Germs of analytic families of diffeomorphisms unfolding a parabolic point (II)

Christiane Rousseau

Work done with C. Christopher, P. Mardešić, R. Roussarie and L. Teyssier following pioneering work by A. Douady, A. Glutsyuk, P. Lavaurs, R. Oudkerk

Structure of the lecture

- Statement of the problem
- The preparation of the family in the codimension k case
- Construction of a modulus of analytic classification in the codimension k case

Statement of the problem

We consider germs of generic analytic k-parameter families f_ϵ of diffeomorphisms unfolding a parabolic point of codimension k

$$f_0(z) = z + z^{k+1} + o(z^{k+1})$$

When are two such germs conjugate?

Conjugacy of two germs of families

Two germs of families of diffeomorphisms f_ϵ and $\tilde{f}_{\tilde{\epsilon}}$ are conjugate it there exists $r, \rho > 0$ and analytic functions

$$h : \mathbb{D}_\rho \to \mathbb{C}, \qquad H : \mathbb{D}_r \times \mathbb{D}_\rho \to \mathbb{C}$$

such that

- h is a diffeomorphism and for each fixed ϵ, $H_\epsilon = H(\cdot, \epsilon)$ is a diffeomorphism;
- for all $\epsilon \in \mathbb{D}_\rho$ and for all $z \in \mathbb{D}_r$, then

$$\tilde{f}_{h(\epsilon)} = H_\epsilon \circ f_\epsilon \circ (H_\epsilon)^{-1}$$

The difficulty is the change of parameters... Hence, we *prepare* the families to a *canonical parameter* so that a conjugacy between them preserves the parameter (i.e. h is the identity.)

Preparation of the family

Let $\tilde{f}_{\tilde{\epsilon}}$ be a k-parameter analytic unfolding of a germ of diffeormorphism

$$f_0(\tilde{z}) = \tilde{z} + \tilde{z}^{k+1} + O(\tilde{z}^{k+2})$$

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By the Weierstrass preparation theorem

$$f_{\tilde{\epsilon}}(\tilde{z}) - \tilde{z} = \widetilde{P}_{\tilde{\epsilon}}(\tilde{z})\tilde{h}(\tilde{z}, \tilde{\epsilon})$$

with $\tilde{h}(\tilde{z}, \tilde{\epsilon}) = 1 + O(|\tilde{z}, \tilde{\epsilon}|)$ and

$$P_{\tilde{\epsilon}}(\tilde{z}) = \tilde{z}^{k+1} + \eta_k(\tilde{\epsilon})\tilde{z}^k + \cdots + \eta_1(\tilde{\epsilon})\tilde{z} + \eta_0(\tilde{\epsilon})$$

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The family is *generic* if $\left|\frac{\partial(\eta_{k-1},\dots,\eta_0)}{\partial(\tilde{\epsilon}_{k-1},\dots,\tilde{\epsilon}_0)}\right| \neq 0$, and hence we can make the change of parameter $\check{\epsilon}_j = \eta_j(\tilde{\epsilon})$.

Preparation of the family

In these new $(\check{z}, \check{\epsilon})$ the family becomes

$$f_{\check{\epsilon}}(\check{z}) = \check{z} + P_{\check{\epsilon}}(\check{z})(1 + Q_{\check{\epsilon}}(\check{z})h(\check{z}, \check{\epsilon})$$

We find a vector field

$$\dot{\check{z}} = P_{\check{\epsilon}}(1 + S_{\check{\epsilon}}(\check{z})$$

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To obtain the prepared form we change $(\check{z}, \check{\epsilon}) \mapsto (z, \epsilon)$ in the expression of the diffeomorphism.

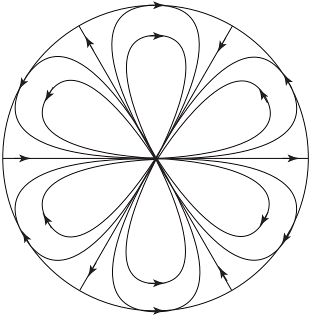
The parameter is almost canonical

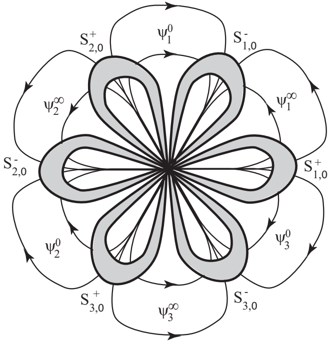
Theorem [RT] Let $(z, \epsilon) \mapsto (\check{z}, \check{\epsilon})$ map a vector field $\dot{z} = v_\epsilon(z) = \frac{P_\epsilon}{1+a(\epsilon)z^k}$ to $\dot{\check{z}} = \check{v} = \frac{\check{P}_{\check{\epsilon}}}{1+\check{a}(\check{\epsilon})\check{z}^k}$. Then there exists $\tau = \exp(2\pi i m/k)$ and $t(\epsilon)$ such that the change has the form

$$\begin{cases} \check{z} = \tau \Phi_{v_\epsilon}^{t(\epsilon)}(z), \\ \check{\epsilon}_j = \tau^{1-j}\epsilon_j, \end{cases}$$

where $\Phi_{v_\epsilon}^{t(\epsilon)}$ is the flow of v_ϵ for the time $t(\epsilon)$.

The modulus space for $\epsilon = 0$

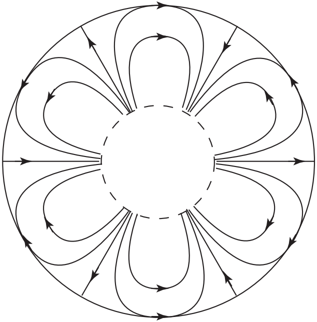




The diffeomorphism now has $2k$ petals near the parabolic point. Hence we need $2k$ fundamental domains and the modulus space has $2k$ components $(\psi^0_1, \psi^\infty_1, \ldots, \psi^0_k, \psi^\infty_k)$

We need to unfold that

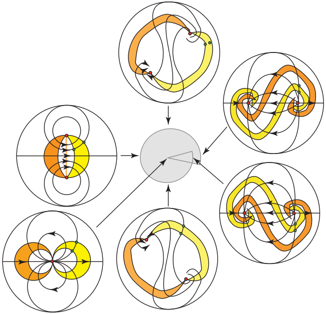
The behavior stays the same near the boundary.

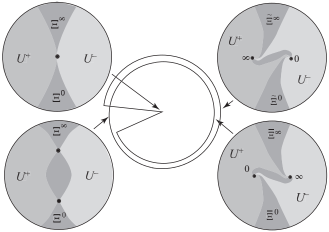


How do we take the fundamental domains inside?

Coming back to the case $k = 1$

Taking fundamental domains as crescent shapes is the same as covering a disk with generalized sectors on which we can almost uniquely conjugate the diffeomorphim to the normal form.





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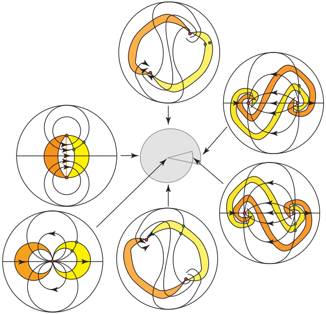
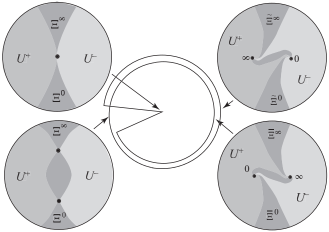




The boundaries of the sectors are trajectories of $\dot{z} = e^{i\alpha}(z^2 - \epsilon)$.

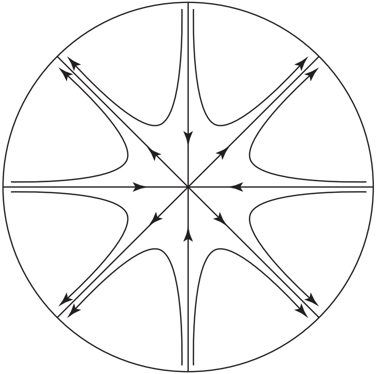
This gives the strategy for $k > 1$

The boundaries of the sectors will be taken as trajectories of $\dot{z} = e^{i\alpha} P_\epsilon(z)$. Such vector fields have been studied by Douady and Sentenac.

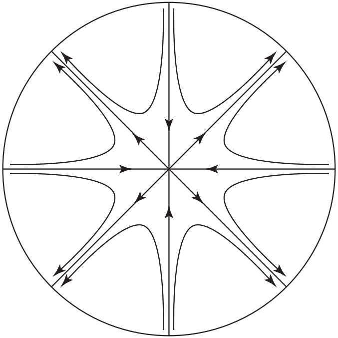
We can no more cover the parameter space with a unique domain, even in a ramified way.

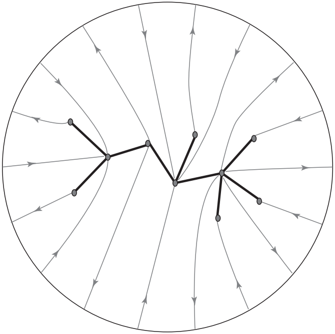
The geometry of the vector field $\dot{z} = P_\epsilon(z)$.

They have been studied by Douady and Sentenac on $\mathbb{CP}^1$ when all singular points are simple (the discriminant is nonzero). The organizing center is the pole of order $k-1$ at infinity and its $2k$ separatrices.



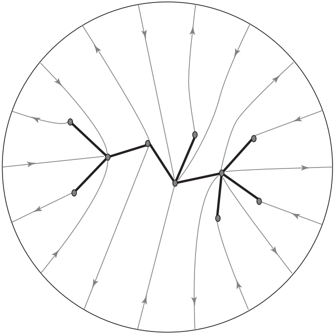
Generically the separatrices land at singular points.





Taking trajectories between the singular points divides a disk into $2k$ *generalized sectors* each attached to two singular points.

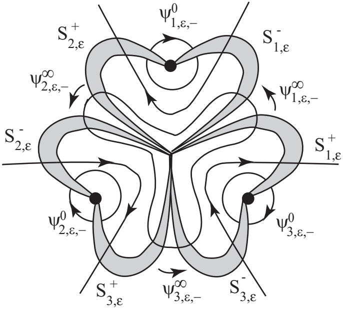
The modulus of analytic classification



On each sector we have an almost unique change of coordinate to the normal form, because the sector contains a fundamental domain conformally equivalent to $\mathbb{CP}^1 \setminus \{0, \infty\}$.

(The sectors are drawn non spiraling but they can be spiraling when approaching the singular points.)

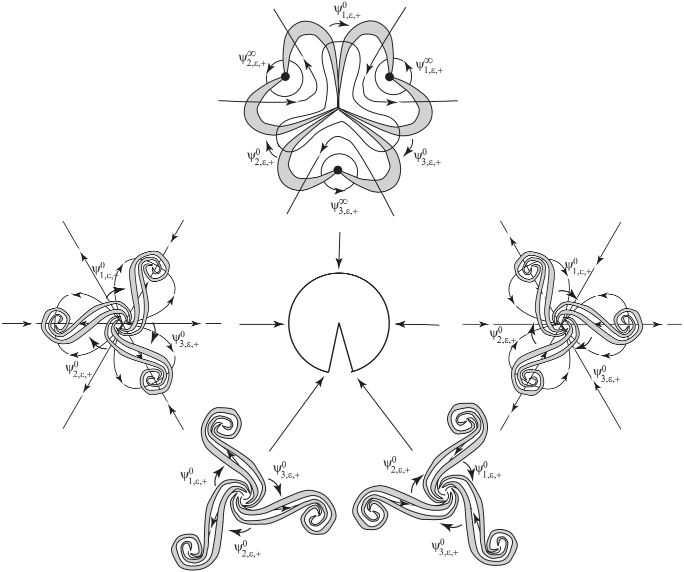
The modulus of analytic classification on a DS-domain in parameter space



It is given by an unfolding

$$(\psi^0_{1,\epsilon}, \psi^\infty_{1,\epsilon}, \dots, \psi^0_{k,\epsilon}, \psi^\infty_{k,\epsilon})$$

Other examples of unfolding



How many sectors in parameter space do we need to cover all parameter values?