Germs of analytic families of diffeomorphisms unfolding a parabolic point (I)

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Work done with C. Christopher, P. Mardešić, R. Roussarie and L. Teyssier following pioneering work by A. Douady, A. Glutsyuk, P. Lavaurs, R. Oudkerk

Structure of the lecture

- Statement of the problem
- The preparation of the family in the codimension 1 case
- Construction of a modulus of analytic classification in the codimension 1 case
- What we learn from the modulus

Statement of the problem

We consider germs of generic analytic k-parameter families f_ϵ of diffeomorphisms unfolding a parabolic point of codimension k

$$f_0(z) = z + z^{k+1} + o(z^{k+1})$$

When are two such germs conjugate?

Conjugacy of two germs of families

Two germs of analytic families of diffeomorphisms f_ϵ and $\tilde{f}_{\tilde{\epsilon}}$ are conjugate it there exists $r, \rho > 0$ and analytic functions

$$h : \mathbb{D}_\rho \to \mathbb{C}, \qquad H : \mathbb{D}_r \times \mathbb{D}_\rho \to \mathbb{C}$$

such that

- h is a diffeomorphism and, for each fixed ϵ, $H_\epsilon = H(\cdot, \epsilon)$ is a diffeomorphism;
- for all $\epsilon \in \mathbb{D}_\rho$ and for all $z \in \mathbb{D}_r$, then

$$\tilde{f}_{h(\epsilon)} = H_\epsilon \circ f_\epsilon \circ (H_\epsilon)^{-1}$$

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The difficulty is the change of parameters... Hence, we *prepare* the families to a *canonical parameter* so that a conjugacy between them preserves the parameter (i.e. h is the identity.)

Preparation of the family

Let $\tilde{f}_{\tilde{\epsilon}}$ be an unfolding of a germ of diffeormorphism

$$f_0(\tilde{z}) = \tilde{z} + \tilde{z}^2 + O(\tilde{z}^3)$$

Then $f_{\tilde{\epsilon}}$ is generic if $\frac{\partial f_{\tilde{\epsilon}}}{\partial \tilde{\epsilon}} \neq 0$.

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By the Weierstrass preparation theorem

$$f_{\tilde{\epsilon}}(\tilde{z}) - \tilde{z} = \widetilde{P}_{\tilde{\epsilon}}(\tilde{z})\tilde{h}(\tilde{z}, \tilde{\epsilon})$$

with $P_{\tilde{\epsilon}}(\tilde{z}) = \tilde{z}^2 + \eta_1(\tilde{\epsilon})\tilde{z} + \eta_0(\tilde{\epsilon})$ and $\tilde{h}(\tilde{z}, \tilde{\epsilon}) = 1 + O(|\tilde{z}, \tilde{\epsilon}|)$

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A translation $\tilde{z} \mapsto \check{z} = \tilde{z} + \frac{1}{2}\eta_1(\tilde{\epsilon})$ allows bringing

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Since the family is generic we have $\tilde{\eta}'(0) \neq 0$, and hence we can make the change of parameter $\check{\epsilon} = -\tilde{\eta}(\tilde{\epsilon})$.

Preparation of the family

In these new $(\check{z}, \check{\epsilon})$, the family becomes

$$f_{\check{\epsilon}}(\check{z}) = \check{z} + (\check{z}^2 - \check{\epsilon})h(\check{z}, \check{\epsilon})$$

We write

$$h(\check{z}, \check{\epsilon}) = c_0(\check{\epsilon}) + c_1(\check{\epsilon})\check{z} + (\check{z}^2 - \check{\epsilon})g(\check{z}, \check{\epsilon})$$

with $c_0(\check{\epsilon}) = 1 + O(\check{\epsilon})$. Then the multipliers at the fixed points are independent of g:

$$\lambda_\pm = 1 \pm 2\sqrt{\check{\epsilon}}\left(c_0(\check{\epsilon}) + c_1(\check{\epsilon})\sqrt{\check{\epsilon}}\right)$$

Preparation of the family

$$\lambda_\pm = 1 \pm 2\sqrt{\check{\epsilon}}\left(c_0(\check{\epsilon}) \pm c_1(\check{\epsilon})\sqrt{\check{\epsilon}}\right)$$

There exists a change of parameter $\check{\epsilon} \mapsto \epsilon = \check{\epsilon} m(\check{\epsilon})$ and a function $a(\epsilon)$ analytic such that

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A corresponding scaling $\check z \mapsto z = \check z m^{1/2}(\check\epsilon)$ brings the singular points to $z^2 - \epsilon$.
Then the formal normal form is the time one map of the vector field $\dot z = \frac{z^2-\epsilon}{1+a(\epsilon)z}$

The parameter is an analytic invariant of a prepared family!

Indeed, if $\mu_\pm = \log \lambda_\pm = \frac{\pm 2\sqrt{\epsilon}}{1 \pm a(\epsilon)\sqrt{\epsilon}}$, then

$$\frac{1}{\mu_+} + \frac{1}{\mu_-} = a(\epsilon)$$

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Indeed $\frac{1}{\mu_+} - \frac{1}{\mu_-} = \frac{n(\check{\epsilon})}{\sqrt{\check{\epsilon}}}$.

This also gives the interpretation of $a(\epsilon)$: it is a shift between the two eigenvalues: $\mu_\pm$ are not exactly inverse one of the other.

Prepared form of the family

A family under the form

$$f_\epsilon(z) = z + (z^2 - \epsilon)h(z, \epsilon)$$

with *canonical parameter* is called *prepared*.

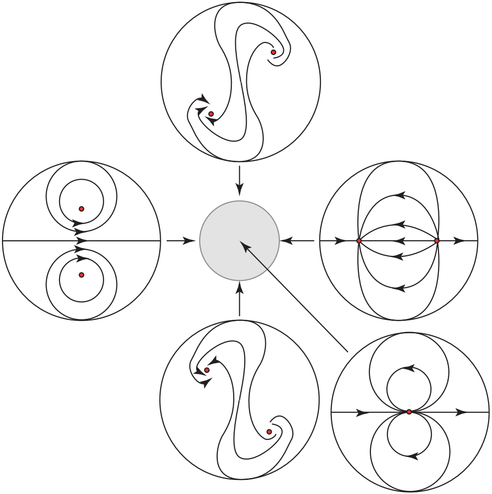
When considering whether two families are conjugate, we can always limit ourselves to prepared families.

We have also identified a normal form, namely the time one map of the vector field $\dot{z} = \frac{z^2-\epsilon}{1+a(\epsilon)z}$

The conjugacy problem for prepared families

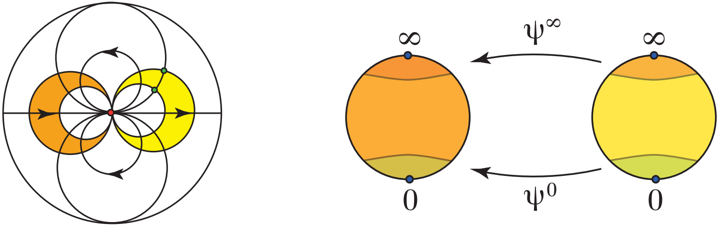
The normal form is unique. Hence if two prepared families have the same normal form, then they would be conjugate if the change of coordinates to the normal form were convergent.

But it is not. However, topologically the family f_ϵ behaves as the time one map of the vector field $\dot{z} = \frac{z^2 - \epsilon}{1 + a(\epsilon) z}$



The classifying object will be the “space of orbits”

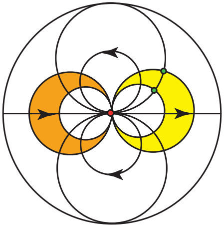
Let us consider the case $\epsilon = 0$

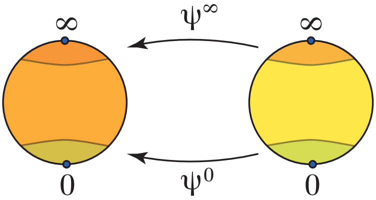


Two fundamental domains are necessary to cover all orbits.

If we identify the two sides of the crescent, the corresponding Riemann surface is conformally equivalent to a sphere minus two points.

The classifying object is called the Ecalle-Voronin modulus

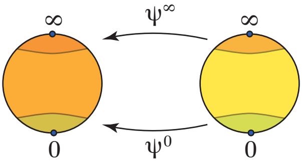




Each sphere ($\mathbb{CP}^1$) has an almost unique coordinate (up to linear map). The *Ecalle-Voronin* modulus is given by the identifying maps (ψ^0, ψ^∞) in the neighborhoods of 0 and ∞.

The maps ψ^0 and ψ^∞ are germs of analytic diffeomorphisms.

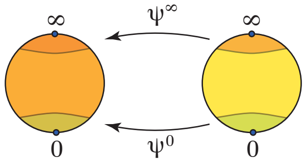
The Ecalle-Voronin modulus



- Two germs of parabolic diffeomorphisms f and $\tilde{f}$ with same formal normal form are conjugate if and only if they have the same Ecalle-Voronin modulus up to linear maps

$$\begin{cases} \psi^0 = L_C \circ \tilde{\psi}^0 \circ L_{C'} \\ \psi^\infty = L_C \circ \tilde{\psi}^\infty \circ L_{C'} \end{cases}$$

The Ecalle-Voronin modulus



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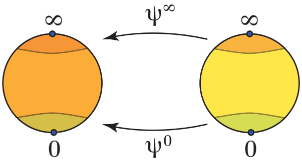
- A germ of parabolic diffeomorphism f is conjugate to its normal form iff ψ^0 and ψ^∞ are both linear.

The Ecalle-Voronin modulus



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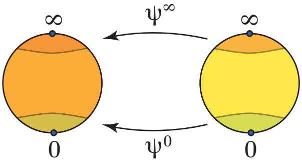
- A germ of parabolic diffeomorphism f is conjugate to its normal form iff ψ^0 and ψ^∞ are both linear.
- Any pair of germs (ψ^0, ψ^∞) in the neighborhoods of 0 and ∞ is realizable as the modulus of a germ of parabolic diffeomorphism.

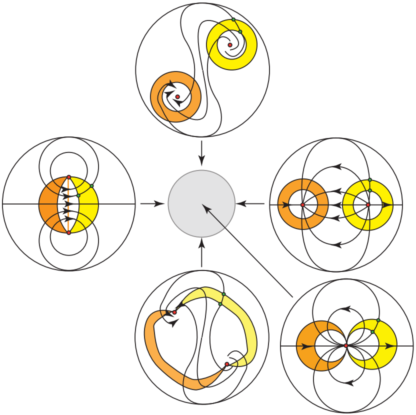
The Ecalle-Voronin modulus



- Any pair of germs (ψ^0, ψ^∞) in the neighborhoods of 0 and ∞ is realizable as the modulus of a germ of parabolic diffeomorphism.

This means that the modulus space is enormous: it is infinite-dimensional. What does this modulus mean?

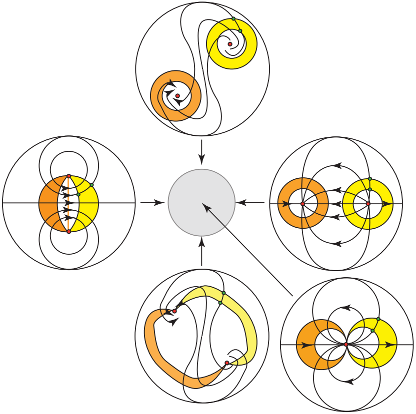
To understand we unfold the modulus



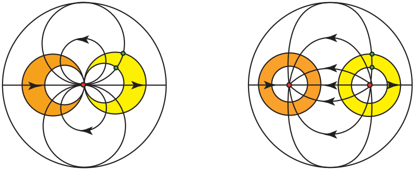
For $\epsilon \neq 0$, there are too natural ways to unfold the crescents:

- as crescents which, once the sides are identified, will have the conformal structure of a sphere;
- as annuli which, once the sides are identified, will have the conformal structure of a torus.

This could suggest two charts in parameter space.



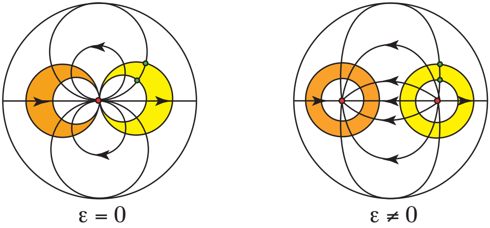
The first chart



For $\epsilon \neq 0$ the diffeomorphism can be conjugated to the model in the neighborhood of each singular point. But generically the two conjucacies are not analytic continuation one of another. If this obstruction persists till the limit $\epsilon = 0$, then the transformation to normal form may be divergent at the limit.

Conversely, if the transformation to normal form is divergent at the limit, then necessarily the two conjucacies are not analytic continuation one of another for small $\epsilon \neq 0$.

The first chart (studied by Glutsyuk)

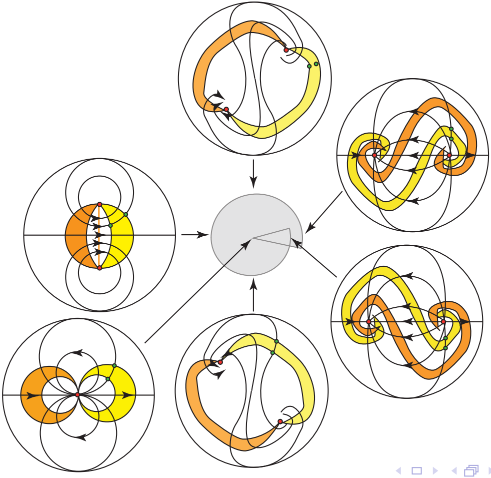


Unfolding has allowed us to understand why we have divergence at the limit.

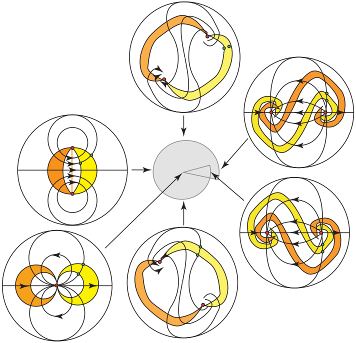
But this point of view does not apply to all values of $\epsilon \neq 0$. Indeed, when $|f'_\epsilon(\pm\sqrt{\epsilon})| = 1$, then the fixed points may not be linearizable… Also, the domains where we can bring to the model may not intersect. Hence the need for a second point of view.

The second chart can be pushed to cover all values of ϵ, but in a ramified way

The idea of unfolding the crescents as crescents goes back to Douady and Lavaurs.



The classifying object for the family of diffeomorphisms



All crescents with sides identified have the conformal structure of spheres. The identifying maps in the neighborhoods of 0 and ∞ form a continuous family $(\psi^0_{\hat\epsilon}, \psi^\infty_{\hat\epsilon})_{\hat\epsilon\in V}$.

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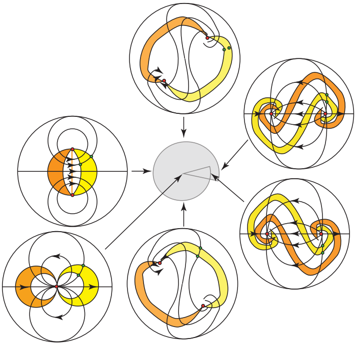

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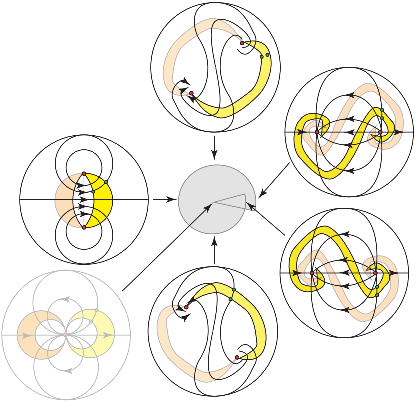
Two families with same formal normal forms are conjugate if and only they have equivalent *modulus* $(\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}})_{\hat{\epsilon}\in V}$ (equivalence under linear changes of coordinates).

The lessons

- The dynamics is closely related to that of the vector field $\dot{z} = z^2 - \epsilon$.
- For each $\epsilon \neq 0$, one crescent is enough to describe the dynamics.
- The parametric resurgence phenomenon also occurs.

One crescent is enough to describe the dynamics



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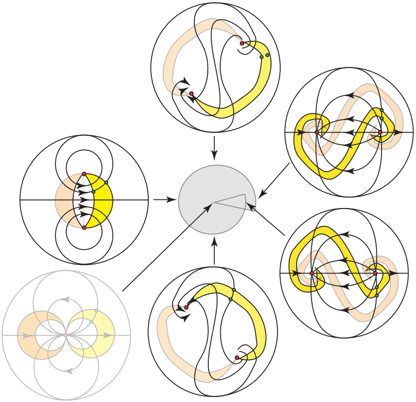
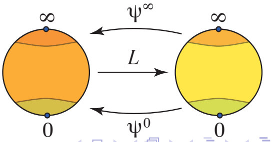
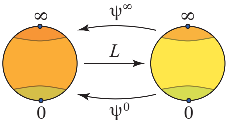


This is because a global diffeomorphism exists between the two crescents, the *Lavaurs map*.



The renormalized return maps



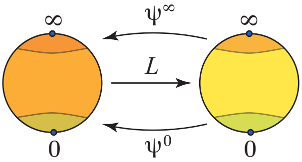
In the spherical coordinates the Lavaurs map is linear. It is possible to study the dynamics of the fixed points by the renormalized return maps

$$\begin{cases} L_{\hat{\epsilon}} \circ \psi^{\infty}_{\hat{\epsilon}}, & \text{near } \sqrt{\hat{\epsilon}}, \\ L_{\hat{\epsilon}} \circ \psi^{0}_{\hat{\epsilon}}, & \text{near } -\sqrt{\hat{\epsilon}}. \end{cases}$$

Another lesson

- $\psi^{\infty}_{\hat{\epsilon}}$ controls the dynamics near $+\sqrt{\hat{\epsilon}}$ and $\psi^{0}_{\hat{\epsilon}}$ controls the dynamics near $+\sqrt{\hat{\epsilon}}$.

The decomposition of the dynamics



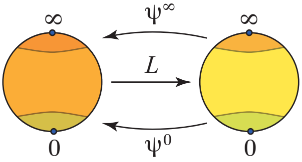
We have decomposed the dynamics into

- a wild linear part, which depends only on $\hat{\epsilon}$ and $a(\epsilon)$ (i.e. the formal part!) and has no limit when $\epsilon \to 0$;
- and a nonlinear part which has a limit when $\epsilon \to 0$.

The Lavaurs map has the form

$$L_{\hat{\epsilon}}(w) = \exp\left(-\frac{2\pi i}{\sqrt{\hat{\epsilon}}}\right) c(\hat{\epsilon}) w$$

The parametric resurgence phenomenon

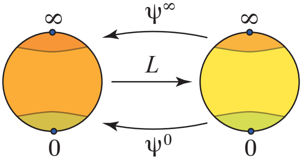


It can occur at any fixed point. Let us consider $-\sqrt{\hat{\epsilon}}$ with renormalized return map

$$\kappa_{\hat{\epsilon}} = L_{\hat{\epsilon}} \circ \psi^0_{\hat{\epsilon}}$$

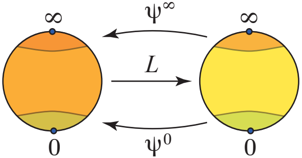
We can choose coordinates on the spheres to that $\psi'_{\hat{\epsilon}}(0) = 1$. We consider sequences ϵ_n such that $\kappa'_{\hat{\epsilon}}(0) = \exp(2\pi i p/q)$. If $\exp(2\pi i p/q)\psi^0_0$ is nonlinearizable (has a nonzero resonant term), then so does κ_{ϵ_n} for n sufficiently large. For ϵ close to ϵ_n orbit(s) of period q appear(s).

The parametric resurgence phenomenon



On sequences of parameter values, the mismatch is carried by the fixed points themselves, which are forced to be nonlinearizable.

What about a fixed point when the multiplier is an irrational rotation?



Suppose that $\kappa'_\epsilon(0) = \exp(2\pi i\alpha)$ with α irrational. Then ϵ is very close to values ϵ' for which $\kappa'_{\epsilon'}(0) = \exp(2\pi i p/q)$. If α is Liouvillian (well approximated by the rationals) this accumulation of periodic points may lead to the nonlinearizability of the fixed point at ϵ.

(This obstruction to linearizability for multipliers of the form $\exp(2\pi i\alpha)$ was studied by Ilyashenko-Pjartli and Yoccoz.)