Singularities of analytic dynamical systems with 1-summable normalizing transformation

Structure of the talk

- The general problem
- Examples
- Common features
- Examples revisited with some results

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The equivalence problem in analytic dynamics

When are two germs of analytic dynamical systems equivalent equivalent in the neighborhood of a singularity under an analytic change of coordinates?

One way of solving the equivalence problem

The use of normal forms

For instance, if there exists an analytic change of coordinates to a linear system.

Two steps

- look for a formal change of coordinates to normal form
- study convergence of normalizing change of coordinates.

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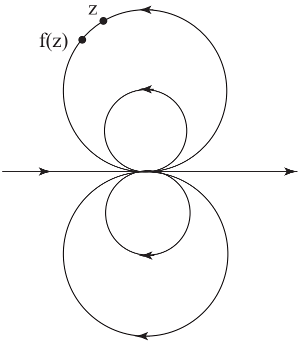
Why?

Examples where the change to normal form diverges

In all these examples the change of coordinate to normal form is 1-summable

- **Example 1.** A germ of analytic diffeomorphism $f : (\mathbb{C},0) \to (\mathbb{C},0)$ such that

$$f(z) = z + z^2 + (1-a)z^3 + o(z^3)$$



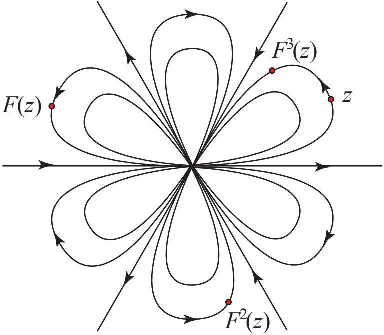
Normal form: the time-one map of the vector field

$$\dot{z} = \frac{z^2}{1+az}$$

Resonant diffeomorphism

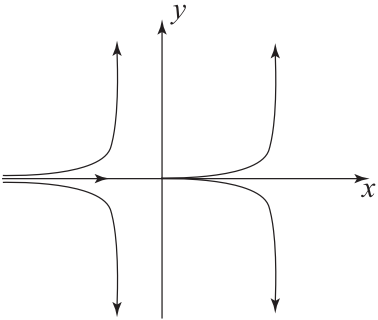
- **Example 2.** A germ of analytic diffeomorphism $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that

$$f(z) = \exp\left(\frac{2\pi i p}{q}\right) z + z^{q+1} + A z^{2q+1} + o(z^{2q+1})$$



Saddle-node

- **Example 3.** A germ of saddle-node of a planar vector field

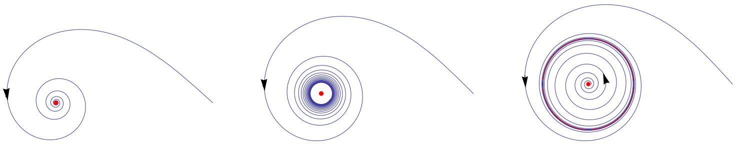


Normal form

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = y(1+ax) \end{cases}$$

Weak focus

- **Example 4.** A Hopf bifurcation



(a) $\epsilon < 0$ (b) $\epsilon = 0$ (c) $\epsilon > 0$

Orbital normal form

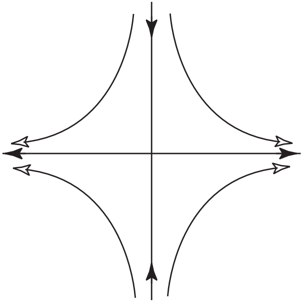
$$\dot{z} = z(i+\epsilon) - z^2\bar{z} + az^3\bar{z}^2$$

which can be rewritten

$$\begin{cases} \dot{r} = \epsilon r - r^3 + ar^5 \\ \dot{\theta} = 1 \end{cases}$$

Resonant saddle

- **Example 5.** A germ of resonant saddle of a planar vector field of order 1 with quotient of eigenvalues $-\frac{p}{q}$

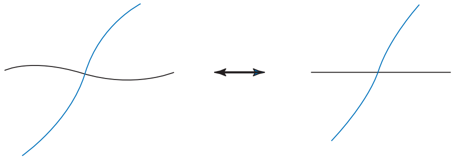


Orbital normal form

$$\begin{cases} \dot{x} = x \\ \dot{y} = y\left(-\frac{p}{q} + x^p y^q + a x^{2p} y^{2q}\right) \end{cases}$$

Curvilinear angle

- **Example 6.** A germ of curvilinear angle



When are two germs of curvilinear angles conformally equivalent?

We will consider the case where the angle is of the form $2\pi\frac{p}{q}$, which we call a *rational angle*.

Linear differential system

- **Example 7.** A nonresonant irregular singular point of Poincaré rank 1 of a linear differential system

$$x^2\frac{dy}{dx} = A(x)y, \qquad y \in \mathbb{C}^n$$

Normal form

$$x^2\frac{dy}{dx} = (D_0 + D_1 x)y$$

with D_0, D_1 diagonal

The common features

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- To understand why we have divergence, we unfold Two objects: codimension 1 will lead to 1-summability in the limit.
- In the unfolding, generically the divergence can be seen as the limit of the gluing of the two local models which are rigid. Hence, divergence is the rule and convergence is exceptional.

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- In all cases we have a finite parameter family representing a formal normal form: “the model family”.
- The extra formal parameter(s) are present to match the need of independent multipliers or eigenvalues in the unfolding.
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- In all cases we observe a "parametric resurgence phenomenon", i.e. the unfolded singular points have pathologies on discrete sequences of parameter values $\{\epsilon_n\}$ converging to $\epsilon = 0$.

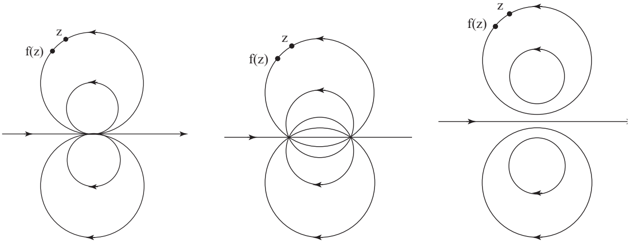
The parabolic point: coallescence of two fixed points

- **Example 1.** Unfolding

$$f_\epsilon(z) = z + (z^2 - \epsilon)(1 + O(z, \epsilon))$$

The model family is the time-one map of

$$\dot{z} = \frac{z^2 - \epsilon}{1 + a(\epsilon) z}$$



Coallescence of a fixed point and a periodic orbit of period q

- **Example 2.** An unfolding of

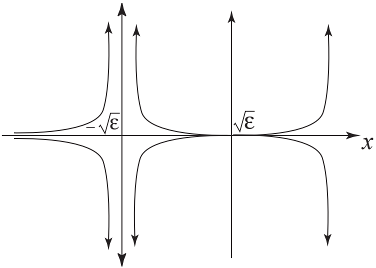
$$f(z) = \exp\left(\frac{2\pi i p}{q}\right) z + \frac{1}{q} z^{q+1} + a z^{2q+1} + o(z^{2q+1})$$

can be taken so that

$$f_\epsilon^{\circ q}(z) = z + z(z^q - \epsilon)(1 + O(z,\epsilon))$$

Coallescence of a saddle and a node

- **Example 3.**



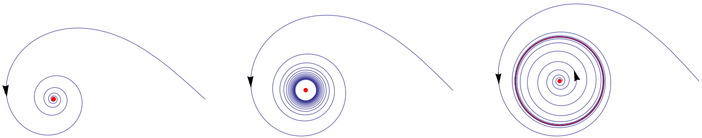
Orbital normal form for the unfolding

$$\begin{cases} \dot{x} = x^2 - \epsilon \\ \dot{y} = y(1 + ax) \end{cases}$$

Coallescence of a focus and a limit cycle

- **Example 4.** A Hopf bifurcation with orbital normal form

$$\dot{z} = z(i + (z\bar{z} - \epsilon)(1 + az\bar{z}))$$



(a) $\epsilon < 0$ (b) $\epsilon = 0$ (c) $\epsilon > 0$

A complex weak focus is orbitally the same as a saddle with ratio of eigenvalues equal to -1

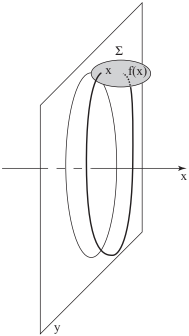
- **Example 4.** Taking $w = \bar{z}$, the system can be rewritten

$$\dot{z} = z(i + (zw - \epsilon)(1 + azw))$$
$$\dot{w} = w(-i + (zw - \epsilon)(1 + azw))$$

which is orbitally the same as a complex saddle. The complex curve $zw = \epsilon$ is a special *leaf*, which has non trivial homology.

Coallescence of the invariants manifolds of a saddle point with a distinguished invariant manifold

- **Example 5.**



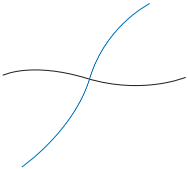
Orbital normal form

$$\begin{cases} \dot{x} = x \\ \dot{y} = y\left(-\frac{p}{q}(1+\epsilon) + x^p y^q + a x^{2p} y^{2q}\right) \end{cases}$$

Two families are orbitally equivalent if and only if the holonomies of their y-separatrices are conjugate.

Curvilinear angles

- **Example 6.** For a curvilinear angle we have a Schwarz symmetry $z \mapsto \Sigma_j(z)$ associated to each curve.



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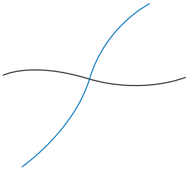

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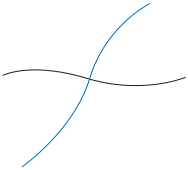


Let

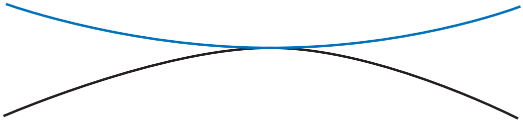
$$f = \Sigma_2 \circ \Sigma_1$$

f is a germ of analytic diffeomorphism:

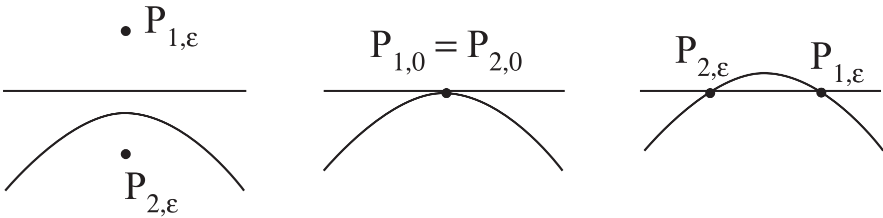
$$f(z) = e^{4\pi i \frac{p}{q}} z + o(z), \qquad \Sigma_1 \circ f = f^{-1} \circ \Sigma_1$$

Case of the horn

It is a special case of Example 1 and the coallescence of two intersection points of the analytic arcs



Unfolding the horn



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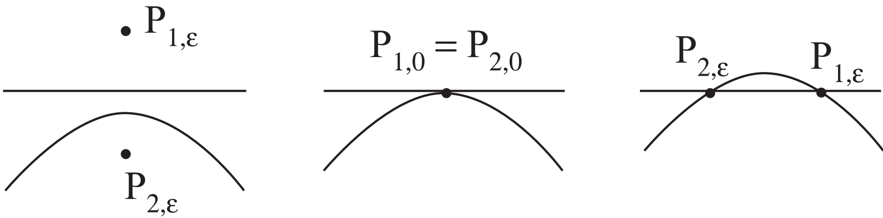

Formal invariant a: a limit of a measure of a shift between the two angles

$$\theta_\pm = \pm\frac{\sqrt{\epsilon}}{1 \pm a(\epsilon)\sqrt{\epsilon}}$$

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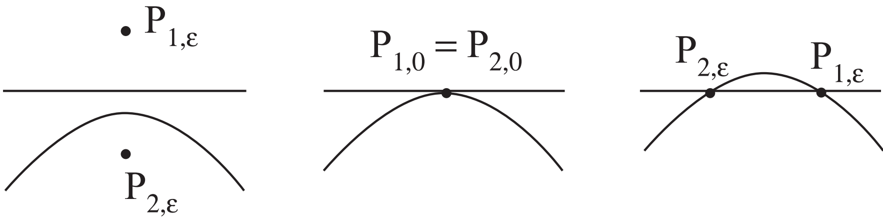

Formal invariant a: a limit of a measure of a shift between the two angles

$$\theta_\pm = \pm \frac{\sqrt{\epsilon}}{1 \pm a(\epsilon)\sqrt{\epsilon}}$$

$$a(\epsilon) = \frac{1}{2}\left(\frac{1}{\theta_+} + \frac{1}{\theta_-}\right)$$

The confluence of two regular singular points

- **Example 7.**

Normal form

$$(x^2-\epsilon)\frac{dy}{dx}=(D_0(\epsilon)+D_1(\epsilon)x)y$$

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In the unfolding we have, generically, at each regular singular point a basis of solutions that are eigenvectors of the monodromy around the regular singular point.

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The divergence in the limit forces that the eigenbases at each singular point mismatch.

For the special resonant values of the parameter at one singular point, it forces the existence of solutions with logarithmic terms.

Going to higher codimension

When we have the confluence of k *special objects* we often observe k-summability of the normalizing changes of coordinates.

Examples

- **Example 1 bis.** A germ of analytic diffeomorphism $f:(\mathbb{C},0)\to(\mathbb{C},0)$ such that
$$f(z)=z+kz^{k+1}+k^2(1-a)z^{2k+1}+o(z^{2k+1})$$

Normal form for the unfolding: the time-one map of
$$\dot{z}=\frac{z^{k+1}+\epsilon_{k-1}z^{k_1}+\cdots+\epsilon_1 z+\epsilon_0}{1+a(\epsilon)z^k}$$

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$k+1$ parameters to control $k+1$ eigenvalues at the $k+1$ singular points.

More examples

- **Example 2 bis.** A germ of analytic diffeomorphism $f : (\mathbb{C},0) \to (\mathbb{C},0)$ such that

$$f(z) = \exp\left(\frac{2\pi i p}{q}\right) z + z^{kq+1} + a z^{2kq+1} + o(z^{2kq+1})$$

More examples

- **Example 3 bis.** A germ of saddle-node of multiplicity $k+1$ with unfolding

$$\begin{cases} \dot{x} = z^{k+1} + \epsilon_{k-1} z^{k-1} + \cdots + \epsilon_1 z + \epsilon_0 \\ \dot{y} = y(1 + a(\epsilon) x^k) \end{cases}$$

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Again $k+1$ parameters to control $k+1$ ratio of eigenvalues at the $k+1$ singular points.

More examples

- **Example 4 bis.** A Hopf bifurcation of codimension k with orbital normal form

$$\dot{z} = z\left[(i+\epsilon_0) + \epsilon_1|z|^2 + \cdots + \epsilon_{k-1}|z|^{2(k-1)} - |z|^{2k} + a|z|^{4k}\right]$$

which can be rewritten

$$\begin{cases} \dot{r} = \epsilon_0 r + \epsilon_1 r^3 + \cdots + \epsilon_{k-1} r^{2k-1} - r^{2k+1} + a r^{4k+1} \\ \dot{\theta} = 1 \end{cases}$$

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$k+1$ parameters to control $k+1$ multipliers of the Poincaré return map at the k limit cycles and at the singular point.

More examples

- **Example 5 bis.** A germ of resonant saddle of codimension k

$$\begin{cases} \dot{x} = x \\ \dot{y} = y\left(-\frac{p}{q} + x^{kp}y^{kq} + ax^{2kp}y^{2kq}\right) \end{cases}$$

More examples

- **Example 6 bis.** A germ of curvilinear angle of codimension k : the associated diffeomorphims $f = \Sigma_2 \circ \Sigma_1$ has a resonant fixed point of codimension k

More examples

- **Example 7 bis.** A nonresonant irregular singular point of Poincaré rank k of a linear differential system

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with normal form of the unfolding

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$(k+1)n$ parameters to control the eigenvalues at $k+1$ singular points