenz system: approaching the Hopf bifurcation



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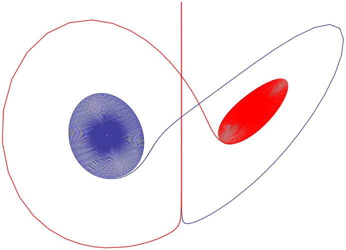
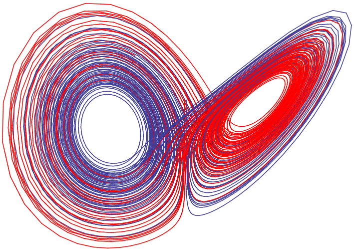


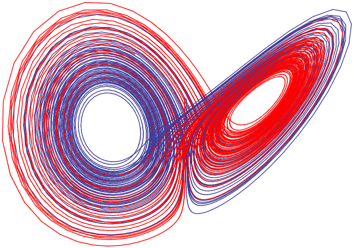
(b) $\rho = 22$

The Lorenz attractor

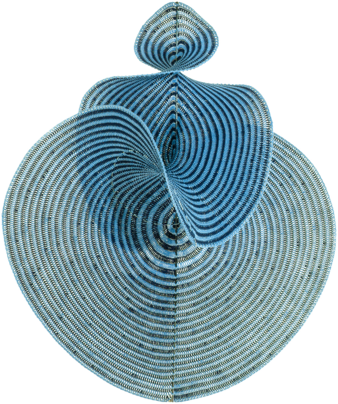


(a) $\rho = 28$

The Lorenz attractor

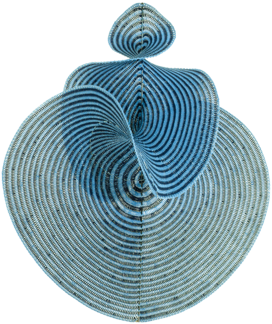


(a) $\rho = 28$



(b) The stable manifold of the origin

The stable manifold of the origin



(a) Crocheted stable manifold by B. Krauskopf and H. Osinga

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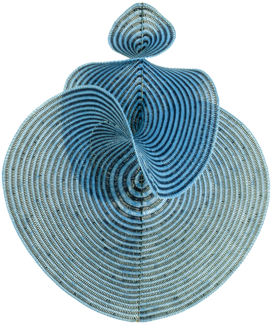

(a) Crocheted stable manifold by B. Krauskopf and H. Osinga

(b) Steel stable manifold by Benjamin Storch

From now on we limit ourselves to analytic dynamical systems.

Can we linearize an ODE in the neighborhood of a singularity?

$$\dot{X} = AX + f_r(X) + O(|X|^{r+1})$$

We look for a change of coordinate $X = Y + h_r(Y)$ allowing to get rid of terms of degree r. On the one hand we have

$$\begin{aligned}\dot{X} &= A(Y + h_r(Y)) + f_r(Y + h_r(Y)) + O((Y + h_r(Y))^{r+1}) \\ &= AY + Ah_r(Y) + f_r(Y) + O(|Y|^{r+1}).\end{aligned}$$

On the other hand we have

$$\begin{aligned}\dot{X} &= (id + Dh_r)\dot{Y} = (id + Dh_r)(AY + O(Y^{r+1})) \\ &= AY + Dh_rAY + O(Y^{r+1}).\end{aligned}$$

This yields to the homological equation

$$L_A(h_r) := Dh_r\, A - Ah_r = f_r$$

The operator L_A is linear where $L_A : \mathscr{H}_r \to \mathscr{H}_r$ is defined on the finite dimensional vector space $\mathscr{H}_r$ of homogeneous vectors of degree r.

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Hence it is surjective iff it is injective.

Otherwise, we can only get rid of the terms that belong to $Im(L_A) \subset \mathscr{H}_r$.

The particular case where A is diagonalizable

The monomials $X^m e_j = x_1^{m_1} \dots x_n^{m_n} e_j$ are eigenvectors of L_A corresponding to the eigenvalue

$$(m,\lambda) - \lambda_j = m_1\lambda_1 + \cdots + m_n\lambda_n - \lambda_j$$

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$$(m, \lambda) - \lambda_j = m_1\lambda_1 + \dots + m_n\lambda_n - \lambda_j$$

Hence we can get rid of all monomials except the *resonant* monomials $X^m e_j$ for which

$$(m, \lambda) = \lambda_j$$

Example: the planar saddle node with eigenvalues 0 and 1

The formal normal form is

$$\dot{x} = \sum_{j\geq k+1} a_j x^j$$

$$\dot{y} = y + y\sum_{j\geq 1} b_j x^j$$

We can do better

If we consider the orbital formal normal form, we can divide by a nonzero function and scale x so as to get

$$\dot{x} = x^{k+1} + \sum_{j>k+1} b_j x^j$$
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We can do even better. We can bring to the form

$$\dot{x} = x^{k+1}(1 + Bx^k)$$
$$\dot{y} = \pm y$$

Proof

In the system

$$\dot{x} = x^{k+1} + \sum_{j>k+1} b_j x^j, \qquad \dot{y} = \pm y$$

we consider a change $x = X + cX^\ell$, $\ell \geq 2$ so as to get rid of the term $b_{k+\ell} x^{k+\ell}$.

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$$\dot{x} = (X + cX^\ell)^{k+1} + \sum_{j>k+1} b_j (X + cX^\ell)^j$$

which we compare to

$$\dot{x} = (1 + c\ell X^{\ell-1}) \left(X^{k+1} + \sum_{j \neq k+\ell} b'_j (X + cX^\ell)^j \right)$$

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Comparison of terms of order $k + \ell$ yield

$$(k+1)c + b_{k+\ell} + \cdots = c\ell + \ldots$$

which we can solve if $\ell \neq k+1$.

Another choice of formal normal form

We can also transform to

$$\dot{x} = x^{k+1}(1 + \sum_{j>0} Bx^{jk})$$

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Another choice of formal normal form

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$$\dot{x} = x^{k+1}\Big(1 + \sum_{j>0} B x^{jk}\Big)$$
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which is orbitally equivalent for $A = -B$ to

$$\dot{x} = x^{k+1}$$
$$\dot{y} = \pm y(1 + Ax^k)$$

Formal normal form at a saddle-node

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Formal normal form at a saddle-node

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- We cannot do better: what is the meaning of A?
- Generically the change to normal form diverges:

Why?

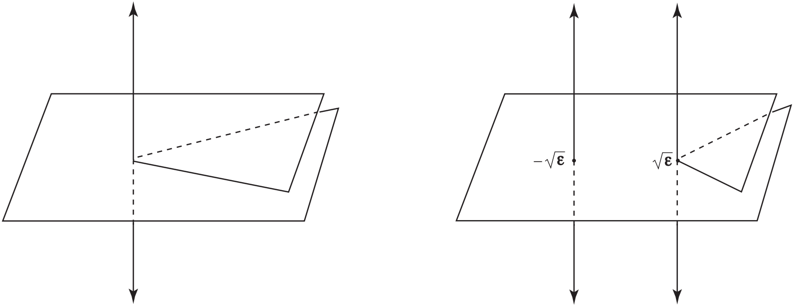
Answers to the divergence

1. We need to extend x, y to be in $\mathbb{C}$.
2. The saddle-node is a multiple singular point. Hence it is natural to unfold. In the unfolding there are rigid *models* near each of the two singular points.

   Generically these models mismatch till the merging of the singular points, yielding divergence at the limit.

One example of mismatch

Consider a saddle-node with normal form $\dot{x} = x^2, \dot{y} = y(1 + Ax)$

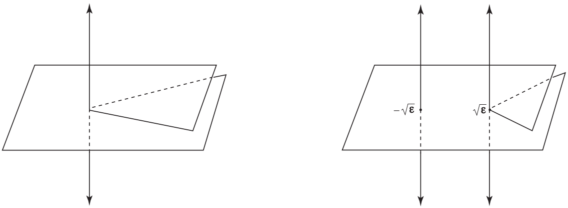


(c) $\epsilon = 0$ (d) $\epsilon \neq 0$

Generically a saddle-node has no analytic center manifold

The parametric resurgence phenomenon

Conclusion 1: When we unfold a system with no analytic center manifold, then the analytic separatrices of the two singular points do not match.



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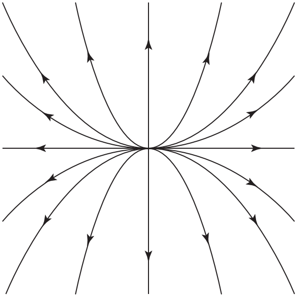
Conclusion 2: When we unfold a system with no analytic center manifold then the node is non linearizable as soon as resonant.

The node is a very simple point!



We have convergence to the orbital normal form

$$\begin{cases} \dot{x} = \lambda x \\ \dot{y} = y \end{cases}$$

as soon as $\lambda \notin 1/\mathbb{N}$.

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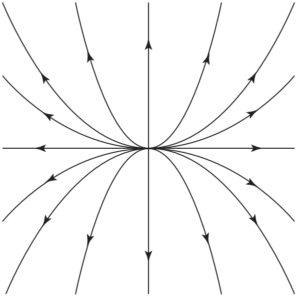

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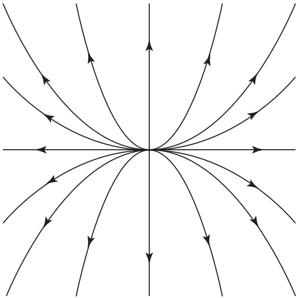

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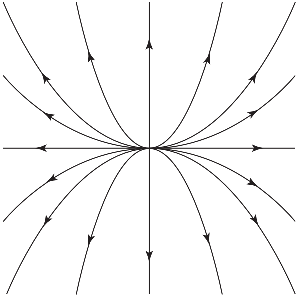
The generic situation



The leaves are given by $y = Cx^{1/\lambda}$. Only $y = 0$ is non ramified.

Hence the generic situation is that the stable manifold of the saddle does not coincide with the non ramified leaf.

Explanation of Conclusion 2

If the node is resonant then the local model at the node is the normal form

$$\begin{aligned} \dot{x} &= \frac{x}{n} \\ \dot{y} &= y + Ax^n. \end{aligned}$$

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If $A = 0$, then all solution curves at the node (except $x = 0$) are analytic of the form $y = Cx^n$.

This case is obviously impossible when unfolding a system with ramification for $\epsilon = 0$ and we are forced to have $A \neq 0$, yielding that all solutions (except $x = 0$) are of the form

$$y = nAx^n \ln x + Cx^n$$

The center-Manifold of a saddle-node

We have understood why divergence is the norm and convergence is the exception.

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This is just one example of the many mismatches that occur within analytic dynamical systems.

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