The modulus of unfoldings of cusps in conformal geometry

C. Rousseau*, DMS and CRM, Université de Montréal

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Abstract

In this paper we give the complete classification of generic 1-parameter unfoldings of germs of real analytic curves with a cuspidal point under conformal equivalence. A cusp is obtained by squaring an analytic curve having contact of order 1 with a line through the origin. We show that this point of view can be extended to the unfolding. This allows to reduce the classification of unfoldings of cusps to the classification of unfoldings of a pair of curves having a contact of order 1 at the origin, one being obtained from the other through a reflection with respect to the origin. This unfolding can be studied in the same way as an unfolding of a curvilinear angle with zero angle, called a horn. We then classify the unfoldings of the special horns corresponding to cusps by means of the associated diffeomorphisms. We interpret the results geometrically.

1 Introduction

In dynamical systems, singularities are studied through normal forms. It is known that in analytic dynamics the change of coordinates to normal form may diverge, even for several of the simplest singularities. Then, there are many equivalence classes of singularities with a given normal form, classified by the moduli space. Quite often, the moduli space is infinite dimensional. This paper is part of a large program to understand the conditions under which two analytic objects are equivalent, and to explain why the moduli spaces are so large. This has been performed to study the unfolding of a parabolic point of a germ of diffeomorphism ([9] and [3]). Indeed, a

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parabolic point is a double fixed point of the diffeomorphism. If we unfold the diffeomorphism, we may have two hyperbolic fixed points. The generic case is when the normalizations do not match up to the limit of confluence. This original idea of Martinet [8] was pushed in more details by Glutsyuk [5]. The works [9] and [3] complete the study for parameter values where the fixed points are not hyperbolic, or when the normalization domains do not intersect.

In this paper we perform the same program with the cusp singularity. We are interested to classify the germs of analytic parameterized curves with a cusp singularity under conformal equivalence. A cusp singularity is a real analytic parameterized curve $\Gamma(t)$ of the form

$$\Gamma(t) = (t^2, t^3 + o(t^3)),$$

up to a conformal transformation. It is common in the literature to consider $\Gamma(t)$ as the square of a curve $\gamma(t)$ with a contact of order 1 with the real axis through the origin. Of course, $\Gamma(t)$ is also the square of $-\gamma(t)$ and the two curves $\gamma(t)$ and $-\gamma(t)$ have a contact of order 1 at the origin. They form a curvilinear angle called a *horn*. Let Σ (resp. Σ') be the Schwarz reflection generated by $\gamma(t)$ (resp. $-\gamma(t)$). Both Σ and Σ' are anti-holomorphic involutions. Their composition $f = \Sigma \circ \Sigma'$ is called the diffeomorphism associated to the curvilinear angle. This diffeomorphism is reversible under the symmetry R_2 with respect to the origin ($R_2(z) = -z$) and under Σ and Σ', namely

$$\begin{cases} f \circ R_2 = R_2 \circ f^{-1}, \\ f \circ \Sigma = \Sigma \circ f^{-1}, f \circ \Sigma' = \Sigma' \circ f^{-1}. \end{cases}$$

It is shown in [10] and [1] that classifying cusps up to conformal equivalence is equivalent to classifying germs of diffeomorphisms f under conjugacy preserving the reversibility properties.

We deal with three ways of considering the problem. In Section 3 we study the cusp curve as a parameterized curve $\Gamma(t)$ in $\mathbb{C}$-space and the curve $\Gamma(t)$ as the square of a curve $\gamma(t)$ with a contact of order 1 with a line through the origin. In Section 4 we study the diffeomorphism f associated to the horn formed by the curves $\gamma(t)$ and $-\gamma(t)$. In each case, we consider the unfolding and we give a prenormal form for this unfolding that is adequate for that particular point of view. In Section 5 we derive the complete modulus space for the germs of generic unfoldings of cusps singularities through the modulus of their associated diffeomorphisms. Section 6 contains a geometric interpretation of the results. For that part, we use that, on adequate domains, there exist unique vector fields of which the diffeomorphism f_ϵ is the time-1 map and we interpret the modulus on the geometry of these flow lines.

2 Preliminaries

2.1 Notation

The following notation is used through the whole paper

- $\sigma(z) = \overline{z}$ is the complex conjugation in $\mathbb{C}$.
- $R_2(z) = -z$ is the rotation of order 2.
- $\tau(z) = -\overline{z} = \sigma \circ R_2$ is the reflection with respect to the imaginary axis in $\mathbb{C}$.
- $S(z) = z^2$.
- $T_C(z) = z + C$ is the translation by C.
- $\sigma_A = T_A \circ \sigma$.
- $\tau_B = T_B \circ \tau = T_B \circ \sigma \circ R_2$
- $\alpha = \frac{\pi}{\sqrt{\hat{\epsilon}}}$, where $\hat{\epsilon}$ belongs to the universal covering of ϵ-space punctured at 0.

2.2 The diffeomorphism associated to a cusp and adequate neighborhood of a cusp point

We consider a germ of analytic parameterized curve Γ_0 in Z-space with a cusp at the origin. Such a parametrization has the form

$$\Gamma_0(t) = t^2 + i(t^3 + o(t^3)),$$

and is hence the square of a smooth parameterized analytic curve γ_0 in z-space, where

$$\gamma_0(t) = t + o(t) + i(\frac{t^2}{2} + o(t^2)).$$

It is of course also the square of the curve $-\gamma_0(t)$. In a neighborhood of the origin $U = \mathbb{D}_r$, the curve γ_0 (resp. $-\gamma_0$) can be sent to the real axis by a conformal diffeomorphism h. On $h(U)$ we have the symmetry $\sigma(z) = \overline{z}$.

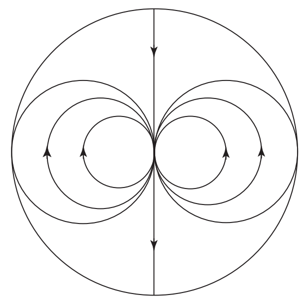


Figure 1: The two petals on the boundary of U.

Hence, this allows to define the Schwarz reflection, $\Sigma = h^{-1} \circ \sigma \circ h$, with respect to the curve γ_0.

Then $\Sigma' = R_2 \circ \Sigma \circ R_2$ is the Schwarz reflection with respect to the curve $-\gamma_0$. We consider the diffeomorphism

$$f_0 = \Sigma \circ \Sigma'.$$

Since γ_0 and $-\gamma_0$ are tangent to the real axis at the origin, then f_0 is a holomorphic diffeomorphism, with a parabolic point at the origin of codimension 1, (i.e. a double fixed point with multiplier equal to 1). We call f_0 the *diffeomorphism associated to the cusp*.

We will suppose that U is sufficiently small so that

- 0 is the unique fixed point of f_0 inside U;
- U is symmetric under R_2;
- U is contained in the domain of the conformal diffeomorphism h defined above;
- the behavior of f_0 on the boundary of U is given by two petals (Figure 1).

Such a neighborhood is called an *adequate* neighborhood for γ_0. (Note that U is also adequate for $-\gamma_0$.) Later, we will consider analytic perturbations γ_ϵ of γ_0. For the whole paper we choose U whose radius is not maximal. We then restrict to sufficiently small values of ϵ ($|\epsilon| < \rho$), so that the two fixed points of the unfolding f_ϵ of f_0 belong to U, when $\epsilon \in \mathbb{D}_\rho$.

Let $S(z) = z^2$. The image $U' = S(U) = B(0, r^2)$ is called an adequate neighborhood for the cusp point of Γ_0. We use $Z = \Gamma_0(t)$ and $z = \gamma_0(t)$.

In the z-plane, we consider the reflections Σ (resp. Σ') with respect to γ_0 (resp. $-\gamma_0$) and the diffeomorphism $f_0 = \Sigma \circ \Sigma'$. If we now switch to the Z-plane by means of $z \mapsto Z = z^2$ then we get a map F_0, which is ramified. So, it is only a diffeomorphism on the 2-covering of Z-plane punctured at the origin. What is the meaning of this map F_0? We may define F_0 as the composition of the Schwarz reflections with respect to the two branches of the cusp. This is perfectly well defined far from the cusp point, and then we use analytic extension. The ramified character comes from the fact that one branch is an extension of the other.

A complete modulus for the cusp will be nothing else than the Ecalle-Voronin modulus of the diffeomorphism f_0 which has a parabolic point at the origin [1]. But, of course, not all diffeomorphisms f_0 with a parabolic curve are diffeomorphisms associated to a cusp. Indeed, the diffeomorphism f_0 satisfies the two additional properties:

$$\begin{cases} R_2 \circ f_0 = (f_0)^{-1} \circ R_2, \\ \Sigma \circ f_0 = (f_0)^{-1} \circ \Sigma. \end{cases}$$

This implies in particular that the formal invariant a of f_0 vanishes. Hence, the formal normal form of $f_0 = \Sigma \circ \Sigma'$ is simply $f_0(z) = \frac{z}{1-iz}$, where Σ (resp Σ') is the reflection with respect to the circle $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$ (resp. $x^2 + (y + \frac{1}{2})^2 = \frac{1}{4}$).

This description extends to the unfolding in a straightforward manner.

2.3 The modulus of conformal classification of a cusp

Let us define for any $A \in \mathbb{C}$,

$$T_A(W) = W + A,$$

$$\tau_A = T_A \circ R_2 \circ \sigma.$$

It has been shown by Nakai [10] and Ahern and Gong [1] that the modulus of conformal classification of a cusp is the Ecalle-Voronin modulus of the associated diffeomorphism f_0. In general, the Ecalle-Voronin modulus of a diffeomorphism with a double fixed point is given by an equivalence class of pairs of diffeomorphisms $[\Psi_0^0, \Psi_0^\infty]/\sim$, where

- Ψ_0^0 (resp. Ψ_0^∞) is defined in a region $\text{Im}\, W < -Y_0$ (resp. $\text{Im}\, W > Y_0$) for some $Y_0 > 0$,
- Ψ_0^0 and Ψ_∞^∞ commute with T_1,

and the equivalence relation is defined as follows

$$(\Psi_0^0, \Psi_0^\infty) \sim (\breve{\Psi}_0^0, \breve{\Psi}_0^\infty) \Longleftrightarrow \exists B, B' \in \mathbb{C} \begin{cases} \Psi_0^0 = T_B \circ \breve{\Psi}_0^0 \circ T_{B'}, \\ \Psi_0^\infty = T_B \circ \breve{\Psi}_0^\infty \circ T_{B'}. \end{cases}$$

In the case of a diffeomorphism associated to a cusp, it is possible to choose a canonical representative of the modulus and eliminate the equivalence classes. This canonical representative satisfies

- $\Psi_0^0 = R_2 \circ (\Psi_0^\infty)^{-1} \circ R_2$, where $R_2(W) = -W$;
- $\begin{cases} \Psi_0^0 = \tau_{\frac{1}{2}} \circ (\Psi_0^0)^{-1} \circ \tau_{\frac{1}{2}}, \\ \Psi_0^\infty = \tau_{\frac{1}{2}} \circ (\Psi_0^\infty)^{-1} \circ \tau_{\frac{1}{2}}. \end{cases}$

Moreover, any pair of germs of diffeomorphisms satisfying these conditions is realizable as the modulus of a diffeomorphism associated to a cusp. (A proof of these results will be included in our study of the unfolding below.)

We will show that the modulus of a family of germs f_ϵ unfolding f_0 is simply an unfolding of the modulus $(\Psi_\epsilon^0, \Psi_\epsilon^\infty)$ satisfying similar conditions as above.

2.4 The model family for the unfolding of a cusp

We have seen that for $\epsilon = 0$ the model family is given by the two circles $x^2+(y-2)^2 = 4$ and $x^2+(y+2)^2 = 4$ and that $f_0 = \Sigma\circ\Sigma'$ is simply $f_0(z) = \frac{z}{1-iz}$. In practice, $f_0(z)$ the time-one map of the vector field $v_0(z) = iz^2\frac{\partial}{\partial z}$. A natural model for the unfolding $f_\epsilon(z)$ is simply the time-one map of the vector field $v_\epsilon(z) = i(z^2-\epsilon)\frac{\partial}{\partial z}$, which yields

$$f_\epsilon(z) = \frac{\sqrt{\epsilon}z\left(1+e^{2i\sqrt{\epsilon}}\right)+\epsilon\left(1-e^{2i\sqrt{\epsilon}}\right)}{z\left(1-e^{2i\sqrt{\epsilon}}\right)+\sqrt{\epsilon}\left(1+e^{2i\sqrt{\epsilon}}\right)}.$$

(f_ϵ is invariant under $\sqrt{\epsilon}\mapsto-\sqrt{\epsilon}$ and hence an analytic function of ϵ.) The map f_ϵ is the composition $\Sigma_\epsilon\circ\Sigma'_\epsilon$, where Σ_ϵ (resp. Σ'_ϵ) is the reflection with respect to the circle $x^2+(y-h)^2 = \epsilon+h^2$ (resp. $x^2+(y+h)^2 = \epsilon+h^2$), where $h = h(\epsilon) = \frac{\sqrt{\epsilon}}{\tan\frac{\sqrt{\epsilon}}{2}}$. The two circles intersect in $\pm\sqrt{\epsilon}$ and we have that $f'_\epsilon(\pm\sqrt{\epsilon}) = \exp(\pm 2i\sqrt{\epsilon})$.

3 Reduction to the study of the unfolding of an analytic curve

In this section we consider normal forms (under conformal equivalence) for a generic unfolding $\Gamma_\epsilon(t)$ of a curve $\Gamma_0(t)$ of cusp type and we show that $\Gamma_\epsilon(t)$ is of the form $\gamma_\epsilon^2(t)$ for some unfolding $\gamma_\epsilon(t)$ of γ_0. We also study the converse direction, thus showing that to classify germs of generic families unfolding a cusp, it suffices to classify germs of generic families unfolding an analytic arc having a contact point of order 1 with a line through the origin under conformal equivalence commuting with R_2.

3.1 Generic unfolding of a curve of cusp type

Proposition 3.1 *We consider a germ of real analytic curve of cusp type of the form $\Gamma_0(t) = t^2 + i(t^3 + o(t^3))$, defined for t in a small neighborhood of the origin in $\mathbb{R}$, and an unfolding $\Gamma_\epsilon(t)$. Modulo affine change of coordinate in $Z = x + iy$ and in t, we can bring the unfolding to the simple form*

$$(x(t), y(t)) = (t^2 + o(t^2), t^3 + \eta(\epsilon)t + o(t^3)). \tag{3.1}$$

Proof. We consider the image of Γ_0 in the Z-plane. We can of course enlarge the problem, and consider t in a small neighborhood of the origin in $\mathbb{C}$. We unfold $\Gamma_0(t)$ in a 1-parameter family $\Gamma_\epsilon(t)$. The family has the form:

$$\begin{aligned}(x(t), y(t)) = (&t^2 + a(\epsilon)t + b(\epsilon) + t^3 f(t, \epsilon),\\ &t^3 + c(\epsilon)t^2 + d(\epsilon)t + e(\epsilon) + t^4 g(t, \epsilon)),\end{aligned}$$

where $a(0) = b(0) = c(0) = d(0) = e(0) = 0$.

We consider a change of coordinates $(x_1, y_1) = (x - \delta_1 y + \delta_2, \delta_1 x + y + \delta_3)$ in the (x, y)-plane, together with a change $t = s + \delta_4$. We then have $(x_1(t), y_1(t)) = (A_0(\epsilon) + A_1(\epsilon)s + A_2(\epsilon)s^2 h(s, \epsilon), \sum_{n=0}^{\infty} B_n(\epsilon)s^n)$, with $h(0, 0) = 1$. We solve the system $A_0(\epsilon) = A_1(\epsilon) = B_0(\epsilon) = B_2(\epsilon) = 0$ in the neighborhood of $\epsilon = 0$. The Jacobian matrix with respect to $(\delta_1, \delta_2, \delta_3, \delta_4)$ at $\epsilon = 0$ is given by the invertible matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}.$$

Hence, the system has a unique solution such that $\delta_j(0) = 0$ by the implicit function theorem, yielding (3.1). □

Definition 3.2 *The unfolding Γ_ϵ of the form* (3.1) *is generic if $\eta'(0) \neq 0$.*

This allows to reparameterize so that $\eta(\epsilon) = -\epsilon$, and we consider the family

$$\Gamma_\epsilon(t) = (x(t), y(t)) = (t^2 + o(t^2), t^3 - \epsilon t + o(t^3)). \tag{3.2}$$

As a complex function of t, $\Gamma_0(t) = x(t) + iy(t)$ has a critical point at $t = 0$. Hence $\Gamma_0'(0) = 0$ and $\Gamma_0''(0) \neq 0$. By the implicit function theorem, Γ_ϵ has a unique critical point for $t = \xi(\epsilon) = \frac{i\epsilon}{2} + o(\epsilon)$, with $\xi(0) = 0$, and Γ_ϵ is $2:1$ in a neighborhood of $\xi(\epsilon)$.

We change to the coordinate $Z \mapsto Z_1 = Z - \Gamma_\epsilon(\xi(\epsilon))$ and we let $\Gamma_{1,\epsilon}(t) = \Gamma_\epsilon(t) - \Gamma_\epsilon(\xi(\epsilon))$.

Lemma 3.3 *There exists $r, \rho > 0$, and an analytic function $\gamma_\epsilon(t)$ such that γ_ϵ is defined for $t \in \mathbb{D}_r$ for all $\epsilon \in \mathbb{D}_\rho$, and*

$$\Gamma_{1,\epsilon}(t) = \gamma_\epsilon^2(t) = (-\gamma_\epsilon(t))^2.$$

Moreover, $\gamma_\epsilon(\xi(\epsilon)) = 0$.

PROOF. From the hypothesis, we have $\Gamma_{1,\epsilon}(t) = (t - \xi(\epsilon))^2(1 + O(t - \xi(\epsilon)))$ from which the result follows. $\square$

If we restrict t to real values, we hence obtain two germs of analytic curves $\gamma_\epsilon(t)$ and $-\gamma_\epsilon(t)$, which are considered to be sitting in z-plane, and whose images under $z \mapsto S(z) = z^2 = Z_1$ is the curve $\Gamma_{1,\epsilon}$. At $\epsilon = 0$, the two curves have a contact point at the origin of order 1. So, for $\epsilon \neq 0$ real, either they do not intersect, or they intersect transversally in two points z_0 and $-z_0$. If so, z_0 and $-z_0$ are sent in Z-plane to a double point of $\Gamma_{1,\epsilon}$. From the form of $\Gamma_{1,\epsilon}$, we see that we have a double point for the real curve when $\epsilon > 0$, and a smooth real curve when $\epsilon < 0$.

3.2 Generic unfolding of a curve having a contact of order 1 with a line through the origin

Conversely, we want to show that all generic unfoldings γ_ϵ of a curve tangent to a line passing through the origin lead, by squaring, to the unfolding of a cusp.

Proposition 3.4 *We consider a regular curve γ in the z-plane tangent to the x-axis. This curve can be written $z(t) = x(t) + iy(t)$ with*

$$(x(t), y(t)) = (t, t^2 + t^3 g(t)).$$

Any 1*-parameter unfolding can be brought to the form*

$$(x(t), y(t)) = (t, t^2 + \eta_0(\epsilon) + t^3 g(t,\epsilon)), \tag{3.3}$$

under linear transformations in z *(keeping* $z = 0$ *fixed) and reparameterization in* t*.*

Proof. A general unfolding has the form

$$(x(t), y(t)) = (t + \eta_0(\epsilon) + t^2 f(t,\epsilon), t^2 + \eta_1(\epsilon)t + \eta_2(\epsilon) + t^3 g(t,\epsilon)),$$

where $f(t, 0) \equiv 0$.

To simplify, we just treat η_j as independent parameters.

We apply simultaneously a transformation $(x_1, y_1) = (x - \delta_1 y, \delta_1 x + y)$ and a transformation moving t to t_1:

$$t = t_1 k(t_1, \epsilon) + \delta_2,$$

with $k(t_1, \epsilon) = 1 + O(t_1) + O(\epsilon)$.

Note that the functions

$$g_1(s, \epsilon, \delta_2) = \frac{(s + \delta_2)^2 g(s + \delta_2, \epsilon) - \delta_2^2 g(\delta_2)}{s},$$

and

$$f_1(s, \epsilon, \delta_2) = \frac{(s + \delta_2)^2 f(s + \delta_2) - \delta_2^2 f(\delta_2)}{s}$$

are analytic and $f_1(s, \delta_2) = O(|(s, \delta_2)|^2)$.

The transformed curve has the form $(x_1(t), y_1(t))$ with

$$\begin{aligned} x_1(t_1) = {} & [\eta_0 + \delta_2 + \delta_2^2 f(\delta_2) - \delta_1\delta_2^2 - \delta_1\eta_1\delta_2 - \delta_1\delta_2^2 g(\delta_2)] \\ & + \left[t_1 k(1 - \delta_1\eta_1 - 2\delta_1\delta_2) + (\delta_2 + t_1 k)^2 f(\delta_2 + t_1 k) - \delta_2^2 f(\delta_2)\right. \\ & \left. -\delta_1 t_1^2 k^2 - \delta_1 \left((t_1 k + \delta_2)^2 g(t_1 k + \delta_2) - \delta_2^2 g(\delta_2)\right)\right], \\ y_1(t_1) = {} & y_1(0) + \left[\eta_1 + \delta_1 + 2\delta_2 + 2\delta_2 g(\delta_2) + \delta_2^2 g'(\delta_2)\right] t_1 \\ & + t_1^2(1 + O(|(\epsilon, \delta)|)) + O(t_1^3). \end{aligned} \tag{3.4}$$

Note that

$$\begin{aligned} & t_1 k(1 - \delta_1\eta_1 - 2\delta_1\delta_2) + [\delta_2 + t_1 k)^2 f(\delta_2 + t_1 k) - \delta_2^2 f(\delta_2)] \\ & \qquad - \delta_1 t_1^2 k^2 - \delta_1 \left[(t_1 k + \delta_2)^2 g(t_1 k + \delta_2) - \delta_2^2 g(\delta_2)\right] \\ & = t_1 \left[k(1 - \delta_1\eta_1 - 2\delta_1\delta_2) + f_1(t_1 k, \delta_2) - \delta_1 t_1 k^2 - \delta_1 g_1(t_1 k)\right]. \end{aligned} \tag{3.5}$$

We solve by the implicit function theorem the two equations

$$\begin{aligned} &\eta_0 + \delta_2 + \delta_2^2 f(\delta_2) - \delta_1\delta_2^2 - \delta_1\eta_1\delta_2 - \delta_1\delta_2^2 g(\delta_2) = 0, \\ &\eta_1 + \delta_1 + 2\delta_2 + 2\delta_2 g(\delta_2) + \delta_2^2 g_1'(\delta_2) = 0, \end{aligned} \tag{3.6}$$

with respect to δ_1, δ_2. Since the Jacobian does not vanish, this system has an analytic solution with δ_1, δ_2 depending analytically on ϵ.

We substitute in the equation

$$k(1 - \delta_1\eta_1 - 2\delta_1\delta_2) + f_1(t_1 k, \delta_2) - \delta_1 t_1 k^2 - \delta_1 g_2(t_1 k) = 1, \tag{3.7}$$

which we can solve for k, near $k = 1$, as a function of (t_1, ϵ).

Hence, we end up with an unfolding of the curve in the form (we forget the indices)

$$(x(t), y(t)) = (t, t^2 + \eta_0(\epsilon) + t^3 g(t, \epsilon)). \tag{3.8}$$

□

Definition 3.5 *The unfolding γ_ϵ of γ in* (3.3) *is generic if $\eta_0'(0) \neq 0$.*

This allows to reparameterize $\eta_0(\epsilon) = -\epsilon$.

Corollary 3.6 *The square of any generic 1-parameter unfolding of the form*

$$(x(t), y(t)) = (t, t^2 - \epsilon + t^3 g(t, \epsilon)) \tag{3.9}$$

is, up to a translation, a generic unfolding of a cusp of the form (3.2).

4 The associated diffeomorphism

We consider a generic γ_ϵ. Let Σ_ϵ and Σ_ϵ' be the Schwarz reflections with respect to γ_ϵ and $-\gamma_\epsilon$ and let R_2 be the rotation of order 2. Of course,

$$\Sigma_\epsilon' = R_2 \circ \Sigma_\epsilon \circ R_2.$$

We consider the analytic diffeomorphism

$$f_\epsilon = \Sigma_\epsilon \circ \Sigma_\epsilon'.$$

Then f is reversible with respect to R_2, namely

$$R_2 \circ f_\epsilon \circ R_2 = (f_\epsilon)^{-1}. \tag{4.1}$$

The diffeomorphism f_0 has a parabolic fixed point at the origin for $\epsilon = 0$. For $\epsilon \neq 0$, it has two fixed points z_0 and $-z_0$. From Proposition 3.4, we suppose that $\gamma_\epsilon(t)$ has the form (3.9). Then z_0 is $\gamma_\epsilon(t_0)$ where t_0 is solution of $\gamma_\epsilon(t) = -\gamma_\epsilon(t)$, namely

$$2(t^2 - \epsilon) + t^3 g(t, \epsilon)) - t^3 g(-t, \epsilon) = 0 = (t^2 - \eta(\epsilon))h(t, \epsilon),$$

the last equality following from Weierstrass preparation theorem. Since $\eta'(0) \neq 0$, we can of course make the change of parameter $\epsilon \mapsto \eta$ (note that this changes somewhat the expression of γ_ϵ).

When $\gamma_\epsilon(t)$ has this form then, modulo a scaling in z

$$f_\epsilon(z) = z + (z^2 - \epsilon)(A(\epsilon) + O(z)).$$

Because of (4.1), then the formal invariant of f is zero. This means that $f_\epsilon'(\sqrt{\epsilon}) = (f_\epsilon'(-\sqrt{\epsilon}))^{-1}$. We can make a final analytic change

$$(z, \epsilon) \mapsto (\mu(\epsilon)z, \epsilon\mu^2(\epsilon)),$$

linear in z, with $\mu(\epsilon)$ real for real ϵ, so as to achieve

$$f_\epsilon'(\sqrt{\epsilon}) = (f_\epsilon'(-\sqrt{\epsilon}))^{-1} = \exp(2i\sqrt{\epsilon}). \tag{4.2}$$

In this final form, the parameter is now canonical and an analytic invariant. Again, this has changed somewhat the expression of γ_ϵ. The corresponding form of f_ϵ is now

$$f_\epsilon(z) = z + (z^2 - \epsilon)[i + B(\epsilon) + (z^2 - \epsilon)k(z, \epsilon)]. \tag{4.3}$$

Definition 4.1 *A germ of generic family of diffeomorphisms of the form* (4.3)*, and satisfying* (4.1) *and* (4.2)*, is called* prepared*.*

Proposition 4.2 *We consider two germs of families of curves $\gamma_{j,\epsilon}$, $j = 1, 2$, as above, both parameterized by their canonical parameter ϵ. They generate Schwarz reflections $\Sigma_{j,\epsilon}$. Then the germs of families $\gamma_{1,\epsilon}$ and $\gamma_{2,\epsilon}$ are conformally equivalent under a germ of family of analytic diffeomorphisms h_ϵ commuting with R_2 if and only if their associated diffeomorphisms $f_{j,\epsilon}$ are conjugate by a germ of family of diffeomorphisms h_ϵ satisfying*

$$\begin{cases} h_\epsilon \circ f_{1,\epsilon} = f_{2,\epsilon} \circ h_\epsilon, \\ h_\epsilon \circ \Sigma_{1,\epsilon} = \Sigma_{2,\epsilon} \circ h_\epsilon, \\ h_\epsilon \circ R_2 = R_2 \circ h_\epsilon. \end{cases} \tag{4.4}$$

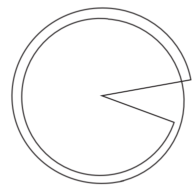


Figure 2: The sectorial domain V.

PROOF. Let h_ϵ be a germ of family of conformal diffeomorphisms such that $h_\epsilon(\gamma_{1,\epsilon}) = \gamma_{2,\epsilon}$ and $R_2 \circ h_\epsilon = h_\epsilon \circ R_2$. Then, of course $\Sigma_{2,\epsilon} = h_\epsilon \circ \Sigma_{1,\epsilon} \circ h_\epsilon^{-1}$. Also, if $\Sigma'_{j,\epsilon} = R_2 \circ \Sigma_{j,\epsilon} \circ R_2$, $j = 1, 2$, then $\Sigma'_{2,\epsilon} = h_\epsilon \circ \Sigma'_{1,\epsilon} \circ h_\epsilon^{-1}$. Hence,

$$f_{2,\epsilon} = \Sigma_{2,\epsilon} \circ \Sigma'_{2,\epsilon} = h_\epsilon \circ f_{1,\epsilon} \circ h_\epsilon^{-1}.$$

Conversely, we suppose that (4.4) is satisfied. Then, h_ϵ is a conformal equivalence sending $\gamma_{1,\epsilon}$ to $\gamma_{2,\epsilon}$ and commuting with R_2. □

By Proposition 4.2, it suffices to classify the germs of prepared families of diffeomorphisms f_ϵ of the form $f_\epsilon = \Sigma_\epsilon \circ \Sigma'_\epsilon$, where Σ_ϵ is a family of Schwarz reflections, and $\Sigma'_\epsilon = R_2 \circ \Sigma_\epsilon \circ R_2$. Such diffeomorphisms of course satisfy $f_\epsilon \circ R_2 = R_2 \circ f_\epsilon^{-1}$ and their formal invariant vanishes identically.

5 The modulus of conformal classification

A complete modulus of analytic classification for a germ of generic family unfolding a parabolic diffeomorphism has been given in [9] and is called the *Lavaurs modulus*. This modulus is an unfolding of the Ecalle-Voronin modulus. It is an equivalence class of pairs of diffeomorphisms $(\Psi^0_{\hat\epsilon}, \Psi^\infty_{\hat\epsilon})$ defined for $\hat\epsilon$ in a sector V_δ in the universal covering of ϵ-space, punctured at 0, of the form (see Figure 2):

$$V_\delta = \{\hat\epsilon : |\hat\epsilon| < \rho, \arg(\hat\epsilon) \in (-\pi - \delta, \pi + \delta)\}, \tag{5.1}$$

for some $\delta \in (0, \pi)$. Here, we are essentially interested in real values of ϵ. The problem with the Lavaurs modulus for ϵ negative is that it provides two different moduli and it is not well adapted to the "real character" of the problem. So, when $\epsilon < 0$, we will provide another modulus, the Glutsyuk modulus, which will also be used to identify the realizable moduli.

The underlying idea of the Lavaurs modulus is to conjugate f_ϵ with the model family on sectorial domains with vertices at the fixed points, and to compare the conjugacies on the intersection of the sectorial domains. In the Glutsyuk modulus, we rather conjugate f_ϵ with the model family in the neighborhood of each fixed point and we compare the conjugacies in the intersection of the two neighborhoods.

5.1 The Lavaurs modulus

As described in [9], the modulus of a family f_ϵ is the unfolding of the Ecalle-Voronin modulus. Since we need to derive its special property, we will construct it.

We suppose that for all $\epsilon \in \mathbb{D}_\rho$, the map f_ϵ is defined on $\mathbb{D}_r$.

We move to coordinates

$$W = p_\epsilon^{-1}(z) = \begin{cases} \frac{1}{2i\sqrt{\epsilon}} \ln \frac{z-\sqrt{\epsilon}}{z+\sqrt{\epsilon}}, & \epsilon \neq 0, \\ \frac{i}{z}, & \epsilon = 0. \end{cases} \tag{5.2}$$

This multi-valued map has period T_α, where

$$\alpha = \frac{\pi}{\sqrt{\hat{\epsilon}}}. \tag{5.3}$$

For $\epsilon = 0$, the image of the domain $U = \mathbb{D}_r$ is the exterior of a disk. The lift of f_ϵ in W-coordinate is F_ϵ, which commutes with T_α, and it is known that F_ϵ is a small perturbation of T_1 (see for instance [9]). In this new coordinate W, the model family is now T_1. We want to construct changes of coordinates conjugating F_ϵ to the model family T_1. Such changes of coordinates are called Fatou coordinates and will be denoted Φ. Usually they cannot be globally defined. They are defined on *translation domains* in W-space (Figure 3).

Definition 5.1 *1. An* admissible line *in W-space is a line ℓ so that ℓ and $F_\epsilon(\ell)$ are disjoint, and such that the strip $S_\epsilon(\ell)$ between ℓ and $F_\epsilon(\ell)$ is included in $p_\epsilon^{-1}(\mathbb{D}_r)$.*

2. The translation domain *$Q_\epsilon(\ell)$ associated with an admissible line is the saturation of the strip $S_\epsilon(\ell)$ under F_ϵ (Figure 3):*

$$\begin{aligned} Q_\epsilon(\ell) = \{W \in p_\epsilon^{-1}(\mathbb{D}_r) \mid \exists n \in \mathbb{Z}, F_\epsilon^{\circ n}(W) \in S_\epsilon(\ell) \\ \text{and} \quad \forall j \in [0,n] \subset \mathbb{Z}, F_\epsilon^{\circ j}(W) \in p_\epsilon^{-1}(\mathbb{D}_r)\}. \end{aligned}$$

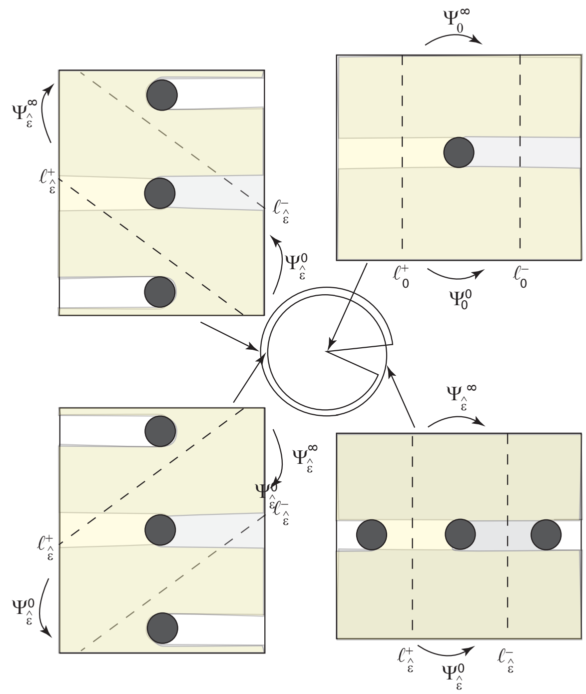


Figure 3: Translation domains of Fatou coordinates in Lavaurs point of view.

The original diffeomorphism f_ϵ is reversible with respect to R_2. Since R_2 has two centers of symmetry, 0 and ∞, we get that F_0, the lift of f_0 in W-coordinate, is also reversible with respect to R_2 in W-space. The lift of R_2 when $\epsilon \neq 0$ is a little more subtle. Note that 0 is sent to the points $\frac{(2k+1)\pi}{2\sqrt{\hat{\epsilon}}}$, for $k \in \mathbb{Z}$. Then R_2 is lifted to a multi-valued map corresponding to a rotation of order 2 around any of these points. Let us call R_α (resp. $R_{-\alpha}$) the rotation of order 2 around $\frac{\pi}{2\sqrt{\hat{\epsilon}}}$ (resp. $-\frac{\pi}{2\sqrt{\hat{\epsilon}}}$). Then

$$\begin{cases} R_\alpha = T_{\frac{\alpha}{2}} \circ R_2 \circ T_{-\frac{\alpha}{2}} = T_\alpha \circ R_2, \\ R_{-\alpha} = T_{-\frac{\alpha}{2}} \circ R_2 \circ T_{\frac{\alpha}{2}} = T_{-\alpha} \circ R_2. \end{cases}$$

We then have

$$R_\alpha \circ F_\epsilon = (F_\epsilon)^{-1} \circ R_\alpha.$$

But, since F_ϵ commutes with T_α we finally also get that

$$R_2 \circ F_\epsilon = (F_\epsilon)^{-1} \circ R_2.$$

The construction of a Fatou coordinate, Φ, defined on a translation domain with image in $\mathbb{C}$ and satisfying $\Phi \circ F_\epsilon = T_1 \circ \Phi$ is quite standard (see for instance [9]). Moreover, a Fatou ccordinate is unique on a translation domain, up to left composition with translations. We consider a translation domain $Q_{\hat{\epsilon}}^+$ (resp. $Q_{\hat{\epsilon}}^-$) associated to a line $\ell_{\hat{\epsilon}}^+$ (resp. $\ell_{\hat{\epsilon}}^-$) located on the left (resp. right) of the principal hole, and let $\Phi_{\hat{\epsilon}}^\pm$ be a Fatou coordinate on $Q_{\hat{\epsilon}}^\pm$.

Definition 5.2 *Considering two translations domains $Q_{\hat{\epsilon}}^\pm$ associated to lines transversal to the line of holes and located on both sides of the principal hole and Fatou coordinates $\Phi_{\hat{\epsilon}}^\pm$ on $Q_{\hat{\epsilon}}^\pm$, the Lavaurs modulus is defined as*

$$\Psi_{\hat{\epsilon}} = \Phi_{\hat{\epsilon}}^- \circ (\Phi_{\hat{\epsilon}}^+)^{-1}.$$

It has two components $\Psi_{\hat{\epsilon}}^0$ (resp. $\Psi_{\hat{\epsilon}}^\infty$) on the two connected components of its domain of definition containing half-planes $\mathrm{Im(W)} < -\mathrm{Y}_0$ (resp. $\mathrm{Im(W)} > \mathrm{Y}_0$). In the limit $\epsilon = 0$, the Lavaurs modulus is simply the Ecalle-Voronin modulus.

Theorem 5.3 *For $\hat{\epsilon} \in V_\delta$, there exist unique Fatou coordinates $\Phi_{\hat{\epsilon}}^\pm$ on $Q_{\hat{\epsilon}}^\pm$ such that the associated Lavaurs modulus $\Psi_{\hat{\epsilon}}^{0,\infty}$ has the following properties*

- $$\Psi_{\hat{\epsilon}}^0 = R_2 \circ (\Psi_{\hat{\epsilon}}^\infty)^{-1} \circ R_2. \tag{5.4}$$
- *$\Psi_{\hat{\epsilon}}^{0,\infty}$ have zero constant terms in their Fourier series expansions (see* (5.7) *below).*
- *Let $\overline{\hat{\epsilon}}$ be the conjugate of $\hat{\epsilon}$ defined by $\arg(\overline{\hat{\epsilon}}) = -\arg(\hat{\epsilon})$ and*
$$\tau_B = T_B \circ R_2 \circ \sigma. \tag{5.5}$$

  Then,
$$\begin{cases} \Psi_{\hat{\epsilon}}^{0,\infty} = \tau_{\frac{1}{2}} \circ (\Psi_{\overline{\hat{\epsilon}}}^{0,\infty})^{-1} \circ \tau_{\frac{1}{2}}, \\ \Psi_{\hat{\epsilon}}^{0,\infty} = \tau_{-\frac{1}{2}} \circ (\Psi_{\overline{\hat{\epsilon}}}^{0,\infty})^{-1} \circ \tau_{-\frac{1}{2}}. \end{cases} \tag{5.6}$$

PROOF. Because of the reversibility of F_ϵ with respect to R_2, and because of the shape of the domains of the Fatou coordinates, it follows that we can take

$$\Phi_{\hat{\epsilon}}^- = R_2 \circ \Phi_{\hat{\epsilon}}^+ \circ R_2,$$

from which (5.4) follows because R_2 sends the domain of $\Psi_{\hat{\epsilon}}^0$ to that of $\Psi_{\hat{\epsilon}}^\infty$.

Since $\Psi_{\hat{\epsilon}}^{\infty}$ commute with T_1 and from the form of their domain of definition, they can be expanded as Fourier series:

$$\begin{cases}\Psi_{\hat{\epsilon}}^{0} = W + \sum_{n\leq 0} b_n^0(\hat{\epsilon}) \exp(2\pi i n W),\\ \Psi_{\hat{\epsilon}}^{\infty} = W + \sum_{n\geq 0} b_n^{\infty}(\hat{\epsilon}) \exp(2\pi i n W).\end{cases} \tag{5.7}$$

Because of (5.4), we have $b_0^0(\hat{\epsilon}) = b_0^{\infty}(\hat{\epsilon})$. Considering a special choice $\Phi_{\hat{\epsilon}}^{+}$, which we call $\breve{\Phi}_{\hat{\epsilon}}^{+}$ and which is obtained by composing $\Phi_{\hat{\epsilon}}^{+}$ on the left with an appropriate translation, we can manage that $b_0^0(\hat{\epsilon}) = b_0^{\infty}(\hat{\epsilon}) = 0$ while keeping (5.4). This fixes completely the Fatou coordinates which depend analytically on $\hat{\epsilon}$ with continuous limit at $\epsilon = 0$.

Let us now explain the treatment of the reversibility with respect to the lift of Σ_ϵ in W-space, which we call $\Sigma_{1,\epsilon}$. Note that $\Sigma_{1,\epsilon}$ is uniform in ϵ and that we have $\Sigma_{1,\epsilon} \circ \Sigma_{1,\overline{\epsilon}} = id$. We choose $\breve{\Phi}_{\hat{\epsilon}}^{+}$ defined above as one of the Fatou coordinates. Let

$$\tau(W) = -\overline{W} = R_2 \circ \sigma(W).$$

We let

$$\breve{\Phi}_{\hat{\epsilon}}^{-} = \tau \circ \breve{\Phi}_{\overline{\hat{\epsilon}}}^{+} \circ \Sigma_{1,\overline{\epsilon}}.$$

If we define

$$\breve{\Psi}_{\hat{\epsilon}} = \breve{\Phi}_{\hat{\epsilon}}^{-} \circ (\breve{\Phi}_{\hat{\epsilon}}^{+})^{-1},$$

then we have

$$\breve{\Psi}_{\hat{\epsilon}}^{0,\infty} = \tau \circ (\breve{\Psi}_{\overline{\hat{\epsilon}}}^{0,\infty})^{-1} \circ \tau,$$

from which it follows that the constant terms in the expansions of $\breve{\Psi}_{\hat{\epsilon}}^{0}$ and $\breve{\Psi}_{\overline{\hat{\epsilon}}}^{0}$ (resp. $\breve{\Psi}_{\hat{\epsilon}}^{\infty}$ and $\breve{\Psi}_{\overline{\hat{\epsilon}}}^{\infty}$) are conjugate.

Of course, we have that

$$\breve{\Phi}_{\hat{\epsilon}}^{-} = T_{-B(\hat{\epsilon})} \circ \Phi_{\hat{\epsilon}}^{-},$$

for some $B(\hat{\epsilon})$ satisfying $\overline{B(\hat{\epsilon})} = B(\overline{\hat{\epsilon}})$. Hence,

$$\breve{\Psi}_{\hat{\epsilon}}^{0,\infty} = T_{-B(\hat{\epsilon})} \circ \Psi_{\overline{\hat{\epsilon}}}^{0,\infty}.$$

It follows that we have

$$\Psi_{\hat{\epsilon}}^{0,\infty} = \tau_{B(\hat{\epsilon})} \circ (\Psi_{\overline{\hat{\epsilon}}}^{0,\infty})^{-1} \circ \tau_{B(\overline{\hat{\epsilon}})}. \tag{5.8}$$

We also have

$$\Psi_{\hat{\epsilon}}^{0,\infty} = \tau_{-B(\hat{\epsilon})} \circ (\Psi_{\overline{\hat{\epsilon}}}^{0,\infty})^{-1} \circ \tau_{-B(\overline{\hat{\epsilon}})}. \tag{5.9}$$

Indeed,

$$
\begin{aligned}
\Psi^0_{\hat\epsilon} &= R_2 \circ (\Psi^\infty_{\hat\epsilon})^{-1} \circ R_2 \\
&= R_2 \circ \tau_{B(\hat\epsilon)} \circ \Psi^\infty_{\overline{\hat\epsilon}} \circ \tau_{B\left(\overline{\hat\epsilon}\right)} \circ R_2 \\
&= \tau_{-B(\hat\epsilon)} \circ R_2 \circ \Psi^\infty_{\overline{\hat\epsilon}} \circ R_2 \circ \tau_{-B\left(\overline{\hat\epsilon}\right)} \\
&= \tau_{-B(\hat\epsilon)} \circ (\Psi^0_{\overline{\hat\epsilon}})^{-1} \circ \tau_{-B\left(\overline{\hat\epsilon}\right)}.
\end{aligned}
$$

Combining (5.8) with (5.9) yields

$$
\Psi^{0,\infty}_{\hat\epsilon} = T_{2B(\hat\epsilon)} \circ \Psi^{0,\infty}_{\hat\epsilon} \circ T_{-2B(\hat\epsilon)}. \tag{5.10}
$$

Hence, $\Psi^0_{\hat\epsilon}$ and $\Psi^\infty_{\hat\epsilon}$ commute with $T_{2B(\hat\epsilon)}$. In the generic case, where one of $\Psi^0_{\hat\epsilon}$ and $\Psi^\infty_{\hat\epsilon}$ is nonlinear for at least one value of ϵ, then we can conclude that $2B(\hat\epsilon)$ is rational, hence constant in $\hat\epsilon$. Let us now show that $B(\hat\epsilon) \equiv \frac{1}{2}$. Since $B(\hat\epsilon)$ depends analytically on $\hat\epsilon \neq 0$, it suffices to prove it for $\hat\epsilon = \epsilon > 0$. We use that $F = \Sigma_{1,\epsilon} \circ (R_2 \circ \Sigma_{1,\epsilon} \circ R_2)$.

$$
\begin{aligned}
T_1 &= \breve\Phi^+_\epsilon \circ \Sigma_{1,\epsilon} \circ R_2 \circ \Sigma_{1,\epsilon} \circ R_2 \circ (\breve\Phi^+_\epsilon)^{-1} \\
&= \tau \circ \breve\Phi^-_\epsilon \circ R_2 \circ \Sigma_{1,\epsilon} \circ (\Phi^-_\epsilon)^{-1} \circ R_2 \\
&= \tau_{B(\epsilon)} \circ \Phi^-_\epsilon \circ R_2 \circ \Sigma_{1,\epsilon} \circ (\breve\Phi^-_\epsilon)^{-1} \circ T_{-B(\epsilon)} \circ R_2 \\
&= \tau_{B(\epsilon)} \circ \Phi^-_\epsilon \circ R_2 \circ (\breve\Phi^+_\epsilon)^{-1} \circ \tau \circ T_{-B(\epsilon)} \circ R_2 \\
&= \tau_{B(\epsilon)} \circ \Phi^-_\epsilon \circ (\Phi^-_\epsilon)^{-1} \circ R_2 \circ \tau_{B(\epsilon)} \circ R_2 \\
&= \tau_{B(\epsilon)} \circ \tau_{-B(\epsilon)} = T_{2B(\epsilon)},
\end{aligned}
$$

from which $B(\hat\epsilon) \equiv \frac{1}{2}$ follows. (5.6) is of course also true if both Ψ^0_ϵ and $\Psi^\infty_{\hat\epsilon}$ are the identity. □

5.2 The Glutsyuk modulus

In the case $\epsilon < 0$, i.e. when the curves $\pm\gamma_\epsilon$ do not intersect, a second point of view is given by the Glutsyuk modulus. This point of view will be needed to identify the modulus space.

Definition 5.4 *For $\epsilon < 0$, let Φ^l_ϵ (resp. Φ^r_ϵ) be a Fatou coordinate associated to a line parallel to the line of holes on the left (resp. right) of the principal hole (Figure 4). The Glutsyuk invariant is defined as*

$$
\Psi^G_\epsilon = \Phi^r_\epsilon \circ (\Phi^l_\epsilon)^{-1}.
$$

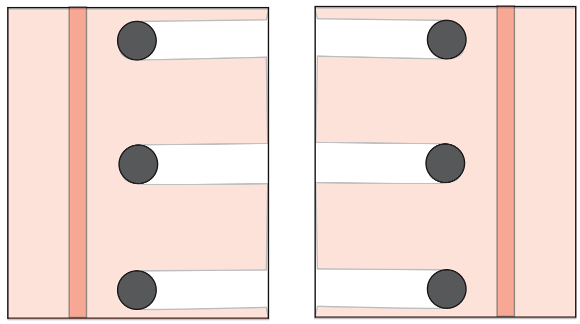


Figure 4: Translation domains of Fatou coordinates in Glutsyuk point of view.

Note that, because the formal invariant is identically zero, the Fatou coordinates $\Phi_\epsilon^{l,r}$ commute with T_α:

$$\Phi_\epsilon^{l,r} \circ T_\alpha = T_\alpha \circ \Phi_\epsilon^{l,r},$$

from which we deduce

$$\Psi_\epsilon^G \circ T_\alpha = T_\alpha \circ \Psi_\epsilon^G. \tag{5.11}$$

Theorem 5.5 *For $\epsilon \in (-\rho, 0)$, there exist unique (Glutsyuk) Fatou coordinates $\Phi_\epsilon^{l,r}$ such that the associated Glutsyuk modulus $\Psi_{\hat{\epsilon}}^{0,\infty}$ has the following properties*

- $$\Psi_\epsilon^G = R_2 \circ (\Psi_\epsilon^G)^{-1} \circ R_2. \tag{5.12}$$

- *The Fourier series expansion of $\Psi_{\hat{\epsilon}}^G$ has zero constant terms in each connected component of its domain.*

- $$\Psi_\epsilon^G = \tau_{\pm\frac{1}{2}} \circ (\Psi_\epsilon^G)^{-1} \circ \tau_{\pm\frac{1}{2}}. \tag{5.13}$$

The construction can be done so as to yield the same Ecalle-Voronin modulus when $\epsilon \to 0$ as in Theorem 5.3.

Proof. (5.12) follows by taking

$$\Phi^r_\epsilon = R_2 \circ \Phi^l_\epsilon \circ R_2.$$

Choosing appropriately Φ^r_ϵ, which we call $\breve{\Phi}^r$ we can suppose that the constant term of Ψ^G_ϵ be 0 above the principal hole. From (5.11), it will also be equal to 0 below the principal hole. Then, Φ^l_ϵ and Φ^r_ϵ are uniquely determined with this property.

Let us now consider the reversibility property with respect to $\Sigma_{1,\epsilon}$, which is the lift of the reflection Σ_ϵ with respect to γ_ϵ. F_ϵ is reversible with respect to $\Sigma_{1,\epsilon}$. As in the case $\epsilon \geq 0$ we take as right Fatou coordinate $\breve{\Phi}^r = \Phi^r$ and we let

$$\breve{\Phi}^l_\epsilon = \tau \circ \breve{\Phi}^r_\epsilon \circ \Sigma_{1,\epsilon}.$$

Hence, this yields a representative of the modulus $\breve{\Psi}^G_\epsilon$ satisfying

$$\breve{\Psi}^G_\epsilon = \tau \circ (\breve{\Psi}^G_\epsilon)^{-1} \circ \tau.$$

Because of this property, if $B(\epsilon)$ is the constant term in the Fourier expansion of $\breve{\Psi}^G_\epsilon$, then $-\overline{B(\epsilon)}$ is the constant term in $(\breve{\Psi}^G_\epsilon)^{-1}$. We will show below that $B(\epsilon) \equiv \frac{1}{2}$. The first step is to show that $B(\epsilon)$ is real. We have $\Phi^l_\epsilon = T_{B(\epsilon)} \circ \breve{\Phi}^l_\epsilon$, and $\Psi^G_\epsilon = \breve{\Psi}^G_\epsilon \circ T_{-B(\epsilon)}$. If we let $\tau_{B(\epsilon)} = T_{B(\epsilon)} \circ \tau = T_{B(\epsilon)} \circ R_2 \circ \sigma$, then it follows that the Glutsyuk modulus Ψ^G_ϵ satisfies :

$$\begin{cases} \Psi^G_\epsilon \circ R_2 = R_2 \circ (\Psi^G_\epsilon)^{-1}, \\ \Psi^G_\epsilon = \tau_{\pm \overline{B}(\epsilon)} \circ (\Psi^G_\epsilon)^{-1} \circ \tau_{\pm \overline{B}(\epsilon)}, \\ \Psi^G_\epsilon \circ T_{B(\epsilon)+\overline{B}(\epsilon)} = T_{B(\epsilon)+\overline{B}(\epsilon)} \circ \Psi^G_\epsilon. \end{cases} \tag{5.14}$$

Because of (5.12) and the normalization chosen for Ψ^G_ϵ, the constant term in the Fourier expansion of $(\Psi^G_\epsilon)^{-1}$ vanishes. Since it is also equal to $\overline{B}(\epsilon) - B(\epsilon)$ and because of (5.14), then $B(\epsilon)$ is real.

Let us now show that $B(\epsilon) \equiv \frac{1}{2}$. Again we use that $F_\epsilon = \Sigma_{1,\epsilon} \circ R_2 \circ \Sigma_{1,\epsilon} \circ R_2$. Then

$$\begin{aligned} T_1 &= \Phi^r_\epsilon \circ \Sigma_{1,\epsilon} \circ R_2 \circ \Sigma_{1,\epsilon} \circ R_2 \circ (\Phi^r_\epsilon)^{-1} \\ &= \tau \circ T_{-B(\epsilon)} \circ \Phi^l_\epsilon \circ R_2 \circ \Sigma_{1,\epsilon} \circ (\breve{\Phi}^l_\epsilon)^{-1} \circ T_{-B(\epsilon)} \circ R_2 \\ &= \tau_{B(\epsilon)} \circ \Phi^l_\epsilon \circ R_2 \circ (\Phi)^r_\epsilon)^{-1} \circ \tau_{B(\epsilon)} \circ R_2 \\ &= \tau_{B(\epsilon)} \circ \Phi^l_\epsilon \circ (\Phi)^l_\epsilon)^{-1} \circ \tau_{-B(\epsilon)} = T_{2B(\epsilon)}. \end{aligned}$$

It is clear that the construction can be done with continuous limit at $\epsilon = 0$. $\square$

5.3 The compatibility condition

The (Lavaurs) modulus $(\Psi^0_{\hat\epsilon}, \Psi^\infty_{\hat\epsilon})$ is defined for $\hat\epsilon \in V_\delta$, where V_δ is defined in (5.1). We hence have two different moduli on the overlap of V_δ in ϵ-space and, in particular, when $\epsilon \in \mathbb{R}^-$. They must describe the same dynamics. This is the *compatibility condition* that was described in [3] and [13], and which turned out to be sufficient for realizability. In [13], it was also established that when the Lavaurs modulus satisfies a reversibility condition with respect to a Schwarz reflection, then the compatibility condition simply expressed that the Glutsyuk modulus derived from the Lavaurs modulus was reversible for real values of the parameter. We analyze the corresponding conditions here.

Since the modulus depends analytically on $\hat\epsilon$, it suffices to analyze the compatibility condition for $\epsilon \in \mathbb{R}^-$. Each such ϵ is represented by two $\hat\epsilon$ with arguments $-\pi$ and π. For that purpose we use the following notation

$$\hat\epsilon = \begin{cases} \check\epsilon, & \arg\hat\epsilon = -\pi, \\ \tilde\epsilon, & \arg\hat\epsilon = +\pi, \end{cases} \tag{5.15}$$

and

$$\Psi^{0,\infty}_{\hat\epsilon} = \begin{cases} \check\Psi^{0,\infty}, & \arg\hat\epsilon = -\pi, \\ \widetilde\Psi^{0,\infty}, & \arg\hat\epsilon = \pi, \end{cases}$$

$$\alpha(\hat\epsilon) = \begin{cases} \check\alpha, & \arg\hat\epsilon = -\pi, \\ \tilde\alpha, & \arg\hat\epsilon = \pi. \end{cases}$$

Recall that $\alpha(\hat\epsilon) = -\frac{\pi}{\sqrt{\hat\epsilon}}$ and that $\check\alpha = -\tilde\alpha = \frac{\pi i}{\sqrt{|\hat\epsilon|}}$.

We define maps $\check H^{0,\infty}$ and $\widetilde H^{0,\infty}$ representing the change from Lavaurs Fatou coordinates (on translation domains corresponding to a line transversal to the line of holes) to Glutsyuk Fatou coordinates (on translation domains corresponding to a line parallel to the line of holes). In the Lavaurs Fatou coordinates, the renormalized return maps are given $\check\Psi^{0,\infty}\circ T_{\check\alpha}$ or $T_{\tilde\alpha}\circ\widetilde\Psi^{0,\infty}$, depending on which translation domain we start. The Glutsyuk Fatou coordinates yield normalizations to the model which are uniform near the singular points: they are characterized by the fact that the renormalized return maps are just T_α. Hence, the maps $\check H^{0,\infty}$ and $\widetilde H^{0,\infty}$ conjugate the renormalized return maps $\check\Psi^{0,\infty}\circ T_{\check\alpha}$ or $T_{\tilde\alpha}\circ\widetilde\Psi^{0,\infty}$ to translations (more details in [3]):

$$\begin{cases} \check H^{0,\infty}\circ\check\Psi^{0,\infty}\circ T_{\check\alpha} = T_{\check\alpha}\circ\check H^{0,\infty}, \\ \widetilde H^{0,\infty}\circ T_{\tilde\alpha}\circ\widetilde\Psi^{0,\infty} = T_{\tilde\alpha}\circ\widetilde H^{0,\infty}. \end{cases} \tag{5.16}$$

(The maps $\check{H}^{0,\infty}$ and $\widetilde{H}^{0,\infty}$ are not uniquely defined by (5.16). In fact, they are defined up to composition on the left with a translation.) The Glutsyuk modulus is then given on one side by

$$\widetilde{\Psi}^G = \widetilde{H}^\infty \circ (\widetilde{H}^0)^{-1},$$

and on the other side by

$$\check{\Psi}^G = \check{H}^0 \circ (\check{H}^\infty)^{-1}.$$

Since it is unique up to left and right composition with translations, this implies that there exists $D, D' \in \mathbb{C}$ such that

$$\widetilde{H}^\infty \circ (\widetilde{H}^0)^{-1} = T_D \circ \check{H}^0 \circ (\check{H}^\infty)^{-1} \circ T_{D'}. \tag{5.17}$$

We call (5.17) the *compatibility condition.* Using (5.4), it is easily checked that we can take

$$\begin{cases} \widetilde{H}^\infty = R_2 \circ \widetilde{H}^0 \circ R_2 \circ T_{-\tilde{\alpha}}, \\ \check{H}^\infty = R_2 \circ \check{H}^0 \circ T_{-\check{\alpha}} \circ R_2. \end{cases} \tag{5.18}$$

This in turn implies

$$\begin{cases} \check{\Psi}^G = R_2 \circ (\check{\Psi}^G)^{-1} \circ R_2, \\ \widetilde{\Psi}^G = R_2 \circ (\widetilde{\Psi}^G)^{-1} \circ R_2. \end{cases}$$

We decide to choose $\widetilde{H}^0$ and $\check{H}^0$ so that the constant terms in the Fourier expansion of $\widetilde{\Psi}^G$ and $\check{\Psi}^G$ vanish. Together with (5.18), this uniquely defines $\widetilde{H}^{0,\infty}$ and $\check{H}^{0,\infty}$. Then, necessarily we take $D = D' = 0$ and we let $\Psi^G = \check{\Psi}^G = \widetilde{\Psi}^G$, which satisfies

$$\Psi^G = R_2 \circ (\Psi^G)^{-1} \circ R_2. \tag{5.19}$$

We also have $\Psi_{\check{\epsilon}}^{0,\infty} = \tau_{\frac{1}{2}} \circ (\Psi_{\tilde{\epsilon}}^{0,\infty})^{-1} \circ \tau_{\frac{1}{2}}$. It is easily checked that we can take

$$\overline{\check{H}}^{0,\infty} = \tau_{\frac{1}{2}} \circ \widetilde{H}^{0,\infty} \circ \tau_{\frac{1}{2}},$$

and define the following representative of the Glutsyuk modulus

$$\overline{\check{\Psi}}^G = \overline{\check{H}}^0 \circ (\overline{\check{H}}^\infty)^{-1} = \tau_{\frac{1}{2}} \circ (\Psi^G)^{-1} \circ \tau_{\frac{1}{2}}.$$

$\check{\bar{\Psi}}^G$ is reversible with respect to R_2. Indeed,

$$\begin{aligned} R_2 \circ \check{\bar{\Psi}}^G \circ R_2 &= R_2 \circ \tau_{\frac{1}{2}} \circ (\tilde{\Psi}^G)^{-1} \circ \tau_{\frac{1}{2}} \\ &= \tau_{-\frac{1}{2}} \circ R_2 \circ (\tilde{\Psi}^G)^{-1} \circ R_2 \circ \tau_{-\frac{1}{2}} \\ &= \tau_{-\frac{1}{2}} \circ \tilde{\Psi}^G \circ \tau_{-\frac{1}{2}} = (\tilde{\Psi}^G)^{-1} \\ &= \tau_{\frac{1}{2}} \circ \tilde{\Psi}^G \circ \tau_{\frac{1}{2}} = (\check{\bar{\Psi}}^G)^{-1}. \end{aligned}$$

Since the constant term in the Fourier expansion of $\tau_{\frac{1}{2}} \circ (\Psi^G)^{-1} \circ \tau_{\frac{1}{2}}$ is zero, then $\check{\bar{\Psi}}^G = \Psi^G$, and we finally have

$$\Psi^G = \tau_{\pm\frac{1}{2}} \circ (\Psi^G)^{-1} \circ \tau_{\pm\frac{1}{2}}.$$

5.4 The moduli space

The following theorem summarizes the results discussed in this section.

Theorem 5.6 *(1) We consider a germ of generic family $\Gamma_\epsilon(t)$ unfolding a germ of analytic curve $\Gamma_0(t)$ with a cusp. A complete modulus of conformal classification is given by the modulus of the germ of family of associated diffeomorphisms f_ϵ to the curvilinear angles formed by $\pm\gamma_\epsilon(t)$ where $\gamma_\epsilon^2(t) = \Gamma_\epsilon(t)$, under conjugacy commuting with R_2 and the corresponding Schwarz reflections. The modulus is given by $(\Psi_\epsilon^0, \Psi_\epsilon^\infty)_{\epsilon\in[0,\rho)}$ and $(\Psi_\epsilon^G)_{\epsilon\in(-\rho,0)}$, which depend analytically of $\epsilon \neq 0$ with same continuous limit at $\epsilon = 0$, and which represent respectively the unique representative of the Lavaurs and Glutsyuk modulus satisfying*

$$\begin{cases} \Psi_\epsilon^0 \circ R_2 = R_2 \circ (\Psi_\epsilon^\infty)^{-1}, \\ \Psi_\epsilon^G \circ R_2 = R_2 \circ (\Psi_\epsilon^G)^{-1}, \end{cases} \tag{5.20}$$

$$\Psi_\epsilon^{0,\infty,G} = \tau_{\pm\frac{1}{2}} \circ (\Psi_{\bar{\epsilon}}^{0,\infty,G})^{-1} \circ \tau_{\pm\frac{1}{2}}. \tag{5.21}$$

Moreover, if $(\Psi_{\hat{\epsilon}}^0, \Psi_{\hat{\epsilon}}^\infty)_{\hat{\epsilon}\in V_\delta}$ is the extension of the Lavaurs modulus to values of $\hat{\epsilon} \in V_\delta$ (defined in (5.1)), and if $\check{H}_\epsilon^{0,\infty}$ and $\widetilde{H}_\epsilon^{0,\infty}$ are defined as in (5.16), then the compatibility condition

$$\widetilde{H}^\infty \circ (\widetilde{H}^0)^{-1} = \check{H}^0 \circ (\check{H}^\infty)^{-1}. \tag{5.22}$$

is satisfied.

(2) $(\Psi^0_{\hat\epsilon}, \Psi^\infty_{\hat\epsilon})_{\hat\epsilon\in V_\delta}$ *is* $1/2$*-summable in* $\hat\epsilon$ *with directions of non-summability corresponding to* $\arg\hat\epsilon = \pm\pi$*.*

(3) If $\Gamma_\epsilon(t)$ *depends analytically on extra parameters, then so does the modulus.*

(4) Any pair $(\Psi^0_\epsilon, \Psi^\infty_\epsilon)_{\epsilon\in[0,\rho)}$ *and* $(\Psi^G_\epsilon)_{\epsilon\in(-\rho,0)}$*, which*

- *depends analytically on* ϵ *with same continuous limit at* $\epsilon = 0$*,*
- *satisfies* (5.20), (5.21)*,*
- *is such that* $(\Psi^0_{\hat\epsilon}, \Psi^\infty_{\hat\epsilon})$ *has a power series expansion in* $\sqrt{\hat\epsilon}$ *which is* $1/2$*-summable in* $\hat\epsilon$ *with sum defined for* $\sqrt{\hat\epsilon}\in V_\delta$ *for some* V_δ *of radius* $\rho' \leq \rho$*,*
- *the sum of the power series satisfies the compatibility condition* (5.22)*,*

is realizable as the modulus of a germ of generic family $\Gamma_\epsilon(t)$ *unfolding a germ of analytic curve* $\Gamma_0(t)$ *with a cusp.*

Proof. We have already proved (1), and (3) follows easily from the construction. (2) is proved in [3]. It follows from the Ramis-Sibuya theorem: the argument is sketched here for the reader who knows about summability. Since it is not needed explicitly here, we do not go into the details of the definition of summability. Indeed, in (4), what we only need is that $(\Psi^0_\epsilon, \Psi^\infty_\epsilon)_{\epsilon\in[0,\rho)}$ has an extension to $\hat\epsilon \in V_\delta$ for some ρ' (which may be smaller than ρ). Then, considering that $\breve\alpha$ and $\tilde\alpha$ are very large pure imaginary numbers, this allows to compute the $\widetilde H^{0,\infty}_\epsilon$ and $\check H^{0,\infty}_\epsilon$ from (5.16). If we do this computation above the principal hole we get

$$\begin{cases} \widetilde H^0_\epsilon = id + O\left(\exp\left(-\frac{C}{\sqrt{|\epsilon|}}\right)\right), \\ \widetilde H^\infty_\epsilon = \widetilde\Psi^\infty_\epsilon + O\left(\exp\left(-\frac{C}{\sqrt{|\epsilon|}}\right)\right), \\ \check H^0_\epsilon = id + O\left(\exp\left(-\frac{C}{\sqrt{|\epsilon|}}\right)\right) \\ (\check H^\infty_\epsilon)^{-1} = \check\Psi^\infty_\epsilon + O\left(\exp\left(-\frac{C}{\sqrt{|\epsilon|}}\right)\right), \end{cases} \tag{5.23}$$

for some positive constant C. This in turn implies

$$\left|\widetilde\Psi^\infty_\epsilon - \check\Psi^\infty_\epsilon\right| = O\left(\exp\left(-\frac{C}{\sqrt{|\epsilon|}}\right)\right),$$

which is sufficient to ensure 1/2-summability of $\Psi_{\hat\epsilon}^\infty$ in $\hat\epsilon$. A similar calculation for the component of Ψ_ϵ^G below the principal hole yields the 1/2-summability of $\Psi_{\hat\epsilon}^0$ in $\hat\epsilon$.

So the only real thing to prove is (4), namely the realization. It is proved in [3] that any $(\Psi_{\hat\epsilon}^0, \Psi_{\hat\epsilon}^\infty)_{\hat\epsilon\in V_\delta}$ which satisfies the compatibility condition is realizable as a germ of generic family f_ϵ of diffeomorphisms unfolding a parabolic diffeomorphism. Hence, it suffices to prove that it is possible to realize some f_ϵ which satisfies the two reversibility properties and, moreover, that f_ϵ can be factored as $\Sigma_\epsilon\circ\Sigma'_\epsilon$ with $\Sigma'_\epsilon = R_2\circ\Sigma_\epsilon\circ R_2$. The realization in [3] was done in two steps. We first realize a pair $(\Psi_{\hat\epsilon}^0, \Psi_{\hat\epsilon}^\infty)$ for a fixed $\hat\epsilon$. This part does not require the compatibility condition, and was called the *local realization*. The local realization could be performed to depend analytically in $\hat\epsilon$ with continuous limit for $\epsilon = 0$. In this way, we realize the modulus by a family $g_{\hat\epsilon}$ which may depend in a ramified way of ϵ. The second step is to correct $g_{\hat\epsilon}$ to a uniform family conjugate to it.

Local realization. It is performed by means of Ahlfors-Bers theorem. We first realize the modulus on an abstract manifold $\overline{M}_{\hat\epsilon}$ which we then recognize to be a neighborhood of the origin in $\mathbb{C}$. The abstract manifold is obtained by gluing together two domains $U^\pm$ whose union is $\mathbb{D}\setminus\pm\sqrt{\hat\epsilon}$ (see Figure 5). We take the two domains $U^\pm$ so that one is the symmetric of the other under R_2. We will make sure that the gluing respects this symmetry. In practice the sectors are obtained as $p_\epsilon(S_{\hat\epsilon}^\pm)$, where $S_{\hat\epsilon}^\pm$ are the strips of Figure 6, the gluing being done by means $z^- = p_\epsilon\circ\Psi_{\hat\epsilon}^{0,\infty}\circ p_\epsilon^{-1}(z^+)$. At the level of the strips, we have that $S_{\hat\epsilon}^+ = R_2(S_{\hat\epsilon}^-)$. So, let $W_1^- = R_2(W_1^+)$ with $W_1^\pm\in S_{\hat\epsilon}^\pm$. Suppose for instance that W_1^+ is identified to $W_2^- = \Psi_{\hat\epsilon}^0(W_1^+)$. Then, we want $W_1^- = R_2(W_1^+)$ to be identified with $R_2(W_2^-)$. But, $W_1^-\in S_{\hat\epsilon}^-$ is identified to $(\Psi_{\hat\epsilon}^\infty)^{-1}(W_1^-) = (\Psi_{\hat\epsilon}^\infty)^{-1}\circ R_2(W_1^+) = R_2\circ\Psi_{\hat\epsilon}^0(W_1^+) = R_2(W_2^-)$, hence the result. For $\hat\epsilon\neq 0$, the intersection $U_{\hat\epsilon}^+\cap U_{\hat\epsilon}^-$ is composed of three parts, $U_{\hat\epsilon}^0$, $U_{\hat\epsilon}^\infty$ and $U_{\hat\epsilon}^C$. The gluing is the following: we identify $z^-\in U_{\hat\epsilon}^-$ with $\Xi_{\hat\epsilon}(z^+)$ for $z^+\in U_{\hat\epsilon}^+$, where

$$\Xi_{\hat\epsilon} = \begin{cases} p_\epsilon\circ\Psi_{\hat\epsilon}^\infty\circ p_\epsilon^{-1}, & \text{on } U_{\hat\epsilon}^\infty,\\ p_\epsilon\circ T_\alpha\circ p_\epsilon^{-1} = id, & \text{on } U_{\hat\epsilon}^C,\\ p_\epsilon\circ\Psi_{\hat\epsilon}^0\circ p_\epsilon^{-1}, & \text{on } U_{\hat\epsilon}^0.\end{cases}$$

Note that the distinguished point $z^+ = 0$ is in $U_{\hat\epsilon}^C$ and hence coincides with $z^- = 0$. This point is simply called 0.

Let us recall the main steps of the construction of [3]: to recognize that $M_{\hat\epsilon}$ is conformally equivalent to a disk minus two points we construct a

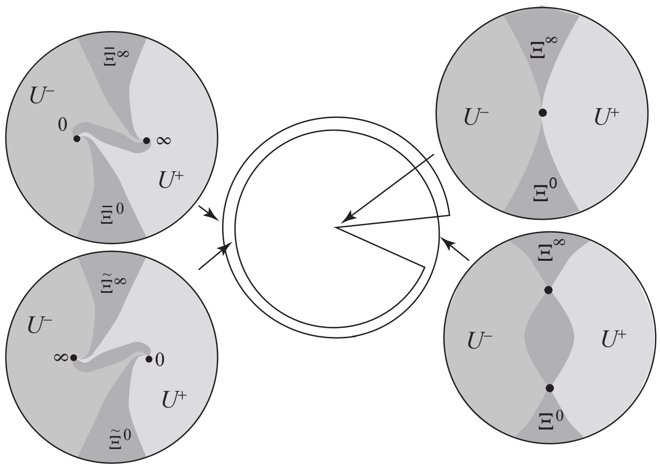


Figure 5: The sectors $U_{\hat{\epsilon}}^{\pm}$ and their intersection.

C^k-coordinate (with k large) of the form

$$\chi_{\hat{\epsilon}}(m) = z^+\Theta_{\hat{\epsilon}}^+ + z^-\Theta_{\hat{\epsilon}}^-, \tag{5.24}$$

where $(\Theta_{\hat{\epsilon}}^+, \Theta_{\hat{\epsilon}}^-)$ is a partition of unity for $\{U_{\hat{\epsilon}}^+, U_{\hat{\epsilon}}^-\}$ and $m \in M_{\hat{\epsilon}}$ has coordinate $z^{\pm}$ in $U_{\hat{\epsilon}}^{\pm}$. It is shown there that the Beltrami field

$$\mu_{\hat{\epsilon}} = \frac{\partial\chi_{\hat{\epsilon}}/\partial\overline{z}^+}{\partial\chi_{\hat{\epsilon}}/\partial z^+}$$

satisfies $|\mu_{\hat{\epsilon}}| < K < 1$. By putting $\mu_{\hat{\epsilon}}(\pm\sqrt{\hat{\epsilon}}) = 0$ we extend $\mu_{\hat{\epsilon}}$ in a C^1 way.

By the Ahlfors-Bers theorem, there exists a 1-1 map $\nu_{\hat{\epsilon}} : \chi_{\hat{\epsilon}}(M_{\hat{\epsilon}}) \to \mathbb{C}$ which is holomorphic in the sense of this structure and whose image is the disk $r\mathbb{D}$. Then

$$\zeta_{\hat{\epsilon}} = \nu_{\hat{\epsilon}} \circ \chi_{\hat{\epsilon}} \tag{5.25}$$

is holomorphic, yielding that the manifold $\overline{M}_{\hat{\epsilon}}$ is conformally equivalent to the disk $r\mathbb{D}$. For $\hat{\epsilon} \neq 0$, the map $\nu_{\hat{\epsilon}}$ is uniquely determined if we ask that $\nu_{\hat{\epsilon}}(0) = 0$ and that the singular point corresponding to $\sqrt{\hat{\epsilon}}$ has the same argument as $\sqrt{\hat{\epsilon}}$. The manifold $\overline{M}_{\hat{\epsilon}}$ was endowed with a diffeomorphism coming from T_1 on the strips. This diffeomorphism was reversible with respect to a holomorphic involution R (coming from R_2 sending one chart to

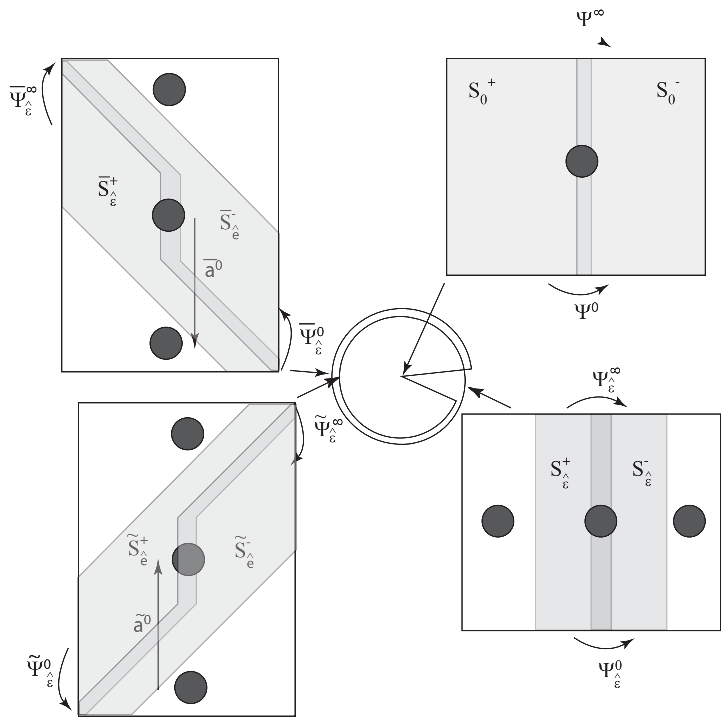


Figure 6: The strips $S^{\pm}_{\hat{\epsilon}}$ projecting on the sectors of Figure 5.

the other) with a fixed point at 0. Through $\nu_{\hat{\epsilon}}$, this involution is transformed into an involution of the disk $\mathbb{D}_r$ preserving the origin. Then this involution is necessarily R_2.

Now, we have the reflection $\tau_{\frac{1}{2}}$ with respect to the line $\mathcal{L} = \text{Re}\, W = \frac{1}{4}$ defined on (part of) the strips $S^{\pm}_{\hat{\epsilon}}$ with image in the other. Again, if $W_1^- = \tau_{\frac{1}{2}}(W_1^+)$, and we have that W_1^+ (resp. W_1^-) is identified to $W_2^- = \Psi(W_1^+)$ (resp. $W_2^+ = (\Psi)^{-1}(W_1^-)$), where Ψ can be any one of $\Psi^0_{\hat{\epsilon}}$, $\Psi^\infty_{\hat{\epsilon}}$ or Ψ^G_{ϵ}, then we want that $W_2^- = \tau_{\frac{1}{2}}(W_2^+)$. This is precisely ensured by the reversibility of Ψ with respect to $\tau_{\frac{1}{2}}$.

Hence, $\tau_{\frac{1}{2}}$ induces a global antiholomorphic involution $\Sigma_{\hat{\epsilon}}$ on the abstract manifold $\overline{M}_{\hat{\epsilon}}$. On each chart we had $\tau_{-\frac{1}{2}} = R_2 \circ \tau_{\frac{1}{2}} \circ R_2$, the diffeomorphism T_1 was decomposed as $T_1 = \tau_{\frac{1}{2}} \circ \tau_{-\frac{1}{2}}$ and, moreover, the gluing is compatible with these properties. This yields that $\tau_{-\frac{1}{2}}$ induces a global antiholomorphic involution $\Sigma'_{\hat{\epsilon}} = R \circ \Sigma_{\hat{\epsilon}} \circ R$ on $\overline{M}_{\hat{\epsilon}}$ and such that that the diffeomorphism $g_{\hat{\epsilon}}$ constructed on $\overline{M}_{\hat{\epsilon}}$ is equal to $g_{\hat{\epsilon}} = \Sigma_{\hat{\epsilon}} \circ \Sigma'_{\hat{\epsilon}}$.

Global realization. We then need to correct to a uniform family. This is where the compatibility condition is needed. Indeed, we have realized a family $g_{\hat{\epsilon}}$ for $\hat{\epsilon} \in V$ and $z \in r\mathbb{D}$. The compatibility condition ensures that, for $\epsilon < 0$, then $g_{\check{\epsilon}}$ is conjugate to $g_{\tilde{\epsilon}}$; there exists a diffeomorphism h_ϵ satisfying

$$h_\epsilon \circ g_{\check{\epsilon}} = g_{\tilde{\epsilon}} \circ h_\epsilon,$$

and also the conditions of (4.4). Of course, h_ϵ can be extended to a sectorial neighborhood of the real negative axis and we can restrict δ in V_δ so that h_ϵ be defined on the whole auto-intersection part (in ϵ-space). We call $\check{z}$ (resp. $\tilde{z}$) the z coordinate for $\hat{\epsilon} = \check{\epsilon}$ (resp. $\hat{\epsilon} = \tilde{\epsilon}$). We construct an abstract 2-dimensional manifold by gluing $V \times r\mathbb{D}$ on the auto-intersection part by means of

$$(\check{z}, \check{\epsilon}) \simeq (\tilde{z}, \tilde{\epsilon}) = (h_\epsilon(\check{z}), \check{\epsilon}).$$

Newlander-Nirenberg's theorem can be used to recognize that this manifold is holomorphically equivalent to a neighborhood of the origin in $\mathbb{C}^2$, once we have glued a disk $r\mathbb{D}$ above $\epsilon = 0$ to fill the hole (details in [3]). Moreover, from the form of the gluing, the new complex coordinates are of the form (w, ϵ), where $w = H(z_{\hat{\epsilon}}, \hat{\epsilon}) = H_{\hat{\epsilon}}(z_{\hat{\epsilon}})$ and $H_{\hat{\epsilon}}$ is a diffeomorphism for fixed $\hat{\epsilon}$ such that $H_{\tilde{\epsilon}}^{-1} \circ H_{\check{\epsilon}} = h_\epsilon$.

The abstract manifold comes equipped with a family of diffeomorphisms which is C^k for $\epsilon = 0$. Newlander-Nirenberg's theorem transforms it into an analytic family of diffeomorphisms. So the only thing to check is that the realized family (on the neighborhood of the origin in $\mathbb{C}^2$) has the required reversibility properties. Since the gluing commutes with both the holomorphic involution R_2, and with the antiholomorphic involution $\Sigma_{\hat{\epsilon}}$, there exist well defined families of holomorphic involutions corresponding to R_2 and of antiholomorphic involutions corresponding to $\Sigma_{\hat{\epsilon}}$, and for each value of ϵ, the diffeomorphism is reversible with respect to both involutions. Indeed, the gluing has sent 0 to 0. Hence, there is a uniform linearizable involution tangent to minus the identity with a fixed point depending analytically on (w, ϵ) and with a C^k limit at $\epsilon = 0$. It is of course possible to take an analytic change of coordinates depending analytically on ϵ and transforming the involution to R_2.

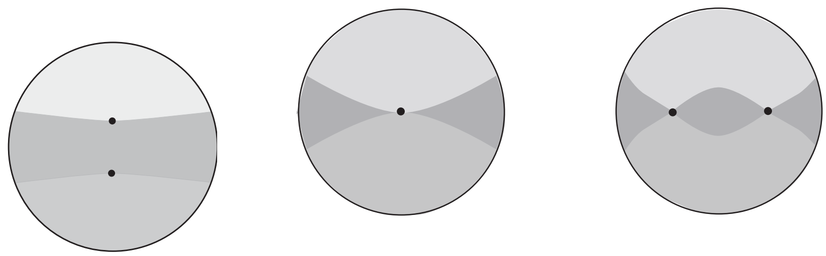
The decomposition of $f_\epsilon = \Sigma_\epsilon \circ \Sigma'_\epsilon$ follows from the corresponding decomposition for each ϵ and the gluing respecting the decomposition. □

6 Geometric interpretation

We have treated the problem of conformal classification of unfolding of cusps on a fixed neighborhood U of the cusp point for real values of the parameters in a small interval around the origin. In the unfolding, the curve is regular for $\epsilon < 0$, and has a regular singular point for $\epsilon > 0$. In both cases, we have a functional modulus (of infinite dimension) which should reflect the underlying geometry of the curve. Of course, the modulus describes the geometry of the associated complex curve. A non trivial modulus reflects obstructions to a conformal equivalence of the complex curve with the model curve (formal normal form). Some of these obstructions have been described in the paper [11] and consist in the birth of singular points through the parametric resurgence phenomenon. But here, we are more interested in the obstructions that can be observed for the real curve inside the fixed neighborhood U. We describe geometric obstructions of two kinds.

The first kind of obstructions are inherited from the obstruction to embed the diffeomorphism f_ϵ into the time-one map of a vector field. However, we can cover a neighborhood of the origin in z-space with two adequate (possibly sectorial) domains. Over each such domain, the diffeomorphism can be embedded into the time-one map of a *unique* vector field. Hence, each domain is canonically endowed with the flow lines of a vector field. The modulus measures the mismatch of the flow lines of the two vector fields on the intersection of the two sectorial domains. This is the unfolding of the point of view described by Gelfreich in [4]. In particular, if two flow lines intersect at a point z they also intersect at all iterates $f_\epsilon^n(z)$. The angle between the flow lines is the same at all intersection points and it can be calculated from the unfolded modulus. When passing to the Z-coordinate, each of the two domains is sent to a full neighborhood of the origin with an auto-intersection. We obtain vector fields on a 2-covering of that neighborhood with a ramification point at the origin.

The second kind of obstructions comes from the presence of the special points in z-space. There are three of them: $z = 0$, and the two fixed points of f_ϵ. When $\epsilon > 0$, these two fixed points are the intersection points of γ_ϵ and $-\gamma_\epsilon$. When $\epsilon < 0$, these points are not located on the curves $\pm\gamma_\epsilon$, but



(a) $\epsilon < 0$ (b) $\epsilon = 0$ (c) $\epsilon > 0$

Figure 7: The two domains covering U.

they organize the dynamics however.

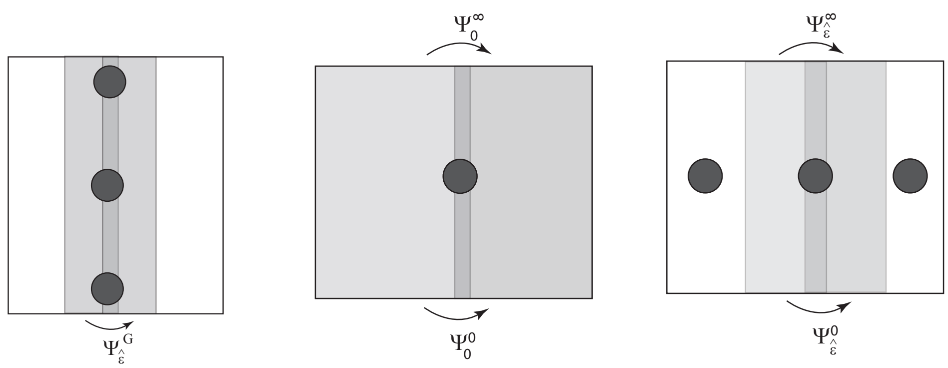
6.1 The geometry of the flow lines

We cover a neighborhood of the origin in z-space with two (possibly sectorial) domains (Figure 7) coming from strips in W-space as in Figure 8. Over these strips we have the Fatou coordinates, conjugating F_ϵ to T_1. In section 5 we have normalized the Fatou coordinates so that they be uniquely defined. On the image of the Fatou coordinates we introduce the vector field $\frac{\partial}{\partial W}$, whose flow lines are the horizontal lines Im $W = Cst$. Note that this vector field is unique even if we had not uniquely defined the normalizations! But the advantage of the normalized Fatou coordinates is that in the equation of a flow line Im $W = Y$, then Y is intrinsically defined. These flow lines induce flow lines on the sectorial domains in z-space, also parameterized by Y. The modulus $(\Psi_\epsilon^0, \Psi_\epsilon^\infty)$ for $\epsilon \geq 0$ (resp. Ψ_ϵ^G for $\epsilon < 0$) measures the angle between the two flow lines parameterized by the same Y on the two Fatou coordinates $\Phi^{r,l}$. Indeed, if two flow lines parameterized by the same Y intersect at z and if we let $W^j = \Phi^j \circ p_\epsilon^{-1}(z)$, $j \in \{l, r\}$, with Im $W^j = Y$ and $\Psi_\epsilon = \Phi^r \circ (\Phi^l)^{-1}$, then the angle between the two flow lines at z is simply given by arg $\Psi_\epsilon(W^l)$.

Let us consider the case $\epsilon \geq 0$ and let

$$\begin{cases} \Psi_\epsilon^0 = W + \sum_{n<-1} b_n(\epsilon) \exp(2\pi i n W), \\ \Psi_\epsilon^\infty = W + \sum_{n>1} b_n(\epsilon) \exp(2\pi i n W). \end{cases} \tag{6.1}$$

Proposition 6.1 *Let $\epsilon \geq 0$. Suppose $b_m(0) \neq 0$ and $b_j(0) = 0$ for $jm > 0$ and $|j| < |m|$. Let $\mathcal{C}_\epsilon^\pm(Y)$ be the flow line in $U_0^\pm$ coming from the line*



(a) $\epsilon < 0$ (b) $\epsilon = 0$ (c) $\epsilon > 0$

Figure 8: The strips which are the domains of the Fatou coordinates.

$\text{Im W} = \text{Y}$ *in the image of the Fatou coordinate* $\Phi_\epsilon^\pm$*, and let* $\theta = \theta(z)$ *be the angle at an intersection point* $z \in \mathcal{C}_\epsilon^+(Y) \cap \mathcal{C}_\epsilon^-(Y)$*. Then, for* $|Y|$ *sufficiently large and* $Y < 0$ *(resp.* $Y > 0$*) for* $m < 0$ *(resp.* $m > 0$*), and for* $\epsilon \geq 0$ *sufficiently small we have* $\theta \neq 0$*. In the case* $\epsilon = 0$ *we have the more precise information:*

$$|\theta| = 2\pi m|b_m(0)|e^{-2\pi mY} + O\left(e^{-2\pi mY}\right).$$

Proof. The case $\epsilon = 0$ was proved by Gelfreich in [4] and the case $\epsilon > 0$ small follows by continuity of $\Psi_\epsilon^{0,\infty}$. The map $\Phi_\epsilon^\pm \circ p_\epsilon^{-1}$ being conformal, the angle $\theta(z)$ is the angle at an intersection point of the curves $\mathcal{E}_1 = \{\text{Im W} = \text{Y}\}$ and $\mathcal{E}_2 = \{\Psi_\epsilon^W \mid \text{Im W} = \text{Y}\}$. For the purpose of completeness we recall the proof of [4]. Let $X + iY \in \mathcal{E}_1 \cap \mathcal{E}_2$, and let $\epsilon = 0$. We let $b_n(0) = b_n$. From (6.1), we deduce

$$\text{Im}\left(\sum_{n \geq m} b_n e^{-2\pi nY} e^{2i\pi nX}\right) = 0.$$

The intersection points of $\mathcal{E}_1$ and $\mathcal{E}_2$ have the form

$$X = -\frac{\arg b_m}{2\pi m} + \frac{k}{2m} + O\left(e^{-2\pi Y}\right), \qquad k \in \mathbb{Z}.$$

Let

$$\Xi(X + iY) = \frac{\partial \Psi_0^\infty(X + iY)}{\partial X}.$$

Then, since X is parameterizing both $\mathcal{E}_1$ and $\mathcal{E}_2$, we have

$$\tan\theta = \frac{\text{Im } \Xi(X+iY)}{\text{Re } \Xi(X+iY)}.$$

For Y sufficiently large, we have

$$\begin{aligned}\theta \sim \tan\theta &= \frac{\text{Im} \sum_{n\geq m} b_n 2i\pi n e^{-2\pi nY+2i\pi nX}}{1+\text{Re} \sum_{n\geq m} b_n 2i\pi n e^{-2\pi nY+2i\pi nX}} \\ &= \pm 2\pi m |b_m| e^{-2\pi mY} + O\left(e^{-2\pi mY}\right).\end{aligned}$$

When $\epsilon > 0$ is sufficiently small we can have the conclusion on a whole strip $|\text{Im W}| > \text{Y}_0$ for some positive Y_0 because all orbits have representatives in a strip of finite width. □

We now consider the case $\epsilon < 0$. The Glutsyuk modulus Ψ_ϵ^G has two representatives, one above the principal hole denoted by Ψ_u^G, and one below denoted by Ψ_d^G. We suppose that $\epsilon = \tilde{\epsilon}$, i.e. that Im $\alpha > 0$. Then Ψ_u^G and Ψ_d^G are related through

$$\Psi_u^G \circ T_\alpha = T_\alpha \circ \Psi_d^G.$$

Let

$$\begin{cases}\Psi_u^G = W + \sum_{n\in\mathbb{Z}} B_n^u(\epsilon)\exp(2\pi i n W), \\ \Psi_d^G = W + \sum_{n\in\mathbb{Z}} B_n^d(\epsilon)\exp(2\pi i n W).\end{cases}$$

Then, it follows from (5.23) (more details in [3]) that there exists $C > 0$ such that

$$\begin{cases} B_n^u(\epsilon) = b_n(0) + O(\epsilon), & n > 0, \\ B_n^u(\epsilon) = O\left(\exp\left(-\frac{C}{\sqrt{|\epsilon|}}\right)\right), & n < 0, \\ B_n^d(\epsilon) = O\left(\exp\left(-\frac{C}{\sqrt{|\epsilon|}}\right)\right), & n > 0, \\ B_n^d(\epsilon) = b_n(0) + O(\epsilon), & n < 0. \end{cases}$$

Proposition 6.2 *Suppose $b_m(0) \neq 0$ for and $b_j(0) = 0$ for $jm > 0$ and $|j| < |m|$. Let $\mathcal{C}_\epsilon^j(Y)$, $j \in \{l, r\}$, be the flow line in $U_0^\pm$ coming from the line* Im W = Y *in the image of the Fatou coordinate Φ_ϵ^j, and let $\theta = \theta(z)$ be the angle at an intersection point $z \in \mathcal{C}_\epsilon^l(Y) \cap \mathcal{C}_\epsilon^r(Y)$. Then,*

- *If $m > 0$, there exists $Y_0 > 0$ such that for $Y \in [2Y_0, 3Y_0]$ and $\epsilon < 0$ sufficiently small, we have $\theta \neq 0$.*

- *If $m < 0$, there exists $Y_0 > 0$ such that for $Y \in [-3Y_0, -2Y_0]$ and $\epsilon < 0$ sufficiently small, we have $\theta \neq 0$.*

PROOF. Let us consider the case $m > 0$. The proof is very similar to that of Proposition 6.1. There, the conclusion was valid for $Y > Y_0$ for some $Y_0 > 0$. We take ϵ sufficiently small so that $|\alpha| > 4Y_0$. Then, for $Y = \text{Im}\, W \in [2Y_0, 3Y_0]$, W belongs to the domain of Ψ_u^G. It is known that $\Psi_u^G \to \Psi_0^\infty$ as $\epsilon \to 0$ and that the convergence is uniform on any compact set. Then it follows by continuity that $\theta \neq 0$ for ϵ sufficiently small. $\square$

Remark 6.3 *Propositions* 6.1 *and* 6.2 *only concern large values of* $|Y|$*, and indeed may not be valid for smaller values of* $|Y|$*.*

Among all flow lines, we will in particular discuss the ones passing through the special point $z = 0$ in Section 6.2.

6.2 The zero-flow lines

In both cases, $\epsilon > 0$ or $\epsilon < 0$, for each domain of the flow we have the special flow line passing through 0, which we call the *zero-flow line* attached to the special point $z = 0$. Even in the case of a non trivial modulus, it may happen that the two zero-flow lines attached to the special point $z = 0$ coincide. We will be interested to identify the families for which this is the case for all real values of ϵ. By continuity, for $\epsilon = 0$, there will exist two flow lines in each sector whose union is an analytic curve: we still call this analytic curve the *zero-flow line.* Since this property is invariant under conformal transformation, it can be read on the modulus. In particular, this will be the case if the curves $\pm\gamma_\epsilon$ are symmetric with respect to the imaginary axis or, of course, conformally equivalent to this case under a conformal diffeomorphism commuting with R_2. We will show that this is the only possibility and see how this is expressed in the modulus.

Theorem 6.4 *We consider a germ of family of curves $\pm\gamma_\epsilon$ as in* (3.3)*. The following are equivalent:*

(i) For all ϵ sufficiently small, the family has a unique zero-flow line.

(ii) There exists a germ of family of conformal diffeomorphism commuting with R_2 and transforming the curves $\pm\gamma_\epsilon$ into curves $\pm\breve{\gamma}_\epsilon$ symmetric with respect to the imaginary axis.

(iii) Let $\sigma_A = T_A \circ \sigma$. The normalized representative of the modulus defined in Section 5 satisfies: there exists $A(\epsilon) \in i\mathbb{R}$ such that

$$\begin{cases} \Psi_\epsilon^0 = R_2 \circ (\Psi_\epsilon^\infty)^{-1} \circ R_2, & \epsilon \geq 0, \\ \Psi_\epsilon^G = R_2 \circ (\Psi_\epsilon^G)^{-1} \circ R_2, & \epsilon < 0, \\ \Psi_\epsilon^0 = \sigma_{A(\epsilon)} \circ \Psi_\epsilon^\infty \circ \sigma_{A(\epsilon)}, & \epsilon \geq 0, \\ \Psi_\epsilon^G = \sigma_{A(\epsilon)} \circ \Psi_\epsilon^G \circ \sigma_{A(\epsilon)}, & \epsilon < 0, \\ \Psi_\epsilon^G \circ T_\alpha = T_\alpha \circ \Psi_\epsilon^G, & \epsilon < 0. \end{cases} \tag{6.2}$$

Proof. Of course, (ii) implies (i).

Let us now show that (i) implies (iii). The zero-flow line is an analytic curve in z-space, which is symmetric under R_2. Hence, it is possible to bring this curve to the imaginary axis in z-space under a conformal change of coordinates commuting with R_2 which transforms f_ϵ to $\breve{f}_\epsilon$. In the new coordinate, this line is invariant under $\breve{f}_\epsilon$, implying that $\breve{f}_\epsilon$ is symmetric with respect to this line:

$$\breve{f}_\epsilon \circ \tau = \tau \circ \breve{f}_\epsilon.$$

Hence, f_ϵ is conjugate to a symmetric $\breve{f}_\epsilon$ under a conjugacy commuting with R_2.

It is then easy, as in Section 5 to show the existence of $A(\epsilon)$ for which (6.2) is satisfied. Indeed, the imaginary axis in z space is sent by p_ϵ^{-1} to the line $\text{Im } W = 0$ for $\epsilon \geq 0$ and to the family of lines $\text{Im } W = \frac{k|\alpha|}{2}$ for $k \in \mathbb{Z}$ when $\epsilon < 0$.

When $\epsilon \geq 0$, because F_ϵ is symmetric with respect to the line $\text{Im } W = 0$, it is easy to show that the Fatou coordinate transforms the symmetry axis into a horizontal line. (A proof in a slightly different context appears in [2].) If the modulus Ψ_ϵ is of the form $\Psi_\epsilon = \Phi_\epsilon^r \circ (\Phi_\epsilon^l)^{-1}$, then it is possible to preserve the reversibility property with respect to R_2 by changing Φ_ϵ^l (resp. Φ_ϵ^r) to $\breve{\Phi}_\epsilon^l = T_A \circ \Phi_\epsilon^l$ (resp. $\breve{\Phi}_\epsilon^r = T_{-A} \circ \Phi_\epsilon^r$). for some $A \in \mathbb{C}$. Because of the reversibility of F_ϵ with respect to R_2, the image of the real axis by the two Fatou coordinates are images one of the other under R_2. Thus, choosing appropriately $A \in i\mathbb{R}$ we can suppose that the symmetry axis is sent to the real axis by the two Fatou coordinates. Hence, both Fatou coordinates $\breve{\Phi}_\epsilon^{l,r}$ commute with σ. If $\breve{\Psi}^{0,\infty} = \breve{\Phi}_\epsilon^r \circ (\breve{\Phi}_\epsilon^l)^{-1}$, this implies $\breve{\Psi}_\epsilon^0 = \sigma \circ \breve{\Psi}_\epsilon^\infty \circ \sigma$. Coming back to $\Psi_\epsilon^{0,\infty}$, this implies that

$$\Psi_\epsilon^0 = \sigma_A \circ \Psi_\epsilon^\infty \circ \sigma_A.$$

When $\epsilon < 0$, we can again show that the horizontal lines are sent to horizontal lines at a distance $\frac{|\alpha|}{2}$ from each other. Hence, by changing Φ_ϵ^l

(resp. Φ_ϵ^r) to $\breve{\Phi}_\epsilon^l = T_A \circ \Phi_\epsilon^l$ (resp. $\breve{\Phi}_\epsilon^r = T_{-A} \circ \Phi_\epsilon^r$), and by choosing $A \in i\mathbb{R}$ appropriately, we can suppose that each line $\text{Im}\, W = \frac{k|\alpha|}{2}$ is sent to itself, and hence invariant under both Fatou coordinates, and thus that each line $\text{Im}\, W = \frac{k|\alpha|}{2}$ is invariant under $\breve{\Psi}_\epsilon^G = \breve{\Phi}_\epsilon^r \circ (\breve{\Phi}_\epsilon^l)^{-1}$. Coming back to Ψ_ϵ^G, this yields the result.

It remains to show that (iii) implies (ii). This comes from the realization process. Indeed, in Section 5.4 we have shown that for a fixed value of ϵ, any modulus can be realized on an abstract 1-dimensional complex manifold, which is further identified to an open neighborhood of the origin in $\mathbb{C}$. On the abstract manifold constructed by pasting two domains endowed with flow lines, the gluing produces an analytic flow line through 0. This remains the case in the identification, which preserves the reversibility with respect to R_2. Moreover, on each chart of the abstract manifold, the diffeomorphism was given by $T_1 = \tau_{\frac{1}{2}} \circ \tau_{-\frac{1}{2}}$ which was symmetric with respect to σ_A. The gluing condition ensures that the invariant lines for σ_A in each chart yield a global invariant line.

The construction can be made to depend analytically on ϵ with a continuous limit at $\epsilon = 0$. □

6.3 Passing to Z-coordinate

In the previous discussion of the geometric interpretation in z-space, we were covering a neighborhood of the origin with two domains, each endowed with flow lines. These two domains and their flow lines were symmetric under R_2. So when we pass to $Z = z^2$, the two domains and their flow lines have the same image. However, the image of each domain should be seen in the 2-covering of Z-space punctured at 0, and the flow lines sit there. If two points $\hat{Z}_1$ and $\hat{Z}_2$ project on the same Z, then the projection of their flow lines intersect at Z. The angle is exactly the same as the one calculated in Section 6.1 between the two flow lines intersecting at any of the two preimages $\pm z$ of Z, as long as $Z \neq 0$.

As for the zero flow-line in Z coordinate, it ends at $Z = 0$. There is exactly one zero-flow line if and only if the unfolded cusp is conformally equivalent to a symmetric one.

The map F_ϵ is multi-valued with a ramification point at the origin. Indeed, far from the cusp point on the positive real axis it is well defined as a composition of the two Schwarz reflections with respect to two distinct parts of the curve. If Z makes a turn around the origin, then the two

Schwarz reflections coincide and exchange order in the composition, so that the analytic extension of F_ϵ is transformed into F_ϵ^{-1}.

6.4 A third geometric interpretation

Another geometric interpretation of the modulus has been described in [11] for positive values of ϵ. It has been shown there that the non triviality of the modulus implies the birth of periodic orbits of f_ϵ for sequences $\{\epsilon_n\}$ of bifurcation values of ϵ converging to the origin for which the fixed points of f_ϵ are resonant. If one makes "copies" of the curvilinear angle at one of the fixed points $\pm\sqrt{\epsilon}$ by Schwarz reflection with respect to one side of the angle and iterating, then all sides of the different angles must pass through these periodic points. (More details in [11].)

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