Addendum to the paper “Modulus of analytic classification for unfoldings of generic parabolic diffeomorphisms”

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Abstract

The paper [2] describes a complete modulus of analytic classification under weak equivalence for generic analytic 1-parameter unfoldings of diffeomorphisms with a generic parabolic point. In this note it is shown that weak equivalence can be replaced by conjugacy.

1 Introduction

We consider a prepared family g_ν

$$g_\nu(z) = z + (z^2 - \nu^2)[1 + \beta(\nu) + A(\nu)z + (z^2 - \nu^2)Q(z, \nu)]. \tag{1.1}$$

as in[2], which is compared to the time-one map of the flow of

$$(v_\nu) \qquad \frac{z^2 - \nu^2}{1 + az}\,\frac{\partial}{\partial z}. \tag{1.2}$$

Note that $g_\nu = g_{-\nu}$. We prove the following theorem:

Theorem 1.1 *If two prepared families g_ν and $\overline{g}_\nu$ as in (1.1) have the same modulus then they are analytically conjugate under an analytic change of coordinate $h(\nu, z) = h_\nu(z)$ depending analytically on (ν, z) in a small neighborhood of the origin and satisfying $h_{-\nu} = h_\nu$. In particular $h(\nu, z) = \tilde{h}(\epsilon, z)$ depends analytically on (ϵ, z), where $\epsilon = \nu^2$.*

2 Symmetries of families unfolding a parabolic fixed point

In this section we discuss briefly the symmetries of the family g_ν as a tool to prove the Theorem 1.1.

Definition 2.1 1. The *group of symmetries* of g_0 is the commutator of g_0 inside the group of germs of analytic diffeomorphisms tangent to the identity at the origin.

2. Similarly, given g_ν defined on a neighborhood containing its fixed points, we will call *symmetry* of g_ν any analytic diffeomorphism on the same neighborhood which commutes with it.

Proposition 2.2 *[1] Depending on the modulus $(\psi_0^0, \psi_0^\infty)$ we get the following cases:*

(1) If g_0 is generic, i.e. ψ_0^0 or ψ_0^∞ does not commute with any linear map, then the symmetry group of g_0 is the group of iterates $\{g_0^n | n \in \mathbb{Z}\}$.

(2) If g_0 is not embedable and $g_0 = k_0^m$ with k_0 tangent to the identity and $m \in \mathbb{N}$ (i.e. $\psi_0^0(w) = w\xi_0^0(w^m)$ and $\psi_0^\infty(w) = w\xi_0^\infty(w^m)$ and one of them is nonlinear of order $m+1$), then the symmetry group of g_0 is the group of iterates $\{k_0^n | n \in \mathbb{Z}\}$.

(3) If g_0 is embedable, i.e. ψ_0^0 and ψ_0^∞ are linear and g_0 is conjugate by m_0 to the time-one map v_0^1 of the flow of the vector field v_0, where v_ν is given in (1.2)*, then all symmetries of g_0 are conjugate by m_0 to time-t maps v_0^t of the flow of v_0 with $t \in \mathbb{C}$. We can think of them as the t-th iterates g_0^t of g_0.*

Proposition 2.3 *We consider a prepared family g_ν unfolding g_0.*

(1) If g_0 is generic, i.e. ψ_0^0 or ψ_0^∞ do not commute with any linear map, then, for sufficiently small ν, any symmetry of g_ν is of the form g_ν^n for $n \in \mathbb{Z}$. In particular if γ_ν is a symmetry of g_ν depending continuously on ν in a sector, and such that $\gamma_0 = id$, then $\gamma_\nu = id$.

(2) If g_0 is not embedable and $g_0 = k_0^m$ with k_0 tangent to the identity and $m \in \mathbb{N}$ (i.e. $\psi_0^0(w) = w\xi_0^0(w^m)$ and $\psi_0^\infty(w) = w\xi_0^\infty(w^m)$ and one of them is nonlinear of order $m+1$), and if γ_ν is a symmetry of g_ν depending continuously on ν in a sector such that $\gamma_0 = id$, then $\gamma_\nu = id$.

(3) If g_0 is embedable, then one of the following cases occurs:

(a) If γ_ν is a symmetry of g_ν depending continuously on ν in a sector, and such that $\gamma_0 = id$, then $\gamma_\nu = id$.

(b) The map g_ν is embedable, i.e. conjugate under m_ν to the time-one map v_ν^1 of the flow of v_ν given in (1.2) *and its symmetries are conjugate by m_ν to the time-$t(\nu)$ maps $v_\nu^{t(\nu)}$ of the flow of v_ν for some continuous map $t(\nu)$ with values in $\mathbb{C}$. The map $t(\nu)$ associated to a symmetry is unique as soon as it unfolds the zero map, in which case it makes sense to call the corresponding symmetry the $t(\nu)$-th iterate $g_\nu^{t(\nu)}$ of g_ν.*

Proof. A symmetry sends orbits to orbits. For $\nu \neq 0$ the orbit structure is completely determined by the quotient of a sphere ($\mathbb{CP}^1$) by the return maps in the neighborhood of 0 and ∞. So a symmetry is given by a diffeomorphism of the sphere preserving 0 and ∞ (i.e. a linear map) which commutes with the return maps.

(1) This case occurs as soon as one of ψ_0^0 and ψ_0^∞ is nonlinear and both are not of the form $\psi_0^0(w) = w\xi_0^0(w^m)$ and $\psi_0^\infty(w) = w\xi_0^\infty(w^m)$. This can be seen on a finite jet. (Indeed if $\psi_0^0(w) = \sum_{i=1}^\infty a_i w^i$ and $\psi_0^\infty(w) = \sum_{i=1}^\infty b_i w^i$ this occurs as soon as there exists $m, n > 1$ with $(m,n) = 1$ such that $a_n \neq 0$ or $b_n \neq 0$ and simultaneously $a_m \neq 0$ or $b_m \neq 0$.) Then the same property is true for ψ_ν^0 and ψ_ν^∞ for ν sufficiently small. So all symmetries γ_ν of g_ν are of the form g_ν^n with $n \in \mathbb{Z}$. If a family γ_ν depends continuously on ν then n needs to be constant and $n = 0$ is the only possibility if we add the condition that $\lim_{\nu \to 0} h_\nu = id$.

(2) is similar. Note that the discrete symmetries may or may not be preserved in the unfolded family. Continuous families of symmetries will be given by some κ_ν^n for a fixed $n \in \mathbb{Z}$ where $\kappa_\nu^d = g_\nu$ and κ_ν is continuous in ν. Of course $d|m$ so that for $\nu = 0$ we have $\kappa_0 = k_0^{m/d}$.

(3)

(a) The first case occurs as soon as one of $\psi_{1,\nu}^0$ or $\psi_{1,\nu}^\infty$ is nonlinear. Indeed suppose that $\psi_\nu^0(w) = a_1(\nu)w + a_s(\nu)w^s + o(w^s)$ with $a_s \not\equiv 0$. As $a_s(\nu)$ depends analytically on $\nu \neq 0$ it is nonzero on an open dense subset on which we can apply the same argument as in (1) or (2) since the only possible symmetries are discrete.

(b) Let us look at an individual symmetry H_ν of v_ν, given by the time-$t(\nu)$ map of its flow. Then $H_\nu'(\pm\nu) = \exp\left(\pm\frac{2\nu}{1\pm a\nu}t(\nu)\right)$. Different times $t(\nu)$ and $\tau(\nu)$ yield the same symmetry H_ν if and only if there exists $k, k' \in \mathbb{Z}$ such that

$$T(\nu) = t(\nu) - \tau(\nu) = \frac{k\pi i(1 + a\nu)}{\nu} = -\frac{k'\pi i(1 - a\nu)}{\nu}.$$

The only continuous solution $T(\nu)$ satisfying $T(0) = 0$ is $T \equiv 0$. □

3 Proof of Theorem 1.1

Proof of Theorem 1.1. We consider two prepared families g_ν and $\overline{g}_\nu$ which have the same modulus. For $\nu \in V_{\delta,0}^L$ defined in (2.15) of [2] we have constructed a conjugacy $h_\nu(z)$ between g_ν and $\overline{g}_\nu$ depending analytically on $\nu \neq 0$ and continuously on ν near $\nu = 0$. If we pass to $\epsilon = \nu^2$ this gives us two conjugacies $h_{1,\epsilon}$ and $h_{2,\epsilon}$ between g_ν and $\overline{g}_\nu$ for $\arg\epsilon \in (-\pi+2\delta, \pi-2\delta)$. We will be done if we show that we can choose the conjugacy $h_\nu(z)$ such that $h_{1,\epsilon} \equiv h_{2,\epsilon}$. Indeed $\gamma_\epsilon = (h_{1,\epsilon})^{-1} \circ h_{2,\epsilon}$ is a symmetry of g_ν (this is the case since $g_\nu = g_{-\nu}$). Moreover γ_ϵ depends analytically on $\epsilon \neq 0$ and has a continuous limit at $\epsilon = 0$. As $h_{i,\epsilon}$, $i = 1, 2$, have the same limit h_0 when $\epsilon \to 0$, then $\gamma_0 = id$. In cases (1), (2) and (3)(a) of Proposition 2.3 then $\gamma_\epsilon = id$.

So we only need to discuss case (3)(b). In this case we stay with the parameter ν and we use the two sectorial domains $V_{\delta,0}^L$ and $V_{\delta,1}^L$ defined in (2.15) of [2]. On each of them we have defined a family of diffeomorphims $h_{0,\nu}$ and $h_{1,\nu}$ between g_ν and $\overline{g}_\nu$. Moreover it is easy to manage that

$$h_{0,\nu} = h_{1,-\nu}. \tag{3.1}$$

(it suffices to take the same normalized Fatou coordinates on each sector). The two sectors intersect on two smaller sectors $V_\delta^+ = \{\nu \in V_{\delta,0}^L | \arg\nu \in (-\pi/2 + \delta, \pi/2 - \delta)\}$ and $V_\delta^- = \{\nu \in V_{\delta,0}^L | \arg\nu \in (\pi/2+\delta, 3\pi/2-\delta)\}$. On each of these sectors the diffeomorphims $\gamma_\nu^\pm = h_{1,\nu}^{-1} \circ h_{0,\nu}$ is a symmetry of g_ν unfolding the identity. By Proposition 2.3 there exist times $\tau^\pm(\nu)$ such that $\gamma_\nu^\pm = g_\nu^{\tau^\pm(\nu)}$ on $V_\delta^\pm$. Moreover from (3.1) we have that

$$\tau^+(\nu) = -\tau^-(-\nu). \tag{3.2}$$

There exists $T_i(\nu)$, $i=0,1$, defined respectively on $V_{\delta,i}^L$ such that

$$T_0(\nu)-T_1(\nu)=\tau^{\pm}(\nu) \qquad \nu\in V_\delta^{\pm}. \tag{3.3}$$

Let

$$\begin{cases}\overline{T}_0(\nu)=\frac{1}{2}(T_0(\nu)+T_1(-\nu))\\ \overline{T}_1(\nu)=\frac{1}{2}(T_0(-\nu)+T_1(\nu)).\end{cases} \tag{3.4}$$

Then $\overline{T}_0(-\nu)=\overline{T}_1(\nu)$ and

$$\overline{T}_0(\nu)-\overline{T}_1(\nu)=\tau^{\pm}(\nu) \qquad \text{for} \quad \nu\in V_\delta^{\pm}. \tag{3.5}$$

We replace the conjugating diffeomorphisms $h_{i,\nu}$ by

$$\hat{h}_{i,\nu}=h_{i,\nu}\circ g_\nu^{-\overline{T}_i(\nu)}. \tag{3.6}$$

Then the two $\hat{h}_{i,\nu}$ coincide on $V_\delta^{\pm}$ and moreover satisfy $\hat{h}_{i,\nu}=\hat{h}_{i,-\nu}$, yielding a conjugacy $\hat{h}_\epsilon$ depending analytically on ϵ. □

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References

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