# The modulus of analytic classification for the unfolding of the codimension-one flip and Hopf bifurcations *

Waldo Arriagada-Silva, DMS Université de Montréal
Christiane Rousseau, DMS Université de Montréal

9 octobre 2010

**Abstract**

In this paper we study equivalence classes of generic 1-parameter germs of real analytic families $\mathcal{Q}_\varepsilon$ unfolding codimension 1 germs of diffeomorphisms $\mathcal{Q}_0 : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ with a fixed point at the origin and multiplier $-1$, under (weak) analytic conjugacy. These germs are generic unfoldings of the flip bifurcation. Two such germs are analytically conjugate if and only if their second iterates, $\mathcal{P}_\varepsilon = \mathcal{Q}_\varepsilon^{\circ 2}$, are analytically conjugate. We give a complete modulus of analytic classification: this modulus is an unfolding of the Ecalle modulus of the resonant germ $\mathcal{Q}_0$ with special symmetry properties reflecting the real character of the germ $\mathcal{Q}_\varepsilon$. As an application, this provides a complete modulus of analytic classification under weak orbital equivalence for a germ of family of planar vector fields unfolding a weak focus of order 1 (*i.e.* undergoing a generic Hopf bifurcation of codimension 1) through the modulus of analytic classification of the germ of family $\mathcal{P}_\varepsilon = \mathcal{Q}_\varepsilon^{\circ 2}$, where $\mathcal{P}_\varepsilon$ is the Poincaré first return map of the family of vector fields.

**Résumé**

Dans cet article, nous étudions la classification sous conjugaison analytique (faible) des germes de familles analytiques génériques à un 1 paramètre, $\mathcal{Q}_\varepsilon$, déployant des germes de difféomorphismes $\mathcal{Q}_0$ : $(\mathbb{R}, 0) \to (\mathbb{R}, 0)$ de codimension 1, ayant un point fixe à l'origine et de multiplicateur $-1$. Ces germes sont des déploiements génériques de la bifurcation de doublement de période. Deux germes sont analytiquement conjugués si et seulement si leurs itérés d'ordre 2, $\mathcal{P}_\varepsilon = \mathcal{Q}_\varepsilon^{\circ 2}$, sont

*This work was supported by NSERC in Canada.

analytiquement conjugués. On donne un module complet de classification analytique : ce module est un déploiement du module d'Écalle du germe résonant $\mathcal{Q}_0$ avec des propriétés de symétrie reflétant le caractère réel du germe $\mathcal{Q}_\varepsilon$. Ceci donne, comme application, un module complet de classification analytique sous équivalence orbitale faible pour un germe de famille de champs de vecteurs du plan ayant une bifurcation de Hopf générique de codimension 1 par le biais du module de classification analytique du germe de famille $\mathcal{P}_\varepsilon = \mathcal{Q}_\varepsilon^{\circ 2}$, où $\mathcal{P}_\varepsilon$ est l'application de premier retour de Poincaré de la famille de champs de vecteurs.

# Contents

# 1 Introduction.

This paper is part of a program to study the analytic classification of generic unfoldings of the simplest singularities of analytic dynamical systems. These dynamical systems can be germs of diffeomorphisms, in which case we study classification under conjugacy, or germs of vector fields, in which case we can study either classification under orbital equivalence or under conjugacy.

The analytic classification of unfoldings of singularities follows the analytic classification of the singularities themselves. The moduli of classification for the simplest 1-resonant singularities have been given by Ecalle, Voronin and Martinet-Ramis ([5], [19] and [12],[13]). Except for the case of the node of a planar vector field, the modulus space is a huge functional space, while the formal invariants are in finite number. This means that there is an infinite number of analytic obstructions for the analytic equivalence of two germs, that cannot be seen at the formal level.

These obstructions can be understood when first, one extends the underlying space from $\mathbb{R}^n$ to $\mathbb{C}^n$, $n = 1, 2$ and then, one unfolds the singularity. Indeed, in the simplest 1-resonance cases, the singularity of the dynamical system comes from the coallescence in a generic unfolding of the dynamical system of a finite number of hyperbolic singularities or special hyperbolic objects (like a periodic orbit or a limit cycle). Each hyperbolic object has its own geometric local model, and the modulus measures the limit of the mismatch of these local models. It is also a measure of the divergence of the normalizing series to the formal normal form. Hence, if a singularity is non equivalent to its formal normal form, then we should expect a mismatch between the local models near the two hyperbolic objects in the unfolding. This was the point of view suggested by Arnold and Martinet [11] and studied systematically by Glutsyuk [8] when the unfolding is considered only in certain conic regions of the parameter space considered in complex space. The treatment had to be adapted by Lavaurs and followers when the bifurcating objects were no more hyperbolic or when the domains of the local models did not intersect.

As far as codimension 1 singularities are concerned, the case of a germ of generic unfolding of a diffeormorphism with a double fixed point, also called *parabolic diffeomorphism* has been studied in [4] and [10], and the case of a germ of generic unfolding of a resonant diffeomorphism (one multiplier being a root of unity) has been studied in [16] and [15]. Germs of generic unfoldings of saddle-node (resp. resonant saddle) singularities of planar vector fields have been studied in [17] (resp. [16]). All these papers consider the unfolding of the corresponding complex singularity. The paper [15] also considers briefly the case of a saddle point of a real vector field. The modulus of the unfolding is always constructed in the same way. The formal normal form for the unfolding is identified and called the *model family.* The germ of family is then compared to the formal normal form on special domains. When one restricts to parameter values for which the special objects are hyperbolic, then these domains are neighborhoods of the special objects. For parameter values for which these neighborhoods intersect, the modulus is given by the comparison of the two normalizations over the intersection of the two domains. This is what is called the *Glutsyuk point of view* and the corresponding modulus is called the *Glutsyuk modulus.* In the papers [16], [4], [15] and [17], another point of view was used, called the *Lavaurs point of view.* The Lavaurs point of view allows to give a modulus for all values of the parameters. The domains on which we compare the germ of family to the formal normal form (model family) are no more neighborhoods of the special objects, but sectorial domains adjacent to two special objects. The corresponding *Lavaurs modulus* depends in a ramified way on the parameter. In particular, there are two different definitions of the Lavaurs modulus for the parameter values for which the Glutsyuk modulus can be defined.

In this paper, we are concerned with the real character of a germ $\mathcal{Q}_\varepsilon$ of an analytic family of diffeomorphisms with a flip bifurcation:

$$\mathcal{Q}_\varepsilon(x) = -x(1-\varepsilon) \pm x^3 + o(x^3).$$

We study how this is reflected in the modulus. So, we are especially interested in the real values of the parameter. In particular, for nonzero values of the parameter, we are in the Glutsyuk point of view. Hence, in this paper, we have decided to make a profound analysis of the Glutsyuk modulus for the case of the unfolding of a periodic diffeomorphism and to determine how the real character of the diffeomorphism is reflected in the modulus. For this reason, our study is restricted to the union of two sectors in the (complexified) parameter space which do not cover a full neighborhood of the origin. As an implication, we only obtain a modulus of classification under weak orbital equivalence.

Our paper was initially motivated by the study of the Hopf bifurcation. In [2], it is shown that two germs of analytic families of planar vector fields with a generic Hopf bifurcation of order 1 are orbitally analytically equivalent if and only if the germs of analytic families of their Poincaré maps are conjugate. The (unfolded) Poincaré map is exactly a real diffeomorphism $\mathcal{P}_\varepsilon : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, which is the second iterate of a semi-Poincaré map $\mathcal{Q}_\varepsilon$

with a flip bifurcation. Hence, through this result, the present paper provides a complete modulus of classification under weak orbital equivalence for germs of families undergoing a generic codimension-one Hopf bifurcation.

## 2 Preparation of the family.

We consider a germ of codimension-one real analytic diffeomorphism $\mathcal{Q}_0$ with a fixed point at the origin and a multiplier equal to $-1$. Under scaling of $x$ and removing the $x^2$-term by a normal form argument, such a germ of diffeomorphism has the form

$$\mathcal{Q}_0(x) = -x \mp \frac{1}{2}x^3 + ax^5 + o(x^5). \tag{2.1}$$

A generic 1-parameter unfolding is a germ of family of diffeomorphisms $Q_\eta(x) = Q(x,\eta)$ such that $\dfrac{\partial^2 Q}{\partial x \partial \eta}(0,0) \neq 0$.

It was shown in [16] that two generic families $Q_\eta$ unfolding germs of the form (2.1) are conjugate if and only if their second iterate $P_\eta = Q_\eta^{\circ 2}$ are conjugate. One direction is obvious. The other direction, namely proving that if the second iterates are conjugate, then the diffeomorphisms are also conjugate requires more work. The proof in [16] was done for complex germs. In the case of real analytic germs a better proof of the other direction is given in [2] for the case of families of real diffeomorphisms $Q_\eta$. Indeed, given any two generic families of real analytic diffeomorphims of the form $P_\eta = Q_\eta^{\circ 2}$, it is proved that they are realizable as Poincaré return maps of analytic vector fields unfolding a weak focus, which are analytically orbitally equivalent (a weak focus of a real vector field is a singular point with two pure imaginary eigenvalues and which is not a centre). Considering their blow-up and the holonomies of a well-chosen separatrix in the blow-up, these holonomies are in turn conjugate ([9]). But these holonomies are nothing else than the corresponding diffeomorphisms $Q_\eta$, so we will mainly discuss $P_\eta$.

Since the families $\mathcal{P}_\eta = \mathcal{Q}_\eta^{\circ 2}$ have real asymptotic expansion, then

$$\begin{array}{rcl} \mathcal{Q}_\eta & = & \mathcal{C} \circ \mathcal{Q}_{\mathcal{C}(\eta)} \circ \mathcal{C}, \\ \mathcal{P}_\eta & = & \mathcal{C} \circ \mathcal{P}_{\mathcal{C}(\eta)} \circ \mathcal{C}, \end{array} \tag{2.2}$$

where $\mathcal{C}$ is the standard complex conjugation $x \mapsto \overline{x}$. In this paper we consider real analytic families unfolding codimension-one diffeomorphisms of the form (2.1), and their second iterates. The following theorem is proved in [16] for a family of complex diffeomorphisms. Its proof can be adapted to respect the real character of $\mathcal{Q}_\eta$. For the sake of completeness we include the main steps here.

**Theorem 2.1** *Given a germ of diffeomorphims $\mathcal{Q}_0$ of the form* (2.1) *and a germ of generic unfolding $\mathcal{Q}_\eta$, there exists a germ of real analytic change of coordinate and parameter $(x,\eta)\mapsto(y,\varepsilon)$ conjugating the family $P_\eta=\mathcal{Q}_\eta^{\circ 2}$ to the prepared form*

$$\widetilde{P}_\varepsilon(y)=y+y(\varepsilon\pm y^2)(1+b(\varepsilon)+a(\varepsilon)y^2+y(\varepsilon\pm y^2)h(y,\varepsilon)), \tag{2.3}$$

*such that $\widetilde{P}_\varepsilon'(0)=\exp(\varepsilon)$. In particular, the parameter $\varepsilon$ is called canonical. It is an invariant. The formal invariant $A(\varepsilon)$ is defined implicitely through the expression*

$$\widetilde{P}_\varepsilon'(\pm\sqrt{-s\,\varepsilon})=\exp\left(-\frac{2\,\varepsilon}{1-sA(\varepsilon)\,\varepsilon}\right), \tag{2.4}$$

*where $A(\varepsilon)$ is real analytic, and $s=\pm1$ is an invariant defining two different cases which are not equivalents by real conjugacy.*

Proof. By the implicit function theorem we can suppose that $x=0$ is a fixed point for all $\eta$. By the Weierstrass-Malgrange preparation theorem, the other two fixed points of $P_\eta$ (which are periodic points of period 2 of $Q_\eta$) are the roots of $p_\eta(x)=x^2+\beta(\eta)x+\gamma(\eta)$, with $\gamma'(0)\neq0$ since the family is generic. Because it is a flip bifurcation, the periodic points of $Q_\eta$ can only coincide when they are equal to $x=0$. Hence, $\beta(\eta)\equiv0$. A reparametrization allows to take $\gamma(\eta)=\pm\eta_1$. Then, the map $\mathcal{P}_\eta$ has the form

$$P_{\eta_1}(x)=x+x(\eta_1\pm x^2)(b_1(\eta_1)+c_1(\eta_1)x+a_1(\eta_1)x^2+x(\eta_1\pm x^2)g(x,\eta_1)),$$

with $b_1(0)\neq0$. Since the fixed points $\pm\sqrt{\eta_1}$ are periodic points of $Q_{\eta_1}$ of order 2, then $\mathcal{P}_{\eta_1}'(\sqrt{\eta_1})=\mathcal{P}_{\eta_1}'(-\sqrt{\eta_1})$. Hence, $c_1(\eta_1)\equiv0$. Then $\mathcal{P}_{\eta_1}'(0)=1+\eta_1 b_1(\eta_1)$ with $b_1(0)\neq0$. An analytic change of parameter $\eta_1\mapsto\varepsilon$ allows to suppose that $\mathcal{P}_\varepsilon'(0)=\exp(\varepsilon)$. A corresponding scaling in $x$ (replacing $x$ by $y=c(\varepsilon)x$) allows to suppose that the fixed points of $\mathcal{P}_\varepsilon$ are given by $y(\varepsilon\pm y^2)=0$ and yields the prepared form. The analyticity of $A(\varepsilon)$, defined in (2.4), is well known and its real character is straightforward. $\square$

From now on, we will limit ourselves to prepared families $\mathcal{Q}_\varepsilon$ such that $\mathcal{P}_\varepsilon=\mathcal{Q}_\varepsilon^{\circ2}$ has the form

$$\mathcal{P}_\varepsilon(x)=x+x(\varepsilon+sx^2)(1+b(\varepsilon)+a(\varepsilon)x^2+x(\varepsilon+sx^2)h(x,\varepsilon)), \tag{2.5}$$

where $s = \pm 1$. We will mostly discuss the case $s = +1$. In particular, all the figures will be drawn only for this case. The case $s = -1$ is obtained through

$$x \mapsto ix.$$

This non-real change of coordinates exchanges the real and imaginary axes.

**Strategy.** The formal normal form of $\mathcal{P}_\varepsilon$ (also called "model family") is the time-one map $\tau_\varepsilon^1$ of the vector field

$$v_\varepsilon(x) = \frac{x(\varepsilon + sx^2)}{1 + A(\varepsilon)x^2} \frac{\partial}{\partial x}, \tag{2.6}$$

where $A(\varepsilon)$ is defined in (2.4). Notice that the real axis is invariant for real $\varepsilon$ . In order to compute the modulus of analytic classification of the Poincaré map, we compare it with $\tau_\varepsilon^1$ over specific sectorial domains of the parameter space.

# 3 The modulus of analytic classification.

In order to solve the conjugacy problem for germs of families of analytic diffeomorphisms (2.5), a complete modulus of analytic classification must be identified, so that two germs of families of analytic diffeomorphisms are analytically conjugate if and only they have the same modulus (Theorem 7.3). We shall see that this modulus is an unfolding of the Ecalle modulus for the germ of diffeomorphism at $\varepsilon = 0$, and so we recall the Ecalle modulus.

## 3.1 The Ecalle modulus of $\mathcal{Q}_0$.

Two germs of analytic diffeomorphisms of the form (2.1) with same sign before the cubic coefficient are real analytically conjugate if and only if they have the same formal invariant $A(\varepsilon)$ and the same orbit space. The Ecalle modulus is one way to describe the orbit space. To explain its construction we first remark that the diffeomorphism $\mathcal{Q}_0$ is topologically like the composition of $x \mapsto -x$ with the time-1/2 map of the vector field (2.6), whose flow lines appear in Figure 1, while the diffeomorphism $\mathcal{P}_0$ is topologically like the time-1 map of (2.6). So, we take a first *fundamental domain* (for $\mathcal{P}_0$) $C_1^+$ limited by a curve $\ell_1$ and its image $\mathcal{P}_0(\ell_1)$. If we identify $x \in \ell_1$ with its image $\mathcal{P}_0(x)$, the fundamental domain is conformally equivalent to a sphere $\mathbb{S}_1^+$. The ends of the crescent $C_1^+$ limited by $\ell_1$ and $\mathcal{P}_0(\ell_1)$ correspond to the

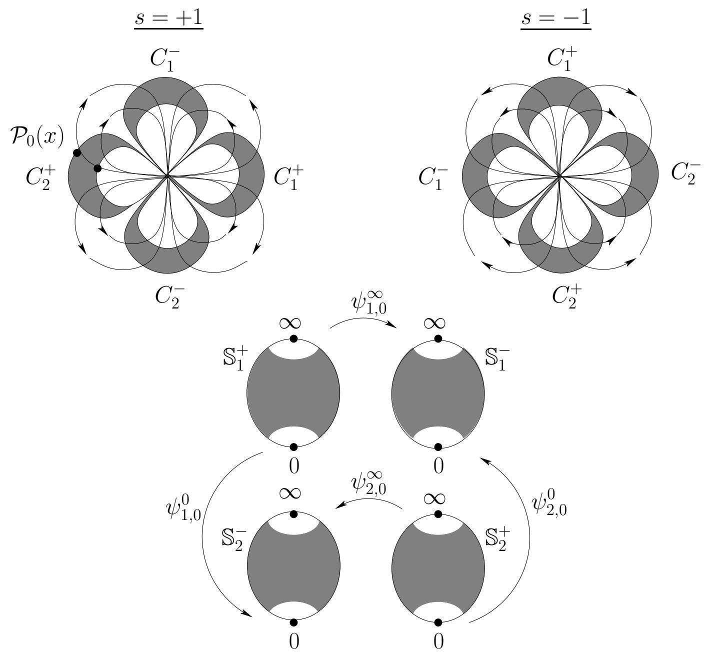


Figure 1: The flow of (2.6) in the cases $s = \pm 1$ and the Ecalle modulus of $\mathcal{Q}_0$.

points 0 and $\infty$ on the sphere. All orbits of $\mathcal{P}_0$ (except that of 0) are represented by a most one point of the sphere $\mathbb{S}_1^+$. However, there exists points in the neighborhood of 0 whose orbits have no representative on the sphere $\mathbb{S}_1^+$. To cover the whole orbit space we need to take three other fundamental neighborhoods $C_1^-, C_2^+, C_2^-$ limited by curves $\ell_j$ and their images $\mathcal{P}_0(\ell_j)$, $j = 2, 3, 4$, respectively. As before, we identify $x \in \ell_j$ with its image $\mathcal{P}_0(x)$ and the union of these fundamental domains is also conformally equivalent to a union of spheres $\mathbb{S}_1^-, \mathbb{S}_2^+, \mathbb{S}_2^-$. But there are points in the neighborhood of 0 (resp. $\infty$) which lie in different spheres but belong to the same orbit. So we need to identify a neighborhood of 0 (resp. $\infty$) with a neighborhood of 0 (resp. $\infty$) in two different spheres. This is done via a collection of analytic diffeomorphisms $\psi_1^0, \psi_2^0$ (resp. $\psi_1^\infty, \psi_2^\infty$) sending 0 to 0 (resp. $\infty$ to $\infty$), so that we get a non-Hausdorff topological manifold endowed with a system of analytic charts given by the collection of spheres glued at the poles by the maps $\psi_j^0$ and $\psi_j^\infty$. The size of the neighborhoods of 0 and $\infty$ depends on the size of the neighborhood of the origin where $\mathcal{P}_0$ is defined, but the germs of analytic diffeomorphims:

$$\begin{array}{lcl} \psi_1^0, \psi_2^0 & : & (\mathbb{C},0) \to (\mathbb{C},0) \\ \psi_1^\infty, \psi_2^\infty & : & (\mathbb{C},\infty) \to (\mathbb{C},\infty) \end{array}$$

are almost intrinsic as maps in the sphere $w$-coordinate. Indeed, the only analytic changes of coordinates on the spheres $\mathbb{S}_j^\pm$ preserving 0 and $\infty$ are the linear maps. If we choose different coordinates on $\mathbb{S}_j^\pm$ we get different germs $(\widehat{\psi}_j^0, \widehat{\psi}_j^\infty)$. The equivalence relation corresponding to changes of coordinates on $\mathbb{S}_j^\pm$ and preserving 0 and $\infty$ is given by

$$(\psi_1^0, \psi_2^0, \psi_1^\infty, \psi_2^\infty) \sim (\widehat{\psi}_1^0, \widehat{\psi}_2^0, \widehat{\psi}_1^\infty, \widehat{\psi}_2^\infty) \Longleftrightarrow \exists \;\; C_1^\pm, C_2^\pm \in \mathbb{C}:$$

$$\left\{ \begin{array}{lcl} \widehat{\psi}_1^0(w) & = & (C_2^-)^{-1} \cdot \psi_1^0(C_1^+ \cdot w), \\ \widehat{\psi}_2^0(w) & = & (C_1^-)^{-1} \cdot \psi_2^0(C_2^+ \cdot w), \end{array} \right. \left\{ \begin{array}{lcl} \widehat{\psi}_1^\infty(w) & = & (C_1^-)^{-1} \cdot \psi_1^\infty(C_1^+ \cdot w), \\ \widehat{\psi}_2^\infty(w) & = & (C_2^-)^{-1} \cdot \psi_2^\infty(C_2^+ \cdot w). \end{array} \right.$$

The identity $\mathcal{P}_0 = \mathcal{Q}_0^{\circ 2}$ is reflected by the fact that it is possible to choose representatives of the modulus such that

$$\left\{ \begin{array}{lcl} \psi_2^0(-w) & = & -\psi_1^0(w), \\ \psi_2^\infty(w) & = & -\psi_2^\infty(w) \end{array} \right.$$

(see Lemma 6.2 for a proof in the unfolded case).

**Definition 3.1** *The Ecalle-modulus of the diffeomorphism $\mathcal{P}_0$ is given by the tuple $(\psi_1^0, \psi_2^0, \psi_1^\infty, \psi_2^\infty)$, modulo the equivalence relation $\sim$ .*

Over a small neighborhood $\mathbb{D}_r$ of the origin (where $\mathbb{D}_r$ is the standard radius-$r$ open disk of the complex plane), the dynamics of $\mathcal{P}_0$ is given along the flow curves of the field (2.6). All the study of the family $\mathcal{P}_\varepsilon$ will be done over that fixed neighborhood $U = \mathbb{D}_r$ for sufficiently small values of $\varepsilon$ .

### 3.2 Glutsyuk point of view in the spherical coordinate.

If $\delta \in (0, \pi/2)$, we define sectorial domains in the universal covering of the the parameter space, see Figure 2:

$$\begin{array}{lcl} V_{\delta,l} & = & \{\varepsilon \in \mathbb{C} : |\,\varepsilon\,| < \rho, \arg(\varepsilon) \in (\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta)\} \\ V_{\delta,r} & = & \{\varepsilon \in \mathbb{C} : |\,\varepsilon\,| < \rho, \arg(\varepsilon) \in (-\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta)\} \end{array} \tag{3.1}$$

and $\rho$ is a small real number depending on $\delta$. We assume that $0 \in V_{\delta,lr}$ and denote

$$V_{\delta,lr}^* = V_{\delta,lr} \backslash \{0\}.$$

The number $\rho$ is chosen so that for values $\varepsilon \in V_{\delta,lr}$, there exist orbits connecting the fixed points in $U$. In this case, we say that we work in the Glutsyuk point of view (Figure 3). When $s = +1$, it is clear that:

- If $\varepsilon \in V^*_{\delta,l}$, the origin is an attractor and the two real singular points $x_\pm = \pm\sqrt{-\varepsilon}$ are repellers in $U$.
- If $\varepsilon = 0$, the origin is the only (non-hyperbolic) fixed point.
- If $\varepsilon \in V^*_{\delta,r}$, the origin is a repeller and two additional imaginary attracting singular points are created in $U$.

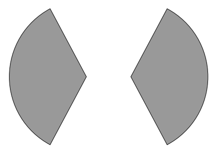


Figure 2: Sectorial domains for the parameter.

For $\varepsilon \in V^*_{\delta,lr}$, the family of diffeomorphisms $\mathcal{P}_\varepsilon$ can be conjugated to the time-one map $\tau^1_\varepsilon$ of the field (2.6) in the neighborhoods of each singular point, whose union is $\mathbb{D}_r$. The modulus measures the obstruction to get a conjugacy on the full neighborhood $\mathbb{D}_r$ in the $x$-space. The vector field (2.6) has singular points $x_0 = 0$, with eigenvalue $\mu_0(\varepsilon) = \varepsilon$, and $x_\pm = \pm\sqrt{-s\,\varepsilon}$ with eigenvalues:

$$\mu_\pm(\varepsilon) = \frac{-2\,\varepsilon}{1 - sA(\varepsilon)\,\varepsilon}. \tag{3.2}$$

Notice that $\mu_0$ and $\mu_\pm$ are analytic invariants of (2.6), which also depend analytically on $\varepsilon$. It follows that $\varepsilon$ and $A(\varepsilon)$ are analytic invariants of the field (2.6). The multipliers of the time-one map $\tau^1_\varepsilon$ of $v_\varepsilon$ are $\lambda_j = e^{\mu_j}$, *i.e.* they are precisely the multipliers of the fixed points of $\mathcal{P}_\varepsilon$. For $\varepsilon \in V^*_{\delta,lr}$, in order to compare $\mathcal{P}_\varepsilon$ with the model diffeomorphism $\tau^1_\varepsilon$ we compare their orbit space. The orbit space of $\mathcal{P}_\varepsilon$ is obtained by taking 3 closed curves $\{\ell_0, \ell_+, \ell_-\}$ around the fixed points, and their images $\{\mathcal{P}_\varepsilon(\ell_\#)\}$ where $\# \in \{0, +, -\}$. Since the fixed points are hyperbolic, the closed regions $\{C_\#\}$ between the curves and their images are isomorphic to three closed annuli. We identify $\ell_\# \sim \mathcal{P}_\varepsilon(\ell_\#)$. Then the quotient $C_\#/\sim$ will be shown to be a conformal torus. Hence, the orbit space turns out to be a non-Hausdorff space conformally equivalent to a quotient of the union of three

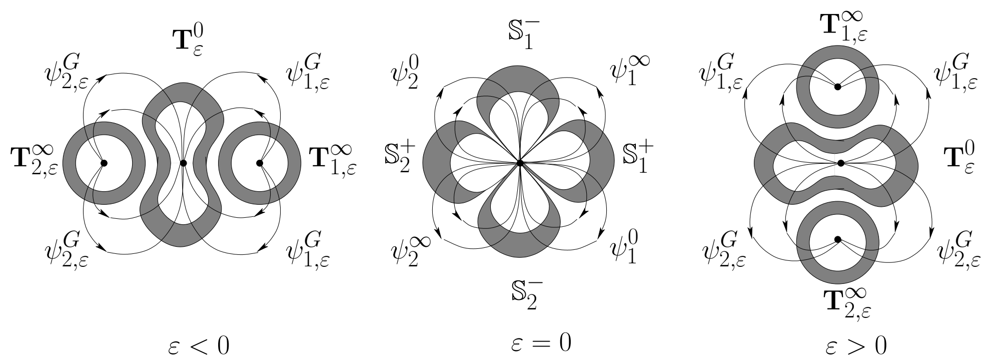


Figure 3: The orbit space of the Poincaré map.

tori $\mathbf{T}_\varepsilon^0, \mathbf{T}_{1,\varepsilon}^\infty, \mathbf{T}_{2,\varepsilon}^\infty$ plus the three singular points (which represent the orbit space of the hyperbolic fixed points), such that

– each orbit has at most one point in each torus,
– each orbit is either a fixed point or is represented in a torus,
– some orbits may have representatives in two different tori.

The Glutsyuk modulus consists in this identification of orbits. For this, we need to introduce (almost) intrinsic coordinates on the tori. One way to introduce coordinates on a torus $\mathbf{T}$ is to consider the latter as a quotient $\mathbf{T} = \mathbb{C}^*/\mathcal{L}_C$ (where $\mathcal{L}_C(x) = Cx$ is the linear map) for some $C \in \mathbb{C}^*$. Then a natural coordinate on $\mathbf{T}$ is the projection of a coordinate on $\mathbb{C}^* = \mathbb{CP}^1 \backslash \{0, \infty\}$, *i.e.* a "spherical" coordinate. In toric coordinates, the identification of orbits in two tori induce germs of families of analytic diffeomorphisms

$$\psi_{j,\varepsilon}^G : \mathbb{C}^* \mapsto \mathbb{C}^*$$

for $j \in \{1, 2\}$, see Figure 3, such that $\psi_{j,\varepsilon} \circ \mathcal{L}_{C_1} = \mathcal{L}_{C_2} \circ \psi_{j,\varepsilon}$ if $\psi_{j,\varepsilon}$ represents a map from $\mathbf{T}_1 = \mathbb{C}^*/\mathcal{L}_{C_1}$ to $\mathbf{T}_2 = \mathbb{C}^*/\mathcal{L}_{C_2}$.

# 4 Lifting of the dynamics.

Fatou coordinates were introduced in 1920 by former P. Fatou ([6]). They are changes of coordinates which allow to transform the prepared family $\mathcal{P}_\varepsilon$ into the "model family" $\tau_\varepsilon^1$ over the sectorial domains (3.1). We construct a special kind of Fatou coordinates: we show that it is possible to choose them respecting the real character of $\mathcal{P}_\varepsilon$. This choice yields a symmetry property on the Glutsyuk invariant in the unfolding.

Although we want to compare the map $\mathcal{P}_\varepsilon$ with its normal form, which is the time-one map of the vector field (2.6), it has been shown (cf. Shishikura [18]) that it is natural to change to the time coordinate of the simpler vector field
$$\dot{x} = x(\varepsilon + sx^2),$$
which is a "small deformation" of (2.6) over $\mathbb{D}_r$.

## 4.1 The unwrapping coordinate.

From now on, the parameter belongs to either of the Glutsyuk sectors (3.1). Consider the "unwrapping" change of coordinates $p_\varepsilon : \mathcal{R}_\varepsilon \to U\backslash\{x_0, x_\pm\}$ defined by:
$$x = p_\varepsilon(Z) = \begin{cases} \Big(\dfrac{s\,\varepsilon}{se^{-2\,\varepsilon\, Z} - 1}\Big)^{\frac{1}{2}}, & \text{for} \quad \varepsilon \neq 0, \\ \Big(-\dfrac{s}{2Z}\Big)^{\frac{1}{2}}, & \text{for} \quad \varepsilon = 0, \end{cases} \tag{4.1}$$
where $\mathcal{R}_\varepsilon$ is the 2-sheeted Riemann surface of the function (see Figure 4)
$$\begin{cases} \Big(\dfrac{1 - se^{-2\,\varepsilon\, Z}}{s\,\varepsilon}\Big)^{\frac{1}{2}}, & \text{for} \quad \varepsilon \neq 0, \\ (\dfrac{sZ}{2})^{\frac{1}{2}}, & \text{for} \quad \varepsilon = 0, \end{cases}$$
and $s = \pm 1$ is the sign of the third order coefficient of the family (2.5). Notice that for all $\varepsilon \in V_{\delta, lr}$, the map $p_\varepsilon$ is periodic with period
$$\alpha(\varepsilon) := -\frac{i\pi}{\varepsilon}, \tag{4.2}$$
that is,
$$p_\varepsilon(Z) = p_\varepsilon(Z - k\frac{i\pi}{\varepsilon}), \quad k \in \mathbb{Z}. \tag{4.3}$$
By (4.3), the image $p_\varepsilon^{\circ -1}(U = \mathbb{D}_r)$ consists in the Riemann surface $\mathcal{R}_\varepsilon$ minus a countable number of holes. The smaller the radius of $U$, the larger the radius of such holes (of order $\dfrac{1}{2r^2}$). Notice that the distance between two consecutive holes, for $\varepsilon \neq 0$, is equal to (4.2). Define the liftings:
$$\begin{aligned} \mathbf{P}_\varepsilon &:= p_\varepsilon^{-1} \circ \mathcal{P}_\varepsilon \circ p_\varepsilon, \\ \mathbf{Q}_\varepsilon &:= p_\varepsilon^{-1} \circ \mathcal{Q}_\varepsilon \circ p_\varepsilon. \end{aligned} \tag{4.4}$$

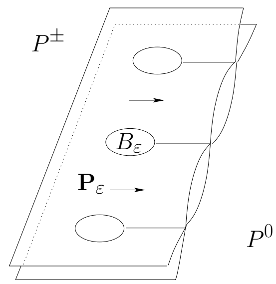


Figure 4: The surface $\mathcal{R}_\varepsilon$, domain of the lifting $\mathbf{P}_\varepsilon$.

By (4.3), the families $\mathbf{P}_\varepsilon, \mathbf{Q}_\varepsilon$ are defined on $\mathcal{R}_\varepsilon$ minus the countable collection of holes. The dynamics of the lifting goes always from left to right on $\mathcal{R}_\varepsilon$. We denote $P^0$ and $P^{\pm}$ the points at infinity located in the direction orthogonal to the line of holes, in such a way that their images by $p_\varepsilon$ be equal to $x_0 = 0$ and $x_{\pm} = \pm\sqrt{-s\,\varepsilon}$, respectively:

$$\begin{array}{rcl} P^0 & = & p_\varepsilon^{\circ -1}(x_0), \\ P^{\pm} & = & p_\varepsilon^{\circ -1}(x_{\pm}). \end{array} \tag{4.5}$$

In a neighborhood of the points $P^{\pm}$ (there are two such points, in correspondence with the leaves of $\mathcal{R}_\varepsilon$) the two sheets go to different singular points in the $x$-coordinate, while on the side of $P^0$ both sheets go to the origin, see Figure 4.

**Definition 4.1** *For any complex number $Z_\infty \in \mathbb{C}$ whose imaginary part is of order $\sim |\alpha|$ in a neighborhood of $P^{\pm}$, we define the translation in $T_{Z_\infty}$ :*

$$T_{Z_\infty}(\cdot) = Z_\infty + \cdot. \tag{4.6}$$

By (4.3), the sequence of equidistant holes can be denoted as:

$$\{T_{\alpha(\varepsilon)}^{\circ k}(B_\varepsilon)\}_{k\in\mathbb{Z}}, \tag{4.7}$$

where $T_{\alpha(\varepsilon)}^{0}(B_\varepsilon) = B_\varepsilon$ corresponds to the integer $k = 0$. It will be called the principal hole, and we will write:

$$\widehat{U}_\varepsilon := p_\varepsilon^{\circ -1}(U) = \mathcal{R}_\varepsilon \setminus \bigcup_{k\in\mathbb{Z}} T_{\alpha(\varepsilon)}^{\circ k}(B_\varepsilon) \tag{4.8}$$

the domain for the dynamics of $\mathbf{P}_\varepsilon, \mathbf{Q}_\varepsilon$. By connexity, the translation (4.6) can be analytically extended along the leaves of $\mathcal{R}_\varepsilon$ to all $Z$ in a neighborhood of the point $P^0$, see Figure 5. The extension is noted $T_{Z_\infty}$ as well.

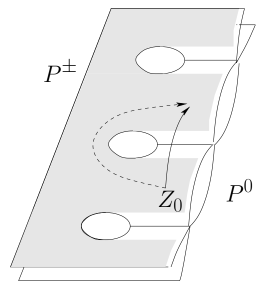


Figure 5: Analytic extension of $T_\alpha$ to a neighborhood of $P^0$, when $\varepsilon > 0$.

We shall use specific values for $Z_\infty$, the first one being $\alpha$, defined in (4.2). Indeed, the families $\mathbf{P}_\varepsilon$ and $\mathbf{Q}_\varepsilon$ commute with $T_\alpha$ :

$$\begin{aligned} \mathbf{P}_\varepsilon \circ T_\alpha = T_\alpha \circ \mathbf{P}_\varepsilon, \\ \mathbf{Q}_\varepsilon \circ T_\alpha = T_\alpha \circ \mathbf{Q}_\varepsilon \end{aligned} \tag{4.9}$$

along the leaves of $\mathcal{R}_\varepsilon$. Indeed, they do so near $P^{\pm}$. Since $T_\alpha$ is globally defined in $\mathcal{R}_\varepsilon$ by analytic continuation, they do so everywhere. Moreover, for small $\varepsilon$, $\mathbf{P}_\varepsilon$ is close to $T_1$ :

**Proposition 4.2** [*10*] *There exist $K > 0$ and $B > 0$, such that for $Z$ and $\varepsilon$ small, one has*

$$\begin{aligned} |\mathbf{P}_\varepsilon(Z) - Z - 1| < KB, \\ |\mathbf{P}'_\varepsilon(Z) - 1| < KB^2, \end{aligned} \tag{4.10}$$

*where $B$ depends on the size of the neighborhood $\mathbb{D}_r$ of the point $x = 0$.*

Notice that the inverse $p_\varepsilon^{\circ -1}$ of the change (4.1) is the multivalued function:

$$Z = p_\varepsilon^{\circ -1}(x) = \begin{cases} \dfrac{1}{2\,\varepsilon} \log\left(\dfrac{x^2}{\varepsilon + s x^2}\right), & \text{for} \quad \varepsilon \neq 0, \\ -\dfrac{s}{2x^2}, & \text{for} \quad \varepsilon = 0, \end{cases} \tag{4.11}$$

where $\log(\cdot)$ is the principal branch of the logarithm.

## 4.2 Glutsyuk point of view and translation domains.

We discuss the case $s = +1$. We will denote

$$\begin{aligned} \Re_\pm &= p_\varepsilon^{\circ -1}(\mathbb{R}_\pm), \\ \Im_\pm &= p_\varepsilon^{\circ -1}(i\mathbb{R}_\pm). \end{aligned} \tag{4.12}$$

By (4.3), there is a countable number of such semi-infinite segments on $\mathcal{R}_\varepsilon$, and by (4.11), $\Re_+$ and $\Re_-$ lie on the same side of $\mathcal{R}_\varepsilon$, but in different leaves. The same holds for $\Im_+$ and $\Im_-$, see Figure 6. The half-lines (4.12) are organized differently in the cases $\varepsilon \leq 0$ and $\varepsilon > 0$.

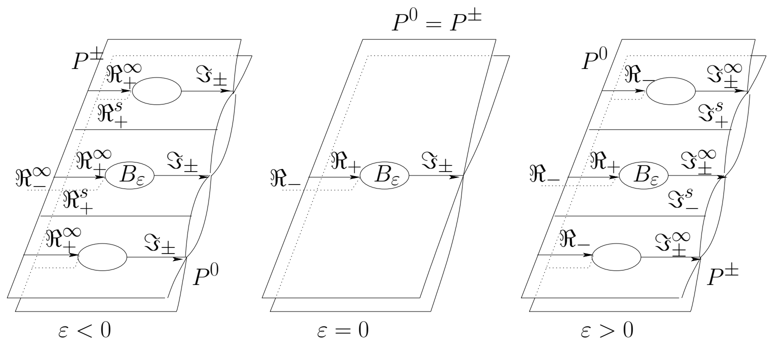


Figure 6: The choice of the cuts on $\mathcal{R}_\varepsilon$ for real values of the parameter.

*Should the parameter be negative,* the location of the fixed points in the $x$ coordinate yields the decomposition

$$\Re_\pm = \Re_\pm^s \cup \Re_\pm^\infty$$

on $\mathcal{R}_\varepsilon$, where $\Re_\pm^s$ is the image by $p_\varepsilon^{\circ -1}$ of the straight real segment joining 0 and $x_\pm$, and $\Re_\pm^\infty$ is the image by $p_\varepsilon^{\circ -1}$ of the straight real segment joining $x_\pm$ and the boundary of the neighborhood $U$ in the $x$ coordinate. Again, one has a countable number of such segments $\Re_\pm^{s,\infty}$ at distance $\alpha(\varepsilon)$ from each other in the $Z$ coordinate. The cuts are located along the half-lines $\Im_\pm$. The half-lines $\Re_\pm^\infty, \Im_\pm$ intersecting the principal hole $B_\varepsilon$ will be noted $\widehat{\Re}_\pm$ and $\widehat{\Im}_\pm$, respectively.

*In the case* $\varepsilon = 0$, there are four half-lines $\Re_\pm$ and $\Im_\pm$ in the $Z$ coordinate. They will be noted $\widehat{\Re}_\pm$ and $\widehat{\Im}_\pm$, respectively. The "hat" means that they intersect the hole $B_0$. The cuts are located along $\widehat{\Im}_\pm$.

*For positive values of the parameter,* on the contrary, the image of the imaginary axis by the map $p_\varepsilon^{\circ -1}$ consists in the union

$$\Im_\pm = \Im_\pm^s \cup \Im_\pm^\infty$$

on $\mathcal{R}_\varepsilon$, where $\Im_\pm^s$ is a countable collection consisting of the image of the straight imaginary segment joining 0 with $x_\pm$, and $\Im_\pm^\infty$ is an infinite collection consisting of the image of the straight imaginary segment joining $x_\pm$

and the boundary of the neighborhood $U$ in the $x$ coordinate. The cuts of $\mathcal{R}_\varepsilon$ are located along the half-lines $\Im_\pm^\infty$. The half-lines $\Re_\pm, \Im_\pm^\infty$ intersecting the principal hole $B_\varepsilon$ will be noted $\widehat{\Re}_\pm$ and $\widehat{\Im}_\pm$, respectively.

**Definition 4.3** *The distinguished line $\widehat{\Re}_\pm$ is called the symmetry axis in the $Z$ coordinate.*

**Translation domains.** Given any $\delta > 0$, there exists $\rho > 0$ such that for $|\,\varepsilon\,| < \rho$, there exists an orbit of the lifting $\mathbf{P}_\varepsilon$ connecting $P^0$ with $P^\pm$. In such a case, we say that we are in the "Glutsyuk point of view" of the dynamics.

A slanted line $\ell \subset \mathcal{R}_\varepsilon$, such that the image $\mathbf{P}_\varepsilon(\ell)$ is placed on the right of $\ell$ and the strip $\widehat{C}_\varepsilon(\ell)$ between $\ell$ and $\mathbf{P}_\varepsilon(\ell)$ belongs to $p_\varepsilon^{\circ-1}(U)$, is called an *admissible line.*

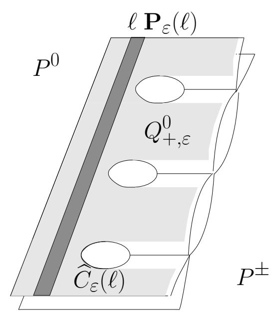


Figure 7: A translation domain $Q^0_{+,\varepsilon}$ and an admissible strip on it.

Let $\ell$ be an admissible line for $\mathbf{P}_\varepsilon$. The translation domain associated to $\ell$ is the set

$$Q_\varepsilon(\ell) = \{Z \in \widehat{U}_\varepsilon : \exists n \in \mathbb{Z}, \mathbf{P}_\varepsilon^{\circ n}(Z) \in \widehat{C}_\varepsilon(\ell), \forall i \in \{0, 1, ..., n\}, \mathbf{P}_\varepsilon^{\circ i}(Z) \in \widehat{U}_\varepsilon\}.$$

In the Glutsyuk point of view, the admissible strips are placed parallel to the line of holes, i.e. according to the $\alpha(\varepsilon)$ direction of the covering transformation $T_{\alpha(\varepsilon)}$. The induced translation domains, called *Glutsyuk translation domains,* are noted $Q_\varepsilon^\infty$ and $Q_\varepsilon^0$ according to whether they contain a neighborhood of $P^\pm$ or $P^0$, respectively, see Figure 7. Among other properties, $Q_\varepsilon(\ell)$ is a simply connected open subset of $\widehat{U}_\varepsilon$; the region $\widehat{C}_\varepsilon(\ell)\backslash\{\ell\}$ is a *fundamental domain* for the restriction of $\mathbf{P}_\varepsilon$ to $Q_\varepsilon(\ell)$ : each $\mathbf{P}_\varepsilon$-orbit in $Q_\varepsilon(\ell)$ has one and only one point in this set. For values of $\varepsilon$ in $V_{\delta,lr}$, there exist four different Glutsyuk translation domains $Q^{0;\infty}_{\pm,\varepsilon}$ in the $Z$-space, which are defined, depending on the sign of $\varepsilon \in \mathbb{R}$, as follows, see Figure 8.

a) If $\varepsilon < 0$, then $Q^{\infty}_{\pm,\varepsilon}$ is a simply connected neighborhood of $P^{\pm}$ containing all the segments $\Re^s_{\pm}$, while $Q^0_{\pm,\varepsilon}$ is a simply connected neighborhood of $P^0$ containing the distinguished half-line $\widehat{\Im}_{\pm}$.

b) If $\varepsilon > 0$, then $Q^{\infty}_{\pm,\varepsilon}$ is a simply connected neighborhood of $P^{\pm}$ containing all the segments $\Im^s_{\pm}$, while $Q^0_{\pm,\varepsilon}$ is a simply connected neighborhood of $P^0$ containing the distinguished half-line $\widehat{\Re}_{\pm}$.

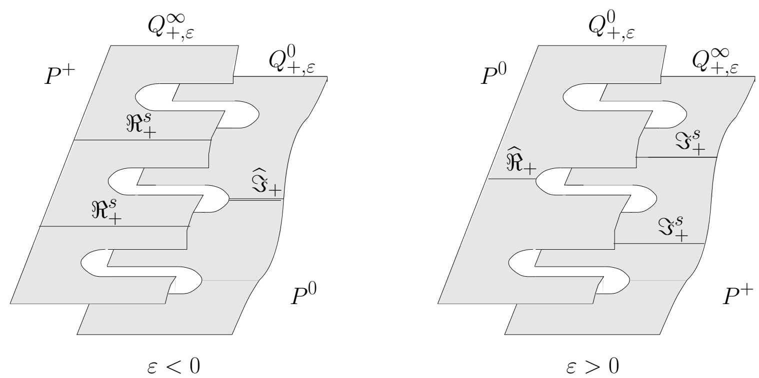


Figure 8: The translation domains $Q^{0,\infty}_{+,\varepsilon}$.

**Lemma 4.4** *The translation $T_\alpha$ satisfies:*

$$
\begin{aligned}
T_\alpha(Q^0_{\pm,\varepsilon}) &= Q^0_{\mp,\varepsilon},\\
T_\alpha(Q^\infty_{\pm,\varepsilon}) &= Q^\infty_{\pm,\varepsilon}.
\end{aligned}
\tag{4.13}
$$

Proof. The second is clear, by definition: $T_\alpha$ is formerly defined in a neighborhood of the point $P^{\pm}$ along the leaves of $\mathcal{R}_\varepsilon$, thus leaving invariant the translation domains $Q^\infty_{\pm,\varepsilon}$. On the other hand, the first equality is certainly true because all the possible paths defining the analytic extension of $T_\alpha$ to a neighborhood of $P^0$ must be contained in $Q^\infty_{\pm,\varepsilon}$. Let us consider for instance $Q^0_{+,\varepsilon}$ above the principal hole. It intersects $Q^\infty_{+,\varepsilon}$ and because of the definition of $T_\alpha$, when we apply $T_\alpha$ (resp. $T_{-\alpha}$) we are below the principal hole if $\varepsilon > 0$ (resp. $\varepsilon < 0$). In that region $Q^\infty_{+,\varepsilon}$ intersects $Q^0_{-,\varepsilon}$. Thus, each translation domain $Q^\infty_{\pm,\varepsilon}$ shares a common region with a translation domain of the kind $Q^0_{\pm,\varepsilon}$. The conclusion follows. $\square$

## 4.3 Conjugation in the $Z$ coordinate.

Choose $Z$ on $\mathcal{R}_\varepsilon$ and fix any simple arc $\Gamma$ joining $Z$ with the axis of symmetry $\widehat{\mathfrak{R}}$, and let $\gamma$ be its image under the map $p_\varepsilon : \gamma = p_\varepsilon(\Gamma)$. Consider the reflection $\overline{\gamma}$ of the path $\gamma$ with respect the real axis $\mathbb{R}$ in the $x$ coordinate. Then define

$$\overline{\Gamma} := p_\varepsilon^{\circ -1}(\overline{\gamma}).$$

**Definition 4.5** *The path $\overline{\Gamma}$ is well defined and is called the reflection of the arc $\Gamma$ with respect to the axis of symmetry $\widehat{\mathfrak{R}}$ in the $Z$ coordinate, see Figure 9. The starting point of $\overline{\Gamma}$ is called the conjugate of $Z$, and is noted $\mathsf{C}(Z)$.*

The conjugation $Z \mapsto \mathsf{C}(Z)$ is well defined: its definition is independent of the arc $\Gamma$. Indeed, if $\Gamma_+$ is any simple path joining $Z$ with the semi-axis of symmetry $\widehat{\mathfrak{R}}_+$ in the $Z$ coordinate, then the reflection of the arc $\Gamma_+$ with respect to $\widehat{\mathfrak{R}}_+$ induces a map

$$Z \mapsto \mathsf{C}_+(Z)$$

along the leaves of $\mathfrak{R}$, which is independent of the free homotopy class with endpoint on $\widehat{\mathfrak{R}}_+$. Choose now any simple arc $\Gamma_-$ joining the point $Z$ with the semi-axis of symmetry $\widehat{\mathfrak{R}}_-$. The reflection of the arc $\Gamma_-$ with respect to $\widehat{\mathfrak{R}}_-$ induces in turn a map

$$Z \mapsto \mathsf{C}_-(Z).$$

Then, it is easily seen that $\mathsf{C}_+(Z) = \mathsf{C}_-(Z)$. Indeed, the arc $\Gamma_+$ induces a path $\gamma_+$ in the $x$ coordinate whose reflection $\overline{\gamma_+}$ with respect the real axis starts at the same starting point as the reflection $\overline{\gamma_-}$ of the path $\gamma_-$ induced by the arc $\Gamma_-$ in the $x$ coordinate, see Figure 9.

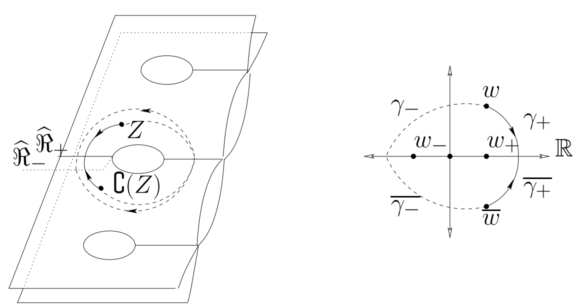


Figure 9: The conjugation in the $Z$ coordinate.

It becomes clear by definition that:

$$\mathtt{C} \circ \mathtt{C} = id \tag{4.14}$$

for real values of the parameter. Moreover, the families $\mathbf{P}_\varepsilon$ and $\mathbf{Q}_\varepsilon$ are invariant under the conjugation in the $Z$ coordinate when $\varepsilon \in \mathbb{R}$ :

$$\begin{aligned} \mathbf{P}_\varepsilon &= \mathtt{C} \circ \mathbf{P}_\varepsilon \circ \mathtt{C}, \\ \mathbf{Q}_\varepsilon &= \mathtt{C} \circ \mathbf{Q}_\varepsilon \circ \mathtt{C}. \end{aligned} \tag{4.15}$$

# 5 Real Fatou Glutsyuk coordinates.

## 5.1 Construction of Fatou coordinates.

**Theorem 5.1** *For values of the parameter in $V_{\delta,lr}$ it is possible to construct four different changes of coordinates $W = \Phi^{0,\infty}_{\pm,\varepsilon,lr}(Z)$ defined on $\mathcal{R}_\varepsilon$ and called Fatou coordinates, conjugating $\mathbf{P}_\varepsilon$ with the translation by one:*

$$\Phi^{0,\infty}_{\pm,\varepsilon,lr}(\mathbf{P}_\varepsilon(Z)) = \Phi^{0,\infty}_{\pm,\varepsilon,lr}(Z) + 1, \tag{5.1}$$

*for every $Z \in Q^{0,\infty}_{\pm,\varepsilon,lr} \cap \mathbf{P}^{\circ -1}_\varepsilon(Q^{0,\infty}_{\pm,\varepsilon,lr})$. These change of coordinates (see Figure 12) are associated with translation domains $Q^{0,\infty}_{\pm,\varepsilon,lr}$ whose admissible strips in $\mathcal{R}_\varepsilon$ lie in a direction parallel to the line of the holes $T^{\circ k}_{\alpha(\varepsilon)}(B_\varepsilon)$. Moreover, if we let $W \mapsto \mathcal{C}(W) := \overline{W}$ be the complex conjugation in the $W$ coordinate, then it is possible to construct these maps so that:*

– *For (real) negative values of the parameter they are related through:*

$$\begin{aligned} \Phi^0_{\pm,\varepsilon,l} &= \mathcal{C} \circ \Phi^0_{\mp,\varepsilon,l} \circ \mathtt{C}, \\ \Phi^\infty_{\pm,\varepsilon,l} &= \mathcal{C} \circ \Phi^\infty_{\pm,\varepsilon,l} \circ \mathtt{C}. \end{aligned} \tag{5.2}$$

– *For (real) positive values of $\varepsilon$ they satisfy:*

$$\begin{aligned} \Phi^0_{\pm,\varepsilon,r} &= \mathcal{C} \circ \Phi^0_{\pm,\varepsilon,r} \circ \mathtt{C}, \\ \Phi^\infty_{\pm,\varepsilon,r} &= \mathcal{C} \circ \Phi^\infty_{\mp,\varepsilon,r} \circ \mathtt{C}. \end{aligned} \tag{5.3}$$

Proof. The construction of the coordinates exists in the literature ([16]) but we wish to show additionally (5.2) and (5.3). So we will describe the construction when the parameter is real. Let $Q_\varepsilon(\ell)$ be a translation domain generated by an admissible line $\ell$ on the left side of the holes (real parameter). Thus, $\ell$ and the axis of symmetry $\widehat{\Re}$ are perpendicular. This

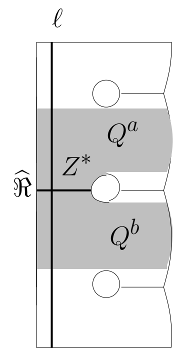


Figure 10: The distinguished curve $\widehat{\Re}$ separates the translation domain.

distinguished line $\widehat{\Re}$ separates the translation domains $Q_\varepsilon(\ell)$ in two connected symmetric regions $Q^a$ (the one above $\widehat{\Re}$) and $Q^b$ (the one below $\widehat{\Re}$), see Figure 10.

Then Equation (4.15) yields:

$$\mathbf{P}_\varepsilon(\widehat{\Re}) \subset \widehat{\Re}. \tag{5.4}$$

Let us write $Z^* = \ell \cap \widehat{\Re}$. Notice that points of $\ell$ can be written as $Z^* + iY$ for $Y \in \mathbb{R}$. Put $C_0 := \{(X,Y) \in \mathbb{R}^2 : 0 \leq X \leq 1\}$ and define $f_\varepsilon : C_0 \to \widehat{C}(\ell)$ as the convex combination

$$f_\varepsilon(X + iY) = (1 - X)(Z^* + iY) + X\mathbf{P}_\varepsilon(Z^* + iY),$$

which can be extended to all of $\mathbb{C}$ by asking that it commutes with $T_1$. It is shown ([16]) that for $Z = X + iY$,

$$\left| \frac{\partial f_\varepsilon}{\partial \overline{Z}} \Big/ \frac{\partial f_\varepsilon}{\partial Z} \right| < 1,$$

so $f_\varepsilon$ is a quasi-conformal map onto the strip $\widehat{C}(\ell)$ and it satisfies $f_\varepsilon^{-1}(\mathbf{P}_\varepsilon(Z)) = f_\varepsilon^{-1}(Z) + 1$ for every $Z \in \ell$. If we identify $\widehat{\Re}_\pm$ with $\mathbb{R}_\pm$, then $f_\varepsilon$ sends the interval $[0,1]$ into a real interval $[Z^*, \mathbf{P}_\varepsilon(Z^*)]$ and then the function defined as

$$\mu := f_\varepsilon^* \widehat{\mu}_0$$

(the pullback of the standard conformal structure $\widehat{\mu}_0$ of $\mathbb{C}$ on the strip $C_0$, defined by the 0 function) is a real measurable function which verifies (due to (5.4)):

$$\mu(Z) = \overline{\mu(\overline{Z})}$$

because of the symmetry of its domain (Schwarz reflection principle). The field $\mu$ is defined on $C_0$ and it is extended to all of $\mathbb{C}$ by $\mu = (\mathcal{T}_1^{\circ n})^*\mu$ on $\{Z = X + iY : -n \leq X \leq -n+1\}$, so the extended $\mu$ has norm $||\mu||_{L^\infty(\mathbb{C})} < 1$ and it is periodic of period 1. Thus, it is a Beltrami field on $\mathbb{C}$ still verifying $\mu(Z) = \overline{\mu(\overline{Z})}$ for all $Z \in \mathbb{C}$. The Ahlfors-Bers Theorem ([1]) yields the existence of a unique quasi-conformal map $g^\mu : \mathbb{C} \to \mathbb{C}$ normalized to $g^\mu(0) = 0$, with complex dilatation $\mu$, *i.e.* satisfying the Beltrami equation

$$g^\mu_{\overline{Z}}/g^\mu_Z = \mu,$$

that leaves $0, 1, \infty$ fixed, and such that $\mu = (g^\mu)^*\widehat{\mu}_0$. In addition, $g^\mu$ commutes with the translation $T_1$ ([16]). Indeed, the homeomorphism $G := g^\mu \circ T_1 \circ g^{\mu\circ-1}$ induces the identity on the sphere $\mathbb{S}^2$ and must thus, be a power of the deck transformation $T_1$ of the universal covering map $\mathcal{E}(\cdot) = e^{-2i\pi(\cdot)}$, namely: $G = T_1^{\circ m}$ for some $m \in \mathbb{Z}$. But $G(0) = g^\mu \circ T_1(0) = g^\mu(1) = 1$, which implies $m = 1$ and then $G = T_1$. Since

$$\mathcal{C} \circ \frac{(g^\mu)_{\overline{Z}}}{(g^\mu)_Z} \circ \mathcal{C} = \mathcal{C} \circ \mu \circ \mathcal{C} = \mu$$

and as $\mathcal{C} \circ (g^\mu)_Z \circ \mathcal{C} = (\mathcal{C} \circ g^\mu \circ \mathcal{C})_Z$ and $\mathcal{C} \circ (g^\mu)_{\overline{Z}} \circ \mathcal{C} = (\mathcal{C} \circ g^\mu \circ \mathcal{C})_{\overline{Z}}$, the map $\mathcal{C} \circ\ g^\mu \circ \mathcal{C}$ is another solution to the Beltrami equation, leaving the same points $0, 1, \infty$ fixed. By unicity of the solution, $\overline{g^\mu(\overline{Z})} = g^\mu(Z)$ for all $Z \in \mathbb{C}$. We define then $\phi : \widehat{C}(\ell) \to \mathbb{C}$ by

$$\phi = g^\mu \circ f_\varepsilon^{\circ-1}.$$

If $Z \in \ell$ one has $T_1 \circ \phi(Z) = \phi \circ \mathbf{P}_\varepsilon(Z)$ (because both $g^\mu$ and $f_\varepsilon$ commute with $T_1$) whence follows that $\phi$ can be extended in a map $\Phi_\varepsilon : Q \to \mathbb{C}$ by

$$\Phi_\varepsilon(Z) = \phi \circ \mathbf{P}_\varepsilon^{\circ n(Z)} - n(Z) \tag{5.5}$$

where $n(Z) \in \mathbb{Z}$ is such that $\mathbf{P}_\varepsilon^{\circ n(Z)}(Z) \in \widehat{C}(\ell)$. The map $\Phi_\varepsilon$ is a holomorphic diffeomorphism which depends analytically on the parameter and which verifies $\Phi_\varepsilon \circ \mathbf{P}_\varepsilon = T_1 \circ \Phi_\varepsilon$. Since $\phi(\widehat{\Re}) \subset \mathbb{R}$, we get

$$\Phi_\varepsilon(\widehat{\Re}) \subset \mathbb{R} \tag{5.6}$$

when the parameter is real. In addition, $Z \in dom(\Phi_\varepsilon)$ yields $\mathsf{C}(Z) \in dom(\Phi_\varepsilon)$, by definition of $\mathsf{C}$. Notice that in the case $\varepsilon < 0$ we have $\widehat{\Re}_\pm \subset$

$Q^{\infty}_{\pm,\varepsilon}$, while if $\varepsilon > 0$, $\widehat{\Re}_{\pm} \subset Q^{0}_{\pm,\varepsilon}$ (*i.e.* translations domains "on the right" do not contain the symmetry axis $\widehat{\Re}$, see Figure 8). Accordingly, if $\varepsilon < 0$ the diffeomorphism (5.5) is noted $\Phi^{\infty}_{\pm,\varepsilon} : Q^{\infty}_{\pm,\varepsilon} \to \mathbb{C}$, and (5.6) yields

$$\Phi^{\infty}_{\pm,\varepsilon,l} = \mathcal{C} \circ \Phi^{\infty}_{\pm,\varepsilon,l} \circ \mathsf{C}\,.$$

On the other hand, if $\varepsilon > 0$ the diffeomorphism (5.5) is noted $\Phi^{0}_{\pm,\varepsilon} : Q^{0}_{\pm,\varepsilon} \to \mathbb{C}$ and the invariance (5.6) implies

$$\Phi^{0}_{\pm,\varepsilon,r} = \mathcal{C} \circ \Phi^{0}_{\pm,\varepsilon,r} \circ \mathsf{C}\,.$$

For the case of a translation domain on the right, we first construct $\Phi^{0}_{+,\varepsilon,l}$ (when $\varepsilon < 0$) or $\Phi^{\infty}_{+,\varepsilon,r}$ (when $\varepsilon > 0$), and note that $\mathcal{C} \circ \Phi^{0}_{+,\varepsilon,l} \circ \mathsf{C}$ (resp. $\mathcal{C} \circ \Phi^{\infty}_{+,\varepsilon,r} \circ \mathsf{C}$) is again a Fatou coordinate when $\varepsilon < 0$ (resp. when $\varepsilon > 0$). Then we define

$$\Phi^{0}_{-,\varepsilon,l} = \mathcal{C} \circ \Phi^{0}_{+,\varepsilon,l} \circ \mathsf{C}$$

for $\varepsilon < 0$, and

$$\Phi^{\infty}_{-,\varepsilon,l} = \mathcal{C} \circ \Phi^{\infty}_{+,\varepsilon,r} \circ \mathsf{C}$$

if $\varepsilon > 0$ and the construction is done. $\square$

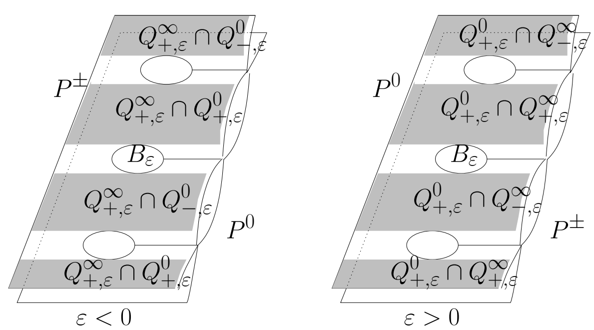


Figure 11: The non-connected intersection of the translation domains.

**Definition 5.2** *Fatou coordinates in Theorem 5.1 are called admissible Real Fatou Glutsyuk coordinates. Theorem 5.1 shows that the symmetry axis* $\widehat{\Re}$ *is invariant under Real Fatou coordinates when the parameter is real.*

*Remarks.*

1. Although Real Fatou Glutsyuk changes of coordinates always exist for $\varepsilon \in V_{\delta,lr}$, the curve $\widehat{\Re}$ is not invariant if $\varepsilon \notin \mathbb{R}$.
2. The subscripts $l, r$ will be dropped when the context allows no confusion.
3. As we will see, the modulus compares the Fatou coordinates on the left and on the right over the intersection of the left and right translations domains. If $\varepsilon \neq 0$, the geometry of $\mathcal{R}_\varepsilon$ yields that the intersection of right and left translations domains is composed of a countable alternating sequence of horizontal strips, see Figure 11.

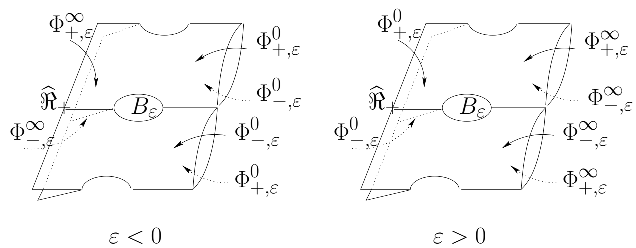


Figure 12: The Real Glutsyuk coordinates around the principal hole.

The remark above yields the organization of the domains of definition for the different Real Glutsyuk coordinates. Due to periodicity, it suffices to describe only these domains around the fundamental hole $B_\varepsilon$, see Figure 12.

**Proposition 5.3** *If $\Phi^1_\varepsilon$ and $\Phi^2_\varepsilon$ are two Fatou Glutsyuk coordinates solving* (5.1) *on the same translation domain, then there exists $C_\varepsilon \in \mathbb{C}$, such that*

$$\Phi^2_\varepsilon(Z) = C_\varepsilon + \Phi^1_\varepsilon(Z).$$

*In particular, for every $Z_0(\varepsilon) \in \mathcal{R}_\varepsilon$ there is a unique Fatou coordinate $\Phi_\varepsilon$ satisfying $\Phi_\varepsilon(Z_0(\varepsilon)) = 0$. Also, it is possible to construct admissible Real Fatou Glutsyuk coordinates in Theorem 5.1 so that they depend analytically on $\varepsilon \in V_{\delta,lr}$ and so that they have the same limit at $\varepsilon = 0$.*

Proof. Since $\Phi^1_\varepsilon, \Phi^2_\varepsilon$ satisfy (5.1) they are related by $\Phi^2_\varepsilon \circ (\Phi^1_\varepsilon)^{\circ -1}(Z+1) = \Phi^2_\varepsilon \circ (\Phi^1_\varepsilon)^{\circ -1}(Z) + 1$, whence the composition $\Phi^2_\varepsilon \circ (\Phi^1_\varepsilon)^{\circ -1}$ is a translation $T_{C_\varepsilon}$. Besides, it suffices to take the base point $Z_0(\varepsilon)$ depending analytically on $\varepsilon$ and with continous limit at $\varepsilon = 0$. □

The choice of the base point $Z_0(\varepsilon)$ provides a degree of freedom in the choice of the Fatou Glutsyuk coordinate. Since there are four Fatou Glutsyuk coordinates we have four degrees of freedom. Later, we shall use 3 of these degrees of freedom to "normalize" these Fatou coordinates. We will take the normalizations on $V_{\delta,l}$ and $V_{\delta,r}$ in such a way that we will get the same limit at $\varepsilon = 0$.

## 5.2 Normalization of Real Fatou Glutsyuk coordinates.

The family $\mathcal{P}_\varepsilon$ is, by definition, the second iterate of a family of germs of diffeomorphisms $\mathcal{Q}_\varepsilon$ unfolding the map $\mathcal{Q}_0$, which is tangent to $-id$. This implies that the orbits of the family $\mathcal{Q}_\varepsilon$ form a 180°-degrees alternating sequence along the orbits of the prepared family of fields at each iteration (*i.e.* the points $w$ and $\mathcal{Q}_\varepsilon(w)$ stand on opposite sides of the origin, see Figure 13). In other words, the lifting $\mathbf{Q}_\varepsilon$ exchanges the two leaves.

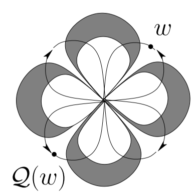


Figure 13: The "jumps" of the orbits of $\mathcal{Q}$ in the case $\varepsilon = 0$.

The fact that the family of diffeomorphisms $\mathcal{P}_\varepsilon$ is a square (namely, $\mathcal{P}_\varepsilon = \mathcal{Q}_\varepsilon^{\circ 2}$) is now exploited. For every $W = \Phi_\varepsilon(Z)$, the map:

$$\mathcal{T}_W(\cdot) = W + \cdot \tag{5.7}$$

is called the translation in $W \in \mathbb{C}$.

**Lemma 5.4** *For each $\varepsilon \in V_\delta^G$, it is possible to construct admissible Real Fatou Glutsyuk coordinates depending analytically on $\varepsilon \in V_{\delta,lr}$, with continuous limit at $\varepsilon = 0$ and such that they are related through:*

$$\begin{aligned} \Phi^0_{\pm,\varepsilon} \circ \mathbf{Q}_\varepsilon &= \mathcal{T}_{\frac{1}{2}} \circ \Phi^0_{\mp,\varepsilon}, \\ \Phi^\infty_{\pm,\varepsilon} \circ \mathbf{Q}_\varepsilon &= \mathcal{T}_{\frac{1}{2}} \circ \Phi^\infty_{\mp,\varepsilon}. \end{aligned} \tag{5.8}$$

Proof. For each $\varepsilon$, the map $\mathcal{Q}_\varepsilon$ commutes with $\mathcal{P}_\varepsilon$. Hence $\mathbf{Q}_\varepsilon = p_\varepsilon^{-1} \circ \mathcal{Q}_\varepsilon \circ p_\varepsilon$ commutes with $\mathbf{P}_\varepsilon$. Let the pairs of Real Fatou Glutsyuk coordinates $\Phi^{0,\infty}_{+,\varepsilon}, \Phi^{0,\infty}_{-,\varepsilon}$ be constructed as in the proof of Theorem 5.1. Then:

$$\Phi^{0,\infty}_{\pm,\varepsilon}(\mathbf{P}_\varepsilon(\mathbf{Q}_\varepsilon(Z))) = \Phi^{0,\infty}_{\pm,\varepsilon}(\mathbf{Q}_\varepsilon(Z)) + 1 = (\Phi^{0,\infty}_{\pm,\varepsilon} \circ \mathbf{Q}_\varepsilon)(\mathbf{P}_\varepsilon(Z)), \tag{5.9}$$

the first equality being consequence of the fact that $\Phi^{0,\infty}_{\pm,\varepsilon}$ is a solution to (5.1), and the second is true because $\mathbf{P}_\varepsilon$ and $\mathbf{Q}_\varepsilon$ commute. Equation (5.9) says that $\Phi^{0,\infty}_{\pm,\varepsilon} \circ \mathbf{Q}_\varepsilon$ is a Fatou Glutsyuk coordinate. By the remark above, the latter is defined on the same translation domain as $\Phi^{0,\infty}_{\mp,\varepsilon}$. Hence, according to Proposition 5.3, there exists $C^{0,\infty}_{\pm,\varepsilon} \in \mathbb{C}$ with the following property:

$$\Phi^{0,\infty}_{\pm,\varepsilon} \circ \mathbf{Q}_\varepsilon = \mathcal{T}_{C^{0,\infty}_{\pm,\varepsilon}} \circ \Phi^{0,\infty}_{\mp,\varepsilon}. \tag{5.10}$$

We will drop the subscript $\varepsilon$ in the constants. Using $\mathbf{Q}_\varepsilon^{\circ 2} = \mathbf{P}_\varepsilon$ and iterating (5.10) yields:

$$\begin{array}{rcl} \Phi^{0,\infty}_{\pm,\varepsilon}(Z) + 1 & \equiv & \Phi^{0,\infty}_{\pm,\varepsilon} \circ \mathbf{P}_\varepsilon(Z) \\ & = & (\Phi^{0,\infty}_{\pm,\varepsilon} \circ \mathbf{Q}_\varepsilon) \circ \mathbf{Q}_\varepsilon(Z) \\ & = & \mathcal{T}_{C^{0,\infty}_{\pm}} \circ (\Phi^{0,\infty}_{\mp,\varepsilon} \circ \mathbf{Q}_\varepsilon)(Z) \\ & = & \mathcal{T}_{C^{0,\infty}_{\pm}} \circ \mathcal{T}_{C^{0,\infty}_{\mp}} \circ \Phi^{0,\infty}_{\pm,\varepsilon}(Z) \\ & = & \Phi^{0,\infty}_{\pm,\varepsilon}(Z) + C^{0,\infty}_{\pm} + C^{0,\infty}_{\mp}, \end{array} \tag{5.11}$$

which means

$$C^{0,\infty}_{+} + C^{0,\infty}_{-} = 1. \tag{5.12}$$

We want to prove that it is possible to choose the Fatou coordinates so that $C^{0,\infty}_{+} = C^{0,\infty}_{-} = 1/2$. That is consequence of $\mathbf{Q}_\varepsilon = \mathtt{C} \circ \mathbf{Q}_\varepsilon \circ \mathtt{C}$ when $\varepsilon \in \mathbb{R}$. Indeed, in the case $\varepsilon < 0$, Equation (5.10) and Theorem 5.1 yield

$$\begin{array}{rcl} \mathcal{T}_{C^0_\pm} \circ \Phi^0_{\mp,\varepsilon} & = & (\mathcal{C} \circ \Phi^0_{\mp,\varepsilon} \circ \mathtt{C}) \circ \mathbf{Q}_\varepsilon \\ & = & \mathcal{C} \circ (\Phi^0_{\mp,\varepsilon} \circ \mathbf{Q}_\varepsilon) \circ \mathtt{C} \\ & = & \mathcal{C} \circ (\mathcal{T}_{C^0_\mp} \circ \Phi^0_{\pm,\varepsilon}) \circ \mathtt{C} \\ & = & \mathcal{C} \circ \mathcal{T}_{C^0_\mp} \circ \mathcal{C} \circ \Phi^0_{\mp,\varepsilon}. \end{array} \tag{5.13}$$

Hence $\overline{C^0_\mp} = C^0_\pm$, and then $Re(C^0_+) = Re(C^0_-)$. We show now that a "correction" is possible by using the degree of freedom, so that $C^0_\pm$ can be taken real (for every $\varepsilon$), while, at the same time, respecting (5.2). If we change the coordinates by

$$\begin{array}{rcl} \Phi^0_{+,\varepsilon} & \mapsto & \mathcal{T}_K \circ \Phi^0_{+,\varepsilon} \\ \Phi^0_{-,\varepsilon} & \mapsto & \mathcal{T}_{\overline{K}} \circ \Phi^0_{-,\varepsilon} \end{array}$$

in (5.10), for $K \in i\mathbb{R}$ to be chosen, then (5.2) remains valid and we get the equations:

$$\begin{aligned}(\mathcal{T}_K \circ \Phi^0_{+,\varepsilon}) \circ \mathbf{Q}_\varepsilon &= \mathcal{T}_{K+C^0_+-\overline{K}} \circ (\mathcal{T}_{\overline{K}} \circ \Phi^0_{-,\varepsilon}) \\ (\mathcal{T}_{\overline{K}} \circ \Phi^0_{-,\varepsilon}) \circ \mathbf{Q}_\varepsilon &= \mathcal{T}_{\overline{K}+C^0_--K} \circ (\mathcal{T}_K \circ \Phi^0_{+,\varepsilon}).\end{aligned}$$

Put $\widehat{C}^0_+ = K + C^0_+ - \overline{K}$ and $\widehat{C}^0_- = \overline{K} + C^0_- - K$. The choice

$$K = -i\frac{Im(C^0_+)}{2} = i\frac{Im(C^0_-)}{2} \in i\mathbb{R}$$

ensures that $\widehat{C}^0_+ = \widehat{C}^0_- = Re(C^0_+) = Re(C^0_-) = 1/2$.

As for the coordinate $\Phi^\infty_\pm$, the proof is straightforward. Indeed, (5.10) and Theorem 5.1 yield this time:

$$\begin{aligned}\mathcal{T}_{C^\infty_\pm} \circ \Phi^\infty_{\mp,\varepsilon} &= (\mathcal{C} \circ \Phi^\infty_{\pm,\varepsilon} \circ \mathsf{C}) \circ \mathbf{Q}_\varepsilon \\ &= \mathcal{C} \circ (\Phi^\infty_{\pm,\varepsilon} \circ \mathbf{Q}_\varepsilon) \circ \mathsf{C} \\ &= \mathcal{C} \circ (\mathcal{T}_{C^\infty_\pm} \circ \Phi^\infty_{\mp,\varepsilon}) \circ \mathsf{C} \\ &= \mathcal{C} \circ \mathcal{T}_{C^\infty_\pm} \circ \mathcal{C} \circ \Phi^\infty_{\mp,\varepsilon}\end{aligned}$$

(compare to (5.13)), thus $C^\infty_\pm = \overline{C^\infty_\pm}$ and $C^\infty_\pm \in \mathbb{R}$. So we can perform a change $\Phi^\infty_{\pm,\varepsilon} \mapsto \mathcal{T}_{K_\pm} \circ \Phi^\infty_{\pm,\varepsilon}$, where $K_\pm = -\dfrac{C^\infty_\pm}{2} \in \mathbb{R}$, in order to bring $C^\infty_+ = C^\infty_- = 1/2$, respecting (5.2).

The case $\varepsilon > 0$ is completely analogous, using (5.3). □

**Definition 5.5** *When Real Fatou Glutsyuk coordinates satisfy* (5.8)*, we shall say that they are normalized.*

## 5.3 Real Fatou Glutsyuk coordinates and translations.

Consider the numbers:

$$\begin{aligned}\alpha_0(\varepsilon) &= \frac{2\pi i}{\mu_0(\varepsilon)} = \frac{2\pi i}{\varepsilon} \\ \alpha_\infty(\varepsilon) &= \frac{2\pi i}{\mu_\pm(\varepsilon)} = -\frac{\pi i(1 - sA(\varepsilon)\,\varepsilon)}{\varepsilon},\end{aligned} \tag{5.14}$$

where $\mu_0(\varepsilon) = \log \mathcal{P}'_\varepsilon(0) = \varepsilon$, and $\mu_\pm(\varepsilon) = \log \mathcal{P}'_\varepsilon(x_\pm) = \dfrac{-2\,\varepsilon}{1 - sA(\varepsilon)\,\varepsilon}$ are the eigenvalues of $v_\varepsilon$ at the singular points $x_0 = 0$ and $x_\pm = \pm\sqrt{-s\,\varepsilon}$, respectively. As usual, we will only describe the case $s=+1$. In the case $s=-1$, each picture in the Figure 3 must be rotated by $90^\circ$ degrees in the clockwise direction and, moreover, the family $\mathcal{P}_\varepsilon^{-1}$ is of the form (2.3).

**Definition 5.6** *The Glutsyuk normalization domains are*

$$U_\varepsilon^{0,\infty} := p_\varepsilon(Q_{\pm,\varepsilon}^{0,\infty}).$$

**Lemma 5.7** *The quotients $U_\varepsilon^0/\mathcal{P}_\varepsilon$ and $U_\varepsilon^\infty/\mathcal{P}_\varepsilon$ are conformally equivalent to non-separated spaces*

$$\mathbf{T}_\varepsilon^0 \cup \{x_0\}, \quad \mathbf{T}_{\pm,\varepsilon}^\infty \cup \{x_\pm\}$$

*which are the union of a point with complex tori $\mathbf{T}_\varepsilon^0$ and $\mathbf{T}_{\pm,\varepsilon}^\infty$, of modulus $\alpha_0(\varepsilon)$ and $\alpha_\infty(\varepsilon)$, respectively.*

Proof. Indeed take, for instance, the fixed point $x_+=\sqrt{-\varepsilon}$. Since we are in the Glutsyuk point of view of the dynamics, on $U_\varepsilon^\infty$ the map $\mathcal{P}_\varepsilon$ admits $x_+$ as a global hyperbolic point. Consider any loop $\gamma$ around $x_+$ and consider its image $\mathcal{P}_\varepsilon(\gamma)$ as well. The region $J$ of the complex plane between these two curves is a fundamental domain (*i.e.* a domain where each orbit of $\mathcal{P}_\varepsilon$ is represented by at most one point) for the dynamics around $x_+$. It is easily seen that

$$U_\varepsilon^\infty/\mathcal{P}_\varepsilon \simeq J/\mathcal{P}_\varepsilon \cup \{x_+\}$$

(they are conformally equivalent). Moreover, we can change $J$ by any iterate $\mathcal{P}_\varepsilon^{\circ n}(J)$ in the quotient, and the resulting space remains the same. By the Poincaré Theorem, the map $\mathcal{P}_\varepsilon$ is linearizable around $x_+$. As $n\to\infty$, the modulus of the quotient complex torus $\mathcal{P}_\varepsilon^{\circ n}(J)/\mathcal{P}_\varepsilon$ converges towards the modulus of the torus $\mathbb{C}^*/\mathcal{L}_{\mu_+(\varepsilon)}$ which is given by $\alpha_\infty=\dfrac{2i\pi}{\mu_+(\varepsilon)}$. Inasmuch as the space $\mathcal{P}_\varepsilon^{\circ n}(J)/\mathcal{P}_\varepsilon\cup\{x_+\}$ is conformally equivalent to $U_\varepsilon^\infty/\mathcal{P}_\varepsilon$, the latter is the union of a complex torus $\mathbf{T}_{+,\varepsilon}^\infty$ of modulus $\alpha_\infty$, and the singular point $\{x_+\}$. This space is non-separated because the point $\{x_+\}$ belongs to the adherence of any orbit of $\mathcal{P}_\varepsilon$. The proofs for $x_-$ and $x=0$ are analogous. $\square$

**Proposition 5.8** *For all $\varepsilon\in V_{\delta,lr}$, it is possible to choose nomalized Real Fatou Glutsyuk coordinates $\Phi_{\pm,\varepsilon}^{0,\infty}: Q_{\pm,\varepsilon}^{0,\infty}\to\mathbb{C}$ satisfying (in addition to (5.2),(5.3) and(5.8)) the equations:*

$$\begin{aligned}\Phi_{\pm,\varepsilon}^0\circ T_\alpha &= \mathcal{T}_{-\frac{\alpha_0}{2}}\circ\Phi_{\mp,\varepsilon}^0,\\ \Phi_{\pm,\varepsilon}^\infty\circ T_\alpha &= \mathcal{T}_{\alpha_\infty}\circ\Phi_{\pm,\varepsilon}^\infty.\end{aligned}\tag{5.15}$$

*In particular, they have the same limit at $\varepsilon=0$.*

PROOF. Consider the translation $T_\alpha$ and a Real Fatou Glutsyuk coordinate $\Phi^\infty_{\pm,\varepsilon} : Q^\infty_{\pm,\varepsilon} \to \mathbb{C}$. By (4.9):

$$\begin{aligned}\Phi^\infty_{\pm,\varepsilon} \circ T_\alpha \circ \mathbf{P}_\varepsilon &= \Phi^\infty_{\pm,\varepsilon} \circ \mathbf{P}_\varepsilon \circ T_\alpha \\ &= \mathcal{T}_1 \circ \Phi^\infty_{\pm,\varepsilon} \circ T_\alpha,\end{aligned}$$

which implies that $\Phi^\infty_{\pm,\varepsilon} \circ T_\alpha$ is a Fatou coordinate. By Lemma 4.4, the latter preserves the translation domains $Q^\infty_{\pm,\varepsilon}$ and then, $\Phi^\infty_{\pm,\varepsilon} \circ T_\alpha$ and $\Phi^\infty_{\pm,\varepsilon}$ are defined on the same translation domain. By Proposition 5.3, there exist constants $C_{\pm,\varepsilon}$ such that:

$$\Phi^\infty_{\pm,\varepsilon} \circ T_\alpha = \mathcal{T}_{C_{\pm,\varepsilon}} \circ \Phi^\infty_{\pm,\varepsilon}. \tag{5.16}$$

Thus, the Fatou coordinate conjugates the pair of commuting diffeomorphisms $\{\mathbf{P}_\varepsilon, T_\alpha\}$ with the pair of translations $\{\mathcal{T}_1, \mathcal{T}_{C_{\pm,\varepsilon}}\}$. Moreover, the Fatou Glutsyuk coordinate induces a holomorphic diffeomorphism:

$$Q^\infty_{\pm,\varepsilon}/\{\mathbf{P}_\varepsilon, T_\alpha\} \cong \mathbb{C}/\{\mathcal{T}_1, \mathcal{T}_{C_{\pm,\varepsilon}}\}$$

between complex surfaces. The latter is, of course, the canonical torus $\mathbb{C}/(\mathbb{C} \times C_{\pm,\varepsilon}\mathbb{C})$. Notice that the quotient $Q^\infty_{\pm,\varepsilon}/T_\alpha$ coincides with the neighborhood $U^\infty_\varepsilon$ with coordinate $x$, where the map $\mathcal{P}_\varepsilon$ is induced by $\mathbf{P}_\varepsilon$. Hence, the quotient $Q^\infty_{\pm,\varepsilon}/\{\mathbf{P}_\varepsilon, T_\alpha\}$ is conformally equivalent to $U^\infty_\pm/\mathcal{P}_\varepsilon$. On the other hand, the translation $T_\alpha$ has been formerly defined on a neighborhood of the points $P^\pm$, thus the positive orientation of the translation $\mathcal{T}_{\alpha_\infty}$ in the $W$ (Fatou) coordinate coincides with the positive orientation of $T_\alpha$, by definition. By (5.16) and Lemma 5.7, the modulus of the torus $\mathbb{C}/\{\mathcal{T}_1, \mathcal{T}_{C_\varepsilon}\}$, *i.e.* the constants $C_{\pm,\varepsilon}$, coincide and must be equal to $\alpha_\infty$ on $Q^\infty_{\pm,\varepsilon}$ :

$$\Phi^\infty_{\pm,\varepsilon} \circ T_\alpha = \mathcal{T}_{\alpha_\infty} \circ \Phi^\infty_{\pm,\varepsilon}. \tag{5.17}$$

The behavior of the Fatou coordinate $\Phi^0_{\pm,\varepsilon} : Q^0_{\pm,\varepsilon} \to \mathbb{C}$ with respect the translation $T_\alpha$ is more involved. Indeed, by Lemma 4.4, $T_\alpha$ sends the translation domains $Q^0_{\pm,\varepsilon}$ into $Q^0_{\mp,\varepsilon}$ and then, reasoning as above, $\Phi^0_{\pm,\varepsilon} \circ T_\alpha$ and $\Phi^0_{\mp,\varepsilon}$ are two Fatou Glutsyuk coordinates defined on the same translation domain. Proposition 5.3 shows then that there exist two constants $C^1_\varepsilon, C^2_\varepsilon$ such that:

$$\begin{aligned}\Phi^0_{+,\varepsilon} \circ T_\alpha &= \mathcal{T}_{C^1_\varepsilon} \circ \Phi^0_{-,\varepsilon} \\ \Phi^0_{-,\varepsilon} \circ T_\alpha &= \mathcal{T}_{C^2_\varepsilon} \circ \Phi^0_{+,\varepsilon},\end{aligned} \tag{5.18}$$

thus yielding:

$$\Phi^0_{\pm,\varepsilon} \circ T_{2\alpha} = \mathcal{T}_{C^1_\varepsilon + C^2_\varepsilon} \circ \Phi^0_{\pm,\varepsilon}. \tag{5.19}$$

The quotients $Q^0_{\pm,\varepsilon}/\{\mathbf{P}_\varepsilon, T_{2\alpha}\}$ are conformally equivalent to $U^0_\varepsilon/\mathcal{P}_\varepsilon$, *i.e.* the union of a complex torus of modulus $\alpha_0$ with the singular point $x_0$. Moreover, in the $W$ (Fatou) coordinate, positive orientation of the translation $\mathcal{T}_{\alpha_0}$ corresponds to negative orientation of $\mathcal{T}_{\alpha_\infty}$. Since the positive orientation of the translation $\mathcal{T}_{\alpha_\infty}$ coincides with that of $T_\alpha$, we get $\Phi^0_{\pm,\varepsilon}\circ T_{2\alpha} = \mathcal{T}_{-\alpha_0}\circ\Phi^0_{\pm,\varepsilon}$, or, in terms of the constants, $C^1_\varepsilon + C^2_\varepsilon = -\alpha_0$. Let us show that $C^1_\varepsilon = C^2_\varepsilon$. Since the Fatou coordinates $\Phi^0_\pm$ are normalized, using (5.8) we have:

$$\begin{aligned}
\Phi^0_{+,\varepsilon}\circ T_\alpha &= (\mathcal{T}_{-\frac{1}{2}}\circ\Phi^0_{-,\varepsilon}\circ\mathbf{Q}_\varepsilon)\circ T_\alpha \\
&= \mathcal{T}_{-\frac{1}{2}}\circ(\mathcal{T}_{C^2_\varepsilon}\circ\Phi^0_+\circ T_{-\alpha})\circ\mathbf{Q}_\varepsilon\circ T_\alpha \quad \text{(by (5.18))} \\
&= \mathcal{T}_{C^2_\varepsilon}\circ\mathcal{T}_{-\frac{1}{2}}\circ\Phi^0_+\circ\mathbf{Q}_\varepsilon \quad (\text{because } \mathbf{Q}_\varepsilon = T_{-\alpha}\circ\mathbf{Q}_\varepsilon\circ T_\alpha) \\
&= \mathcal{T}_{C^2_\varepsilon}\circ\Phi^0_-.
\end{aligned}$$

Comparing with the first equation in (5.18), we get $C^1_\varepsilon = C^2_\varepsilon = -\dfrac{\alpha_0}{2}$. □

Grosso modo, (5.15) says that, in order to make a full turn around the origin in $x$ coordinate, it is necessary to iterate twice the translation around the origin in the unwrapping coordinate. On the contrary, an iteration of the translation around infinity in the $Z$ coordinate yields a full turn around $x_\pm$.

**Lemma 5.9** *When the parameter is (real) positive, the normalized Real Fatou Glutsyuk coordinates of Theorem 5.1, Lemma 5.4 and Proposition 5.8, satisfy as well:*

$$\Phi^0_\varepsilon\left\{Im(Z) = \pm\frac{\alpha}{2i}\right\} \quad\subset\quad \left\{Im(W) = \mp\frac{\alpha_0}{4i}\right\},$$

$$\Phi^\infty_\varepsilon\left\{Im(Z) = \pm\frac{\alpha}{2i}\right\} \quad\subset\quad \left\{Im(W) = \pm\frac{\alpha_\infty}{2i}\right\}.$$

Proof. Both (5.3) and (5.15) imply:

$$\begin{aligned}
\mathcal{T}_{\mp\frac{\alpha_0}{2}}\circ\mathcal{C}\circ\Phi^0_\varepsilon &= \Phi^0_\varepsilon\circ T_{\pm\alpha}\circ\mathsf{C}, \\
\mathcal{T}_{\mp\alpha_\infty}\circ\mathcal{C}\circ\Phi^\infty_\varepsilon &= \Phi^\infty_\varepsilon\circ\mathsf{C}\circ T_{\pm\alpha}.
\end{aligned} \tag{5.20}$$

Put $Im(Z) = \pm\dfrac{\alpha}{2i}$. Thus, $Z = \mathsf{C}(Z)\pm\alpha = T_{\pm\alpha}\circ\mathsf{C}(Z)$ and if we write $\Phi^0_\varepsilon(Z) = A + iB$, for $A, B\in\mathbb{R}$, then (5.20) yields:

$$\begin{aligned}
A + iB &= \Phi^0_\varepsilon(Z) \\
&= \Phi^0_\varepsilon(T_{\pm\alpha}\circ\mathsf{C}(Z)) \\
&= \overline{\Phi^0_\varepsilon(Z)}\mp\frac{\alpha_0}{2} \\
&= A - iB\mp\frac{\alpha_0}{2},
\end{aligned}$$

whence $B = \mp\dfrac{\alpha_0}{4i}$. The second inclusion follows along similar steps. $\square$

# 6 Real and Symmetric Glutsyuk invariants.

Fix four Fatou Glutsyuk coordinates $\Phi^{0,\infty}_{\pm,\varepsilon,lr}$ on the leaves of $\mathcal{R}_\varepsilon$, whose base points depend analytically on the parameter, see Figure 12 and define:

*a*) For $\varepsilon \in V_{\delta,l}$ :

$$\begin{aligned} \Psi^{++}_{\varepsilon,l} &= \Phi^0_{+,\varepsilon,l} \circ (\Phi^\infty_{+,\varepsilon,l})^{\circ -1}, \\ \Psi^{+-}_{\varepsilon,l} &= \Phi^0_{-,\varepsilon,l} \circ (\Phi^\infty_{+,\varepsilon,l})^{\circ -1}, \\ \Psi^{-+}_{\varepsilon,l} &= \Phi^0_{+,\varepsilon,l} \circ (\Phi^\infty_{-,\varepsilon,l})^{\circ -1}, \\ \Psi^{--}_{\varepsilon,l} &= \Phi^0_{-,\varepsilon,l} \circ (\Phi^\infty_{-,\varepsilon,l})^{\circ -1}. \end{aligned} \tag{6.1}$$

*b*) For $\varepsilon \in V_{\delta,r}$ :

$$\begin{aligned} \Psi^{++}_{\varepsilon,r} &= \Phi^\infty_{+,\varepsilon,r} \circ (\Phi^0_{+,\varepsilon,r})^{\circ -1}, \\ \Psi^{+-}_{\varepsilon,r} &= \Phi^\infty_{-,\varepsilon,r} \circ (\Phi^0_{+,\varepsilon,r})^{\circ -1}, \\ \Psi^{-+}_{\varepsilon,r} &= \Phi^\infty_{+,\varepsilon,r} \circ (\Phi^0_{-,\varepsilon,r})^{\circ -1}, \\ \Psi^{--}_{\varepsilon,r} &= \Phi^\infty_{-,\varepsilon,r} \circ (\Phi^0_{-,\varepsilon,r})^{\circ -1}. \end{aligned} \tag{6.2}$$

In either case, this collection will be noted $\Psi^G_\varepsilon$. By periodicity, it suffices to describe the dynamics around the principal hole. Since $\mathcal{P}_\varepsilon = \mathcal{Q}^{\circ 2}_\varepsilon$, it is possible to reduce these four components to two independent ones.

**Definition 6.1** *The Glutsyuk invariant is the family of equivalence classes of $\Psi^G_\varepsilon$ with respect to composition with translations $\mathcal{T}_{C(\varepsilon)}$ in the source and target spaces where the constant $C(\varepsilon)$ is real on real $\varepsilon$ and it depends holomorphically on the parameter over $V_{\delta,l} \cup V_{\delta,r}$ with a continuous limit at $\varepsilon = 0$, such that $C(0) \neq 0$.*

**Lemma 6.2** *By choosing normalized Real Fatou Glutsyuk coordinates, it is possible in turn to choose components $\Psi^{\pm,\pm}_\varepsilon$ of a representative of the Glutsyuk invariant $\Psi^G_\varepsilon$ which are related through:*

$$\begin{aligned} \Psi^{++}_\varepsilon &= \mathcal{T}_{-\frac{1}{2}} \circ \Psi^{--}_\varepsilon \circ \mathcal{T}_{\frac{1}{2}}, \\ \Psi^{-+}_\varepsilon &= \mathcal{T}_{-\frac{1}{2}} \circ \Psi^{+-}_\varepsilon \circ \mathcal{T}_{\frac{1}{2}} \end{aligned} \tag{6.3}$$

*for every $\varepsilon \in V^G_\delta$.*

Proof. It suffices to take normalized Fatou Glutsyuk coordinates, so that (6.3) is satisfied by definition. $\square$

## 6.1 Real Glutsyuk invariant: First Presentation.

When the Glutsyuk invariant is defined using Real Fatou Glutsyuk coordinates, we get a natural property of symmetry under the Schwarz reflection, respecting the real normalization of the Glutsyuk coordinates.

**Theorem 6.3** *There exists a representative $\Psi_\varepsilon^G = (\Psi_\varepsilon^{\pm\pm})$ of the Glutsyuk modulus associated with the family of diffeomorphisms $\mathcal{P}_\varepsilon$ satisfying, in addition to* (6.3)*, the identities:*

– *If $\varepsilon \in V_{\delta,l}$ :*

$$\begin{aligned} \Psi_{\varepsilon,l}^{++} &= \mathcal{T}_{-\frac{\alpha_0}{2}} \circ \Psi_{\varepsilon,l}^{+-} \circ \mathcal{T}_{-\alpha_\infty}, \\ \Psi_{\varepsilon,l}^{--} &= \mathcal{T}_{-\frac{\alpha_0}{2}} \circ \Psi_{\varepsilon,l}^{-+} \circ \mathcal{T}_{-\alpha_\infty}. \end{aligned} \tag{6.4}$$

– *If $\varepsilon \in V_{\delta,r}$ :*

$$\begin{aligned} \Psi_{\varepsilon,r}^{++} &= \mathcal{T}_{\alpha_\infty} \circ \Psi_{\varepsilon,r}^{-+} \circ \mathcal{T}_{\frac{\alpha_0}{2}}, \\ \Psi_{\varepsilon,r}^{--} &= \mathcal{T}_{\alpha_\infty} \circ \Psi_{\varepsilon,r}^{+-} \circ \mathcal{T}_{\frac{\alpha_0}{2}}. \end{aligned} \tag{6.5}$$

– *Moreover, for every $\varepsilon \in V_{\delta,lr}$ :*

$$\begin{aligned} \Psi_{\varepsilon}^{++} &= \mathcal{C} \circ \Psi_{\mathcal{C}(\varepsilon)}^{+-} \circ \mathcal{C}, \\ \Psi_{\varepsilon}^{--} &= \mathcal{C} \circ \Psi_{\mathcal{C}(\varepsilon)}^{-+} \circ \mathcal{C}. \end{aligned} \tag{6.6}$$

*Such a representative can be constructed so as to have a limit at $\varepsilon = 0$, which is the Ecalle modulus.*

Proof. It suffices to take normalized Real Fatou Glutsyuk coordinates depending analytically on the parameter with continuous limit at $\varepsilon = 0$, (this is the same limit for the two cases $\varepsilon \in V_{\delta,l}$ and $\varepsilon \in V_{\delta,r}$). Then (6.4) and (6.5) are immediate consequences of (6.1), (6.2) and Proposition 5.8. On the other hand, (6.6) comes after Theorem 5.1 and the idempotency (4.14) on the conjugation in the $Z$ coordinate, when the parameter is real. Since the dependence of the modulus is analytic in the parameter, the equality extends to values $\varepsilon \in V_{\delta,lr}$. Notice that the symmetry axis still exists in the limit $\varepsilon = 0$, and the invariance exists in the limit as well. $\square$

**Definition 6.4** *The equivalence class of a representative $\Psi_\varepsilon^G$ of the Glutsyuk invariant chosen as in Lemma 6.2 and Theorem 6.3 for values $\varepsilon \in V_{\delta,lr}$ will be called the Real Glutsyuk modulus.*

**Corollary 6.5** *For every $\varepsilon \in V_{\delta,lr}$, a representative of the Real Glutsyuk modulus is completely determined by one of the maps $\Psi_\varepsilon^{\pm\pm}$.*

In this first presentation, the symmetry (conjugation $\mathsf{C}$ in the time $Z$ coordinate) is taken with respect the symmetry axis $\widehat{\Re}$. Since the Real Fatou Glutsyuk coordinates send the symmetry axis $\widehat{\Re}$ into $\mathbb{R}$, the real line is invariant under the Real Glutsyuk invariant when the parameter is real. This means that in the $x$-coordinate the symmetry has been taken with respect the real segment $I_+ \cup I_-$ joining the singular points $x_\pm$ with the boundary of $U$, see Figure 14. Moreover, in the limit $\varepsilon \to 0$ the segment $I_+ \cup I_-$ tends to $\mathbb{R} \cap U$. Thus, in the Fatou coordinate, the conjugation $\mathcal{C}$ is still defined when $\varepsilon = 0$ and the Ecalle invariant inherits the symmetry (6.6).

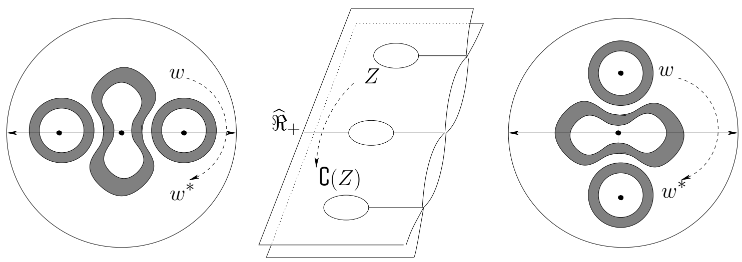


Figure 14: The symmetry in the First Presentation.

**The Ecalle modulus.** Since $\alpha(\varepsilon) = -\dfrac{\pi i}{\varepsilon}$, the distance between two consecutive holes becomes infinite in the limit $\varepsilon \to 0$, and then each diffeomorphism $\Psi_\varepsilon^{\pm\pm}$, for $\varepsilon \in V_{\delta,lr}$, gives rise to a component of the Ecalle invariant, with preimage in a region around the principal hole. Notice that (6.3) and (5.2) or (5.3) remain valid during the limit process, so in the limit there is only one independent component. Figure 15 shows the domains around the principal hole (connected strips) on the surface $\mathcal{R}_\varepsilon$ whose image by the Fatou Glutsyuk coordinates and subsequent quotient by the translation $\mathcal{T}_1$, correspond to annuli-like domains for the different components of the Glutsyuk invariant. However, we can choose the representative of the Real Glutsyuk modulus so as to give rise to the same invariant in the limit $\varepsilon \to 0$, no matter whether $\varepsilon \in V_{\delta,l}$ or $\varepsilon \in V_{\delta,r}$.

**Proposition 6.6** *The Ecalle modulus can be deduced from the Real Glutsyuk invariant. It is given by:*

$$\begin{aligned} \Psi_1^\infty &= \lim_{\varepsilon\to 0} \Psi_{\varepsilon,lr}^{++}, & \Psi_1^0 &= \lim_{\varepsilon\to 0} \Psi_{\varepsilon,lr}^{+-}, \\ \Psi_2^\infty &= \lim_{\varepsilon\to 0} \Psi_{\varepsilon,lr}^{--}, & \Psi_2^0 &= \lim_{\varepsilon\to 0} \Psi_{\varepsilon,lr}^{-+}, \end{aligned} \tag{6.7}$$

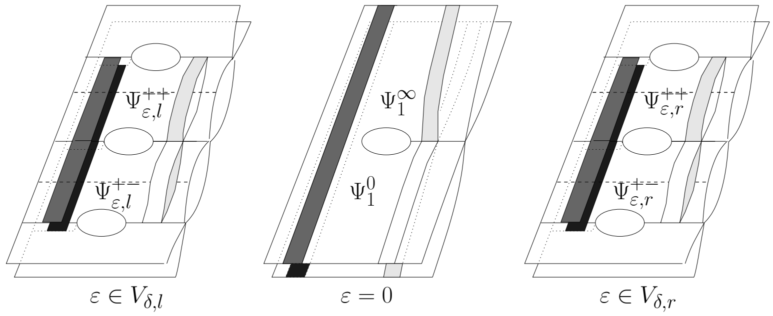


Figure 15: The Glutsyuk invariant in the limit $\varepsilon \to 0$.

*see Figure 15. Moreover, its components may be chosen conjugate as well:*

$$\begin{aligned} \Psi_1^\infty &= \mathcal{C} \circ \Psi_1^0 \circ \mathcal{C} \\ \Psi_2^\infty &= \mathcal{C} \circ \Psi_2^0 \circ \mathcal{C}, \end{aligned} \tag{6.8}$$

*and, in addition,*

$$\begin{aligned} \Psi_1^\infty &= \mathcal{T}_{-\frac{1}{2}} \circ \Psi_2^\infty \circ \mathcal{T}_{\frac{1}{2}} \\ \Psi_1^0 &= \mathcal{T}_{-\frac{1}{2}} \circ \Psi_2^0 \circ \mathcal{T}_{\frac{1}{2}}. \end{aligned} \tag{6.9}$$

Proof. Each component of the modulus at $\varepsilon = 0$ is the limit of two representatives in the two cases $\varepsilon \in V_{\delta,l}$ and $\varepsilon \in V_{\delta,r}$. More specifically, we have:

$$\begin{aligned} \Psi_1^\infty &= \lim_{\varepsilon\to 0^l} \Phi^0_{+,\varepsilon,l} \circ (\Phi^\infty_{+,\varepsilon,l})^{\circ -1} = \lim_{\varepsilon\to 0^r} \Phi^\infty_{+,\varepsilon,r} \circ (\Phi^0_{+,\varepsilon,r})^{\circ -1}, \\ \Psi_1^0 &= \lim_{\varepsilon\to 0^l} \Phi^0_{-,\varepsilon,l} \circ (\Phi^\infty_{+,\varepsilon,l})^{\circ -1} = \lim_{\varepsilon\to 0^r} \Phi^\infty_{-,\varepsilon,r} \circ (\Phi^0_{+,\varepsilon,r})^{\circ -1}, \\ \Psi_2^\infty &= \lim_{\varepsilon\to 0^l} \Phi^0_{-,\varepsilon,l} \circ (\Phi^\infty_{-,\varepsilon,l})^{\circ -1} = \lim_{\varepsilon\to 0^r} \Phi^\infty_{-,\varepsilon,r} \circ (\Phi^0_{-,\varepsilon,r})^{\circ -1} \\ \Psi_2^0 &= \lim_{\varepsilon\to 0^l} \Phi^0_{+,\varepsilon,l} \circ (\Phi^\infty_{-,\varepsilon,l})^{\circ -1} = \lim_{\varepsilon\to 0^r} \Phi^\infty_{+,\varepsilon,r} \circ (\Phi^0_{-,\varepsilon,r})^{\circ -1}, \end{aligned} \tag{6.10}$$

where $\varepsilon \to 0^l$ (resp. $\varepsilon \to 0^r$) means $\varepsilon \to 0$ and $\varepsilon \in V_{\delta,l}$ (resp. $\varepsilon \in V_{\delta,r}$). The symmetries on the Ecalle modulus follow from Theorem 6.3. □

## 6.2 Symmetric Glutsyuk invariant: Second Presentation.

When we use Real Fatou Glutsyuk coordinates and allow a subsequent imaginary translation on them, we break the symmetries (6.6). However, if the translations are well chosen, we get a different form of symmetry corresponding to a symmetry in the $x$-coordinate under the Schwarz reflection with respect to the line segment joining the points $x_\pm$. This presentation is also very interesting and deserves a detailed discussion.

**Theorem 6.7** *There exists a representative* $\Psi^G_\varepsilon = (\Psi^{\pm\pm}_\varepsilon)$ *of the Glutsyuk modulus satisfying* (6.3), (6.4) *and* (6.5), *that carries the real character of the family of vector fields as follows. Let* $\# \in \{++,+-,-+,--\}$ *be a shortcut for the superscripts.*

– *If* $\varepsilon \in V_{\delta,l}\setminus\{0\}$ *then:*

$$\Psi^{\#}_{\varepsilon,l} = \mathcal{C}\circ\Psi^{\#}_{\mathcal{C}(\varepsilon),l}\circ\mathcal{C}, \tag{6.11}$$

*i.e. the representative is "symmetric" with respect to the image of the line* $\Re^s_\pm$.

– *If* $\varepsilon \in V_{\delta,r}\setminus\{0\}$ *then:*

$$\Psi^{\#}_{\varepsilon,r} = \mathcal{T}_{-\frac{1}{2}}\circ\mathcal{C}\circ\Psi^{\#}_{\mathcal{C}(\varepsilon),r}\circ\mathcal{C}\circ\mathcal{T}_{\frac{1}{2}}, \tag{6.12}$$

*i.e. the representative is "symmetric" with respect to the image of the line* $\Im^s_\pm$.

Proof. We start taking Real Fatou Glutsyuk coordinates $\Phi^{0,\infty}_{\pm,\varepsilon}$. By analytic dependence of the Glutsyuk coordinates in $\varepsilon \in V_{\delta,lr}\setminus\{0\}$, it suffices to show the theorem for real values of the parameter.

– *The case* $\varepsilon < 0$. The induced Real Glutsyuk invariant already verifies (6.6), so we must show that a correction is possible so that (6.11) be satisfied. Theorem 6.3 yields:

$$\begin{aligned}\Psi^{++}_{\varepsilon,l} &= \mathcal{T}_{-\frac{\alpha_0}{2}}\circ\Psi^{+-}_{\varepsilon,l}\circ\mathcal{T}_{-\alpha_\infty}\\ &= \mathcal{T}_{-\frac{\alpha_0}{2}}\circ\mathcal{C}\circ\Psi^{++}_{\varepsilon,l}\circ\mathcal{C}\circ\mathcal{T}_{-\alpha_\infty}.\end{aligned}$$

Consider the translations $\mathcal{T}_{d(\varepsilon)}, \mathcal{T}_{d'(\varepsilon)}$, where the constants $d(\varepsilon), d'(\varepsilon)$ are to be chosen later. Replacing $\Psi^{++}_{\varepsilon,l} \mapsto \mathcal{T}_{d(\varepsilon)}\circ\Psi^{++}_{\varepsilon,l}\circ\mathcal{T}_{d'(\varepsilon)}$ in the equation above, we get:

$$\begin{aligned}\mathcal{T}_{d(\varepsilon)}\circ\Psi^{++}_{\varepsilon,l}\circ\mathcal{T}_{d'(\varepsilon)} &= \mathcal{T}_{-\frac{\alpha_0}{2}}\circ\mathcal{C}\circ\mathcal{T}_{d(\varepsilon)}\circ\Psi^{++}_{\varepsilon,l}\circ\mathcal{T}_{d'(\varepsilon)}\circ\mathcal{C}\circ\mathcal{T}_{-\alpha_\infty}\\ &= \mathcal{T}_{-\frac{\alpha_0}{2}+\overline{d(\varepsilon)}}\circ\mathcal{C}\circ\Psi^{++}_{\varepsilon,l}\circ\mathcal{C}\circ\mathcal{T}_{-\alpha_\infty+\overline{d'(\varepsilon)}}.\end{aligned}$$

If $d(\varepsilon) = -\dfrac{\alpha_0}{4} = -\dfrac{i\pi}{2\,\varepsilon}$ and $d'(\varepsilon) = -\dfrac{\alpha_\infty}{2} = \dfrac{i\pi(1-A(\varepsilon)\,\varepsilon)}{2\,\varepsilon}$ (where $A(\varepsilon)$ is the real formal invariant), then we get

$$\Psi^{++}_{\varepsilon,l} = \mathcal{C}\circ\Psi^{++}_{\varepsilon,l}\circ\mathcal{C}.$$

The same procedure shows $\Psi^{\#}_{\varepsilon,l} = \mathcal{C}\circ\Psi^{\#}_{\varepsilon,l}\circ\mathcal{C}$, for $\# \in \{+-,-+,--\}$.

– *The case* $\varepsilon > 0$. By (6.5) and (6.6) we have:

$$\begin{aligned}\Psi^{++}_{\varepsilon,r} &= \mathcal{T}_{\alpha_\infty} \circ \Psi^{-+}_{\varepsilon,r} \circ \mathcal{T}_{\frac{\alpha_0}{2}} \\ &= \mathcal{T}_{\alpha_\infty} \circ \mathcal{C} \circ \Psi^{--}_{\varepsilon,r} \circ \mathcal{C} \circ \mathcal{T}_{\frac{\alpha_0}{2}}.\end{aligned}$$

The procedure used above shows that the corrections $\Psi^{++}_{\varepsilon,r} \mapsto \mathcal{T}_{d(\varepsilon)} \circ \Psi^{++}_{\varepsilon,r} \circ \mathcal{T}_{d'(\varepsilon)}$ and $\Psi^{--}_{\varepsilon,r} \mapsto \mathcal{T}_{d(\varepsilon)} \circ \Psi^{--}_{\varepsilon,r} \circ \mathcal{T}_{d'(\varepsilon)}$, for $d(\varepsilon) = \frac{\alpha_\infty}{2} = -\frac{i\pi(1 - A(\varepsilon)\,\varepsilon)}{2\,\varepsilon}$ and $d'(\varepsilon) = \frac{\alpha_0}{4} = \frac{i\pi}{2\,\varepsilon}$ yield

$$\Psi^{++}_{\varepsilon,r} = \mathcal{C} \circ \Psi^{--}_{\varepsilon,r} \circ \mathcal{C}. \tag{6.13}$$

In the same spirit, we show:

$$\Psi^{-+}_{\varepsilon,r} = \mathcal{C} \circ \Psi^{+-}_{\varepsilon,r} \circ \mathcal{C}. \tag{6.14}$$

Then (6.13), (6.14) and Lemma 6.2 yield the conclusion:

$$\Psi^{\#}_{\varepsilon,r} = \mathcal{T}_{-\frac{1}{2}} \circ \mathcal{C} \circ \Psi^{\#}_{\varepsilon,r} \circ \mathcal{C} \circ \mathcal{T}_{\frac{1}{2}}.$$

Notice that this new "renormalized" representative still respects (6.3), (6.4) and (6.5). □

The composition with translations $\mathcal{T}_{d(\varepsilon)}, \mathcal{T}_{d'(\varepsilon)}$ in the proof above has destroyed the real normalization of the Real Fatou Glutsyuk coordinates $\Phi^{0,\infty}_{\pm,\varepsilon}$, and also the continuity at $\varepsilon = 0$. However, this non-real normalization is very interesting, even if it does not pass to the limit when $\varepsilon \to 0$. Indeed, in the $Z$ coordinate the imaginary translations $\mathcal{T}_{d(\varepsilon)}, \mathcal{T}_{d'(\varepsilon)}$ have displaced the symmetry axis to the line $\Re^s_\pm$ if $\varepsilon < 0$, and to the line $\Im^s_\pm$ if $\varepsilon > 0$, right above the principal hole. In the Fatou coordinate, the two imaginary translations have displaced the real axis to the lines $Im(W) = \frac{\alpha_0}{4i}$ and $Im(W) = \frac{\alpha_\infty}{2i}$, according to Lemma 5.9, thus breaking the real normalization of the Fatou Glutsyuk coordinates. The three real cases deserve explanation.

*The parameter is (real) negative.* The normalization reflects the natural symmetry of the invariant with respect the image (by $p_\varepsilon^{-1}$) of the real segment joining $x_+$ to $x_-$ in the $x$-coordinate, see Figure 16. Inasmuch as the symmetry is taken with respect a "real" line in the Fatou coordinate, the invariant still carries the real character of the foliation, as can be seen from formula (6.11).

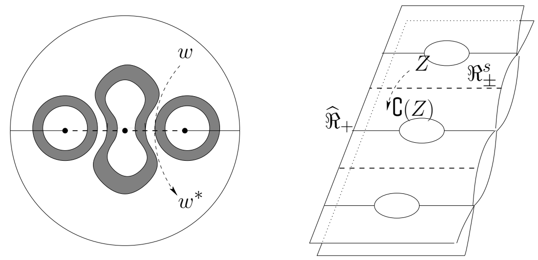


Figure 16: The symmetry when the parameter is negative.

*The parameter is (real) positive.* The imaginary translations have brought the symmetry axis to the image (by $p_\varepsilon^{-1}$) of the imaginary segment $I$ joining the singular points $x_0, x_\pm$, see Figure 17. Thus, the non-real normalization yields an invariant in the $x$-coordinate which is symmetric with respect to $I$. That is exactly the meaning of the formula (6.12). This "imaginary" symmetry is explained by:

- the real symmetry carried by the former Real Fatou Glustyuk coordinates, so that the components of $\Psi^G$ are 2-by-2 symmetric images one of another (this is (6.13) and (6.14));
- the fact that the Poincaré map of the family is a square: $\mathcal{P}_\varepsilon = \mathcal{Q}_\varepsilon^{\circ 2}$. In the $x$-plane this can be viewed as a sort of "symmetry with respect to the origin". Composing this symmetry with the symmetry with respect the real axis, yields a symmetry with respect to the imaginary axis.

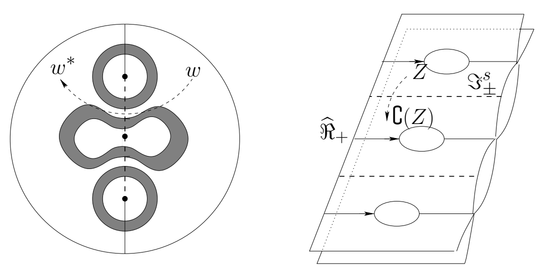


Figure 17: The symmetry when the parameter is positive.

*The parameter is null.* As the lines $\mathfrak{R}^s_\pm, \mathfrak{I}^s_\pm$ no longer exist when $\varepsilon = 0$, this presentation does not pass to the limit when $\varepsilon \to 0$. The Ecalle modulus cannot be deduced from this presentation. Indeed, the real (resp. imaginary) segment in the $x$-coordinate joining the fixed points disappears when $\varepsilon \to 0^-$ (resp. $\varepsilon \to 0^+$).

**Definition 6.8** *Any representative* $\Psi_\varepsilon^G$ *of the Glutsyuk invariant chosen as in Theorem 6.7 will be called Symmetric Glutsyuk modulus.*

**Corollary 6.9** *A representative of the Symmetric Glutsyuk modulus is completely determined by one of its components* $\Psi_\varepsilon^{\pm\pm}$.

# 7 Invariants under weak conjugacy.

We take Fatou Glutsyuk coordinates depending continuously on $\varepsilon \in V_{\delta,lr}$. The domain of $\Psi_\varepsilon^G$ contains a union of four horizontal strips $S_\varepsilon^{\pm\pm}$ located right above (resp. below) the principal hole $B_\varepsilon$. As the Glutsyuk invariant satisfies $\Psi_\varepsilon^G(W+1) = \Psi_\varepsilon^G(W)+1$ we can expand the difference $\Psi_\varepsilon^G - id$ in Fourier series on $S_\varepsilon^{\pm\pm}$ :

$$\begin{aligned}
(\Psi_\varepsilon^{++}(W) - W)\Big|_{S_\varepsilon^{++}} &= \sum_{n\in\mathbb{Z}} c_n^{++}(\varepsilon)\exp(2i\pi nW),\\
(\Psi_\varepsilon^{+-}(W) - W)\Big|_{S_\varepsilon^{+-}} &= \sum_{n\in\mathbb{Z}} c_n^{+-}(\varepsilon)\exp(2i\pi nW),\\
(\Psi_\varepsilon^{-+}(W) - W)\Big|_{S_\varepsilon^{-+}} &= \sum_{n\in\mathbb{Z}} c_n^{-+}(\varepsilon)\exp(2i\pi nW),\\
(\Psi_\varepsilon^{--}(W) - W)\Big|_{S_\varepsilon^{--}} &= \sum_{n\in\mathbb{Z}} c_n^{--}(\varepsilon)\exp(2i\pi nW).
\end{aligned} \tag{7.1}$$

Then, using (6.4) in the case $\varepsilon \in V_{\delta,l}$ we deduce:

$$\begin{cases}
c_0^{++}(\varepsilon) - c_0^{+-}(\varepsilon) = c_0^{--}(\varepsilon) - c_0^{-+}(\varepsilon) = -i\pi sA(\varepsilon),\\
c_n^{++}(\varepsilon) = c_n^{+-}(\varepsilon)e^{-\frac{2n\pi^2(1-sA(\varepsilon)\,\varepsilon)}{\varepsilon}}, \quad \text{for} \quad n \neq 0,\\
c_n^{--}(\varepsilon) = c_n^{-+}(\varepsilon)e^{-\frac{2n\pi^2(1-sA(\varepsilon)\,\varepsilon)}{\varepsilon}}, \quad \text{for} \quad n \neq 0,
\end{cases}$$

and using (6.5) in the case $\varepsilon \in V_{\delta,r}$ we get:

$$\begin{cases}
c_0^{++}(\varepsilon) - c_0^{-+}(\varepsilon) = c_0^{--}(\varepsilon) - c_0^{+-}(\varepsilon) = i\pi sA(\varepsilon),\\
c_n^{++}(\varepsilon) = c_n^{-+}(\varepsilon)e^{-\frac{2n\pi^2}{\varepsilon}}, \quad \text{for} \quad n \neq 0,\\
c_n^{--}(\varepsilon) = c_n^{+-}(\varepsilon)e^{-\frac{2n\pi^2}{\varepsilon}}, \quad \text{for} \quad n \neq 0.
\end{cases}$$

**Corollary 7.1** *The differences* $c_0^{++}(\varepsilon)-c_0^{+-}(\varepsilon)$ *and* $c_0^{--}(\varepsilon)-c_0^{-+}(\varepsilon)$ *when* $\varepsilon\in V_{\delta,l}$ *(resp.* $c_0^{++}(\varepsilon)-c_0^{-+}(\varepsilon)$ *and* $c_0^{--}(\varepsilon)-c_0^{+-}(\varepsilon)$ *when* $\varepsilon\in V_{\delta,r}$*) are analytic invariants of the system. Moreover, if the Glutsyuk modulus is prescribed on* $\varepsilon\in V_{\delta,lr}$*, then the formal parameter* $A(\varepsilon)$ *is known for values of the parameter in* $V_{\delta,lr}$.

**Definition 7.2** *Two germs* $\{\mathcal{P}_{\varepsilon_1}\}_{\varepsilon_1\in V_{\delta,lr}}$, $\{\widehat{\mathcal{P}}_{\varepsilon_2}\}_{\varepsilon_2\in V_{\delta,lr}}$ *of analytic families of diffeomorphisms are "weakly conjugate" as real families if there exists a germ of bijective map* $\mathcal{H}(\varepsilon_1,x)=(\mathbf{k}(\varepsilon_1),\mathbf{h}(\varepsilon_1,x))$ *fibered over the parameter space, where:*

*i)* $\mathbf{k}:\varepsilon_1\to\varepsilon_2=\mathbf{k}(\varepsilon_1)$ *is a germ of real analytic diffeomorphism preserving the origin.*

*ii) There exists* $\rho>0$ *and* $r>0$*, such that for each* $\varepsilon_1\in V_{\delta,l}(\rho)\cup V_{\delta,r}(\rho)$*, there is a representative* $\mathbf{h}_{\varepsilon_1}(x)=\mathbf{h}(\varepsilon_1,x)$ *of the germ depending analytically on* $x\in\mathbb{D}_r$ *and real for real* $\varepsilon_1,x$ *such that* $\mathbf{h}_{\varepsilon_1}$ *conjugates* $\mathcal{P}_{\varepsilon_1},\widehat{\mathcal{P}}_{\mathbf{k}(\varepsilon_1)}$ *:*
$$\mathbf{h}_{\varepsilon_1}\circ\mathcal{P}_{\varepsilon_1}=\widehat{\mathcal{P}}_{\mathbf{k}(\varepsilon_1)}\circ\mathbf{h}_{\varepsilon_1}.\tag{7.2}$$
*The representative* $\mathbf{h}_{\varepsilon_1}$ *depends analytically on* $\varepsilon_1\neq0$ *and it is continuous at* $\varepsilon_1=0$.

**Theorem 7.3** *Two families* $\{\mathcal{P}_{\varepsilon_1}\}_{\varepsilon_1\in V_{\delta,lr}}$ *and* $\{\widehat{\mathcal{P}}_{\varepsilon_2}\}_{\varepsilon_2\in V_{\delta,lr}}$ *(with the same sign* $s$ *before the cubic coefficient) are weakly conjugate by a real conjugacy that depends analytically on the parameter* $\varepsilon\in V_{\delta,lr}\backslash\{0\}$ *and continuously at* $\varepsilon=0$*, if and only if the Glutsyuk moduli of their associated prepared families coincide.*

Proof. Since two families are conjugate if and only if the associated prepared families are conjugate, it suffices to work with prepared families. The preparation shows that the parameters $\varepsilon_1$ and $\varepsilon_2$, the canonical parameters of the families, are analytic invariants, thus we can consider the conjugacy over the identity ($\varepsilon_1=\varepsilon_2:=\varepsilon$), and then it suffices to compare the two families for a given $\varepsilon\in V_{\delta,lr}$.

We can of course suppose that we have equal representatives of the Glutsyuk modulus for the two families. Since for values $\varepsilon\in V_{\delta,lr}$ the singular points $x_0,x_\pm$ are hyperbolic, they are linearizable. Hence, there exists in the neighborhood of each fixed point two sectorial diffeomorphisms $\varphi_\varepsilon^{0,\pm}=\Phi_{\varepsilon,\pm}^{G,0,\infty}\circ p_\varepsilon^{-1}$ and $\widehat{\varphi}_\varepsilon^{0,\pm}=\widehat{\Phi}_{\varepsilon,\pm}^{G,0,\infty}\circ p_\varepsilon^{-1}$ conjugating, respectively, the Poincaré maps $\mathcal{P}_\varepsilon$ and $\widehat{\mathcal{P}}_\varepsilon$ with $\tau_\varepsilon^1$. The diffeomorphisms $\Phi_\varepsilon^{G,0,\infty},\widehat{\Phi}_\varepsilon^{G,0,\infty}$ are Real Fatou Glutsyuk coordinates (see Definition 5.2). The neighborhoods of the singular points in the $x$ coordinate where the normalization is possible are noted, respectively, $U_\varepsilon^-, U_\varepsilon^0$ and $U_\varepsilon^+$. The map:

$$\begin{cases} f_\varepsilon^- = (\widehat{\varphi}_\varepsilon^-)^{-1} \circ \varphi_\varepsilon^- = p_\varepsilon \circ (\widehat{\Phi}_{\varepsilon,-}^{G,\infty})^{-1} \circ \Phi_{\varepsilon,-}^{G,\infty} \circ p_\varepsilon^{-1}, \\ f_\varepsilon^0 = (\widehat{\varphi}_\varepsilon^0)^{-1} \circ \varphi_\varepsilon^0 = p_\varepsilon \circ (\widehat{\Phi}_{\varepsilon}^{G,0})^{-1} \circ \Phi_{\varepsilon}^{G,0} \circ p_\varepsilon^{-1}, \\ f_\varepsilon^+ = (\widehat{\varphi}_\varepsilon^+)^{-1} \circ \varphi_\varepsilon^+ = p_\varepsilon \circ (\widehat{\Phi}_{\varepsilon,+}^{G,\infty})^{-1} \circ \Phi_{\varepsilon,+}^{G,\infty} \circ p_\varepsilon^{-1}, \end{cases}$$

is clearly a well defined change of coordinates conjugating the two families of diffeomorphisms, since the local changes of coordinates are extensions of each other over the neighborhood $U$ when $\varepsilon \in V_{\delta,lr}$ : $f_\varepsilon^0 \equiv f_\varepsilon^-$ on $U_\varepsilon^- \cap U_\varepsilon^0$, and $f_\varepsilon^0 \equiv f_\varepsilon^+$ on $U_\varepsilon^0 \cap U_\varepsilon^+$. The conclusion follows. $\square$

# 8 Application to the Hopf bifurcation.

A germ of one-parameter family of analytic planar vector fields unfolding a weak focus in a neighborhood of the origin, is linearly equivalent to a germ of family of differential equations:

$$\begin{aligned} \dot{x} &= \alpha(\varepsilon)x - \beta(\varepsilon)y + \sum_{j+k\geq 2} b_{jk}(\varepsilon)x^j y^k, \\ \dot{y} &= \beta(\varepsilon)x + \alpha(\varepsilon)y + \sum_{j+k\geq 2} c_{jk}(\varepsilon)x^j y^k, \end{aligned} \tag{8.1}$$

with $\alpha(0) = 0$ and $\beta(0) \neq 0$. After rescaling the time ($t \mapsto \beta(\varepsilon)t$) we can suppose $\beta(\varepsilon) \equiv 1$.

**Definition 8.1** *The family* (8.1) *is called "generic" if* $\alpha'(0) \neq 0$. *The weak focus is of order one if* $L_1(0) \neq 0$, *where* $L_1$ *is the first Lyapounov constant:*

$$L_1 = 3b_{30}+b_{12}+c_{21}+3c_{03}+\frac{1}{\beta}[b_{11}(b_{20}+b_{02})-c_{11}(c_{20}+c_{02})-2b_{20}c_{20}+2b_{02}c_{02}].$$

It is well known that the Poincaré first return map $\mathcal{P}_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ is analytic and can be extended to an analytic diffeomorphism

$$\mathcal{P}_\varepsilon : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$$

which is the square of a diffeomorphism $\mathcal{Q}_\varepsilon$ exchanging $\mathbb{R}^+$ with $\mathbb{R}^-$ and such that $\mathcal{Q}_0'(0) = -1$. The following theorem was proved in [2]

**Theorem 8.2** *[2] Two germs of generic families of real analytic vector fields unfolding a vector field with a weak focus of order one at the origin are real analytically orbitally equivalent if and only if the germs of families unfolding the Poincaré maps of the germs of vector fields are analytically conjugate by a real conjugacy.*

Along with Theorem 7.3, this result yields:

**Theorem 8.3** *Two germs of generic families of real analytic vector fields unfolding a vector field with a weak focus of order one at the origin are weakly real analytically orbitally equivalent if and only if the germs of families unfolding the Poincaré maps of the germs of vector fields have the same sign $s$ and the Glutsyuk modulus of their associated prepared families coincide.*

# 9 Directions for future research.

The present paper opens interesting perspectives which we hope to work on in the future. In particular, let us mention

1. We have described the Glutsyuk modulus on two sectors which do not cover a full neighborhood of the origin. From this modulus, we could recover the Lavaurs modulus which has been studied in the other works on the subject ([16], [10], [14] and [15]). Since the modulus depends analytically on $\varepsilon$, in practice, the Glutsyuk modulus, defined only on a union of two sectors in the parameter space, determines the Lavaurs modulus for parameter values in a full neighborhood of the origin. So we should be able to replace weak equivalence by equivalence in the Theorems 7.3 and 8.3. We hope to address this question in near future. The challenge is of course to show that the equivalence is real. Another interesting question is to determine the dependence of the Glutsyuk modulus on $\varepsilon$ at $\varepsilon = 0$, in order to identify the "realizable" moduli.
2. In this paper we have classified the germs of generic analytic families of vector fields undergoing a Hopf bifurcation of order 1, under orbital equivalence. We hope in the future to address the same problem under conjugacy of vector fields. For this purpose, in [3] we have decomposed each vector field as a vector field with angular velocity equal to 1 (called the orbital part) times a "time part" given by a non vanishing function. For a given orbital part, the time part of the modulus identifies the equivalence classes of time parts. The problem is then reduced to identify the time part of the modulus in the case

of the Hopf bifurcation, and also to identify the "realizable" moduli under conjugacy.

3. Isochronous weak foci of vector fields have been studied in the literature ([3],[7],[17]). Since the property of being isochronous depends only on the conjugacy class of the vector fields, it should be read on the modulus (orbital and time parts) of the weak focus for $\varepsilon = 0$. It is known that there are formal obstructions to isochronicity ([7]) but the analytic obstructions are still unknown.
4. Finally, a natural problem is to generalize to higher codimension (this is done for the saddle-node in [17]). In particular, are there obstructions at the orbital level? Is the triviality of the orbital modulus a necessary condition?

# 10 Acknowledgements.

We thank Colin Christopher for helpful discussions.

# References

[1] L. Ahlfors. Lectures on quasiconformal mappings. *Van Nostrand*, (1966).

[2] W. Arriagada-Silva. Characterization of the generic unfolding of a weak focus. *Preprint J. Diff. Eqs.*, JDEQ10-246R1, (2010).

[3] W. Arriagada-Silva. *Characterization of the unfolding of a weak focus and modulus of analytic classification.* PhD thesis, Université de Montréal, (2010).

[4] C. Christopher and C. Rousseau. The moduli space of germs of generic families of analytic diffeomorphisms unfolding a parabolic fixed point. *Preprint*, (2008).

[5] J. Ecalle. Les fonctions résurgentes. *Publications mathématiques d'Orsay*, (1985).

[6] P. Fatou. Sur les équations fonctionelles. *Bull. Soc. Math. France, Paris*, **47, 48**: 161–271, 33–94, 208–314, (1919-1920).

[7] J. Giné and M. Grau. Characterization of isochronous foci for planar analytic defferential systems. *Proc. Roy. Soc. Edinburgh Sect. A*, **135**: 985–998, (2005).

[8] A.A. Glutsyuk. Confluence of singular points and nonlinear stokes phenomenon. *Trans. Moscow Math. Soc.*, **62**: 49–95, (2001).

[9] J.F.-Mattei and R. Moussu. Holonomie et intégrales premières. *Ann. Scient. Éc. Norm. Sup., 4e série*, **13**: 469–523, (1980).

[10] P. Mardešić, R. Roussarie, and C. Rousseau. Modulus of analytic classification for unfoldings of generic parabolic diffeomorphisms. *Moscow Mathematical Journal*, **4**: 455–498, (2004).

[11] J. Martinet. Remarques sur la bifurcation nœud-col dans le domaine complexe. *Astérisque*, **150-151**: 131–149, (1987).

[12] J. Martinet and J.P. Ramis. Problèmes de modules pour des équations différentielles non linéaires du premier ordre. *Publ. IHES*, **55**: 63–164, (1982).

[13] J. Martinet and J.P. Ramis. Classification analytique des équations différentielles non linéaires résonnantes du premier ordre. *Ann. Scient. Éc. Norm. Sup., 4e série*, **16**: 571–621, (1983).

[14] R. Pérez-Marco and J.-C. Yoccoz. Germes de feuilletages holomorphes à holonomie prescrite. *S.M.F., Astérisque*, **222**: 345–371, (1994).

[15] C. Rousseau. The moduli space of germs of generic families of analytic diffeomorphisms unfolding of a codimension one resonant diffeomorphism or resonant saddle. *J. Differential Equations*, **248**: 1794–1825, (2010).

[16] C. Rousseau and C. Christopher. Modulus of analytical classification for the generic unfolding of a codimension one resonant diffeomorphism or resonant saddle. *Annales de l'Institut Fourier*, **57**: 301–360, (2007).

[17] C. Rousseau and L.Teyssier. Analytical moduli for unfoldings of saddle node vector fields. *Moscow Mathematical Journal*, **8**: 547–614, (2008).

[18] M. Shishikura. Bifurcation of parabolic fixed points. *"The Mandelbrot set, theme and variations", London Math. Society Lecture Notes*, **274**: 325–363, (2000).

[19] S. M. Voronin. Analytic classification of germs of conformal maps $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ with identical linear part. *Funktsional. Anal. i Prilozhen and Func. Anal. Appl.*, **15**: 1–17 (Russian), 1–13 (English), (1981).