NORMALIZABILITY, SYNCHRONICITY AND RELATIVE EXACTNESS FOR VECTOR FIELDS IN $\mathbb{C}^2$

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Abstract. In this paper we study the necessary and sufficient condition under which an orbitally normalizable vector field of saddle or saddle-node type in $\mathbb{C}^2$ is analytically conjugate to its formal normal form (i.e. normalizable) by a transformation fixing the leaves of the foliation locally.

We first express this condition in terms of the relative exactness of a certain 1-form derived from comparing the *time-form* of the vector field with the time-form of the normal form. We then show that this condition is equivalent to a *synchronicity condition*: the vanishing of the integral of this 1-form along certain asymptotic cycles defined by the vector field. This can be seen as a generalization of the classical theorem of Poincaré saying that a center is isochronous (i.e. synchronous to the linear center) if and only if it is linearizable.

The results, in fact, allow us in many cases to compare any two vector fields which differ by a multiplicative factor. In these cases we show that the two vector fields are analytically conjugate by a transformation fixing the leaves of the foliation locally if and only if their time-forms are synchronous.

1. Introduction

In this paper we consider the analytic system

$$\begin{aligned} \dot{x} &= x + f(x,y) = x + o(|(x,y)|) \\ \dot{y} &= -\lambda y + g(x,y) = -\lambda y + o(|(x,y)|), \quad \lambda \geq 0 \end{aligned} \tag{1.1}$$

in $\mathbb{C}^2$ with a saddle point or saddle-node at the origin. If the origin of (1.1) is orbitally normalizable or integrable (*i.e.* orbitally linearizable), we wish to understand what conditions guarantee that the origin is in fact normalizable or linearizable. That is, we are concerned with the "time" element of the dynamics as well as the "orbital" element. However, we will ignore the effects of a purely uniform dilation of time, so that the form (1.1) does not represent a restriction on the possible eigenvalues of the vector field.

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If we choose a 1-form ω which is dual to the vector field above then it is known that, when $d\omega$ does not vanish at the origin, a multiplicative factor can be absorbed into the 1-form by a change of variables. In the case of vector fields the situation is more complex. In fact, to recover a vector field by duality we need not only ω, but a 2-form Ω such that $i_V\Omega = \omega$. Changes of variables which simplify ω will also change the dualizing form Ω, and it is not hard to show that there are obstructions to normalizing the vector field which do not appear in the case of 1-forms.

In the simplest case, where the saddle comes from a real center with eigenvalues $\pm i$, then the classical theorem of Poincaré states that the center is in fact linearizable if and only if the period function is locally a constant. Our aim in this paper is to generalize this result to all orbitally normalizable systems of the form (1.1). That is, we assume the system can be conjugated with a system in normal form up to some non-constant multiple, and seek conditions which guarantee that this multiple can be chosen to be unity.

In fact, the results we give here are not restricted to orbitally normalizable vector fields, but can also be applied to the comparison of two vector fields of the form (1.1) which differ by multiplication by a nonzero function, except that in the case where λ is irrational we require that the critical point be integrable. However, since the applications to normalizability are probably the more interesting, we emphasize these.

In order to generalize Poincaré's theorem to these situations, we restrict the type of coordinate changes considered and assume that they not only fix the orbital normal form, but that they locally fix the leaves of the foliation defined by the vector field. This condition may not seem natural for the problem of linearizing an isochronous center as the period of all orbits is the same. However in general we encounter cycles (or asymptotic cycles) whose period varies. A change of coordinates must preserve these periods. The condition of fixing the leaves of the foliation locally assures of course that all periods are preserved.

Thus, we limit our study to transformations which fix the leaves of the germ of the foliation induced by the vector field. Restricting to this situation we now find that there is a nice equivalence between (i) the existence of such a transformation; (ii) the vanishing of the integral of the difference of the time-forms of the two vector fields; and (iii) the relative exactness of the difference of the time-forms. This last condition can equivalently be expressed as the solvability of a certain homological equation as studied in [T]). We now sketch the contents of the paper in more detail.

Section 2 is a section of generalities: we give the definitions of linearizability, integrability, normalizability etc. which we shall use in this paper, we define the *time-form* of a vector field dt. We also give the transformation of a vector field V under the pull-back by the time-g flow for some non-constant time g. Finally, we show that two vector fields of the form (1.1) are equivalent by a transformation fixing the leaves of the foliation locally, if and only if the transformation can be expressed as the time-g flow for some function g.

In Section 3 we give formal normal forms for vector fields with respect to formal conjugacy. These are expressed as the classical orbital normal forms for vector fields together with a multiplicative factor which represents the resonant terms which cannot be absorbed by a change of variables. In the case when λ is a positive rational, the classification of non integrable vector fields up to conjugacy has been given by Voronin for $\lambda \neq 0$ and Teyssier for $\lambda = 0$. Our aim here is much more specific, however.

In Section 4, we return to the problem of characterizing the analytic conjugacy of two orbitally equivalent vector fields V_1 and $V_2 = hV_1$. Let ω be a 1-form dual to V_1 (and hence also to V_2), and dt_1 and dt_2 be the time-forms for V_1 and V_2. By definition, the time-forms dt_i are meromorphic 1-forms (defined up to a multiple of ω) such that $i_{V_i}dt_i = 1$. We show that the two vector fields V_1 and $V_2 = hV_1$ are analytically conjugate by a transformation fixing the leaves of the foliation locally if the difference of their time-forms $\eta = dt_2 - dt_1$ is relatively exact with respect to ω. That is, we want

$$\eta = dh + k\omega, \tag{1.2}$$

with h analytic and k meromorphic. In particular, if V_1 is a vector field of the form (1.1) and $V_2 = hV_1$, we show that the above condition is necessary and sufficient (cf. Definition 2.5 and Remark 2.6).

In Section 5 we show that the relative exactness of a differential form η with respect to ω is characterized in terms of the vanishing of the integrals of η along all asymptotic cycles contained in the leaves of ω. This is a result of general interest. When λ is rational, the results are particular cases of the work by Berthier and Loray [BL], in the case of a resonant saddle, and by Teyssier [T1] in the saddle-node case. The case for integrable critical points with irrational λ has been covered in a more general context by Berthier and Cerveau [BC], but their results require some diophantine condition on λ. We give the result here for general λ in the planar case. We call this property *synchronicity* in analogy to the term *isochronicity*, used when comparing a center of a vector field with the linear center, and say that dt_1 is *synchronous* to dt_2.

We thus prove the following theorem.

Main Theorem.

(1) *Let V_1 and $V_2 = hV_1$ be two analytic vector fields of the form* (1.1) *which differ by multiplication by a nonzero function h, and let η be the difference of their time-forms. Then the following are equivalent:*
   - *(a) V_1 and V_2 are analytically conjugate by a transformation fixing the leaves of the foliation;*
   - *(b) η is relatively exact with respect to some form ω, dual to the V_i (cf. Definition 2.5).*

(2) *If either λ is rational or λ is irrational and V_1 (and V_2) are orbitally linearizable, then (a) and (b) are equivalent to:*
   - *(c) the integral of the form η vanishes along every asymptotic cycle of the vector fields V_i (that is, the two vector fields are synchronous).*

(3) *In particular any integrable (resp. orbitally normalizable) system* (1.1) *is linearizable (resp. normalizable) if and only if its time-form is synchronous to the time-form of its formal normal form or one of its conjugates.*

More exact details are given in Theorem 4.4 and Theorem 5.4.

It is the authors' hope that the study of time dependence for integrable and normalizable systems will enrich the understanding of the analytic behavior of non-degenerate critical points just as the study of isochronous centers has enriched the study of centers for planar vector fields.

2. Generalities

2.1. Main definitions. We recall some standard definitions on conjugacy and orbital equivalence for germs of vector fields and the notion of relatively exactness that we use here.

Definition 2.1.

i) *Two germs of vector fields are formally (resp. analytically) conjugate if one can be transformed to the other by a formal (resp. analytic) change of coordinates.*
ii) *Two germs of vector fields are formally (resp. analytically) orbitally equivalent if one is formally (resp. analytically) conjugate to a formal (resp. analytic) multiple of the other.*

Vector fields of the form (1.1) have a diagonal linear part. Either there are no resonances or all resonances follow from a single linear relation between the eigenvalues.

Definition 2.2.

i) *A normal form of a germ of a vector field of the form* (1.1)*, is a vector field containing no non-resonant monomials.*
ii) *A germ of a vector field is normalizable if it is analytically conjugate to a normal form.*
iii) *A vector field is orbitally normalizable if it is analytically orbitally equivalent to a normal form.*

The case of a germ of vector field formally conjugate (resp. formally orbitally equivalent) to the linear normal form for rational values of λ is special, as the notions of formal conjugacy (resp. formal orbital equivalence) and analytic conjugacy (resp. analytic orbital equivalence) are equivalent [B].

In more detail, we have the following definitions.

Definitions 2.3.

(i) *The system* (1.1) *is integrable at the origin if and only if it is orbitally linearizable, i.e. there exists an analytic change of coordinates* $(r, s \neq 0)$

$$(X,Y) = (rx+\phi(x,y), sy+\psi(x,y)) = (rx+o(x,y), sy+o(x,y)), \tag{2.1}$$

bringing the system (1.1) *to the form*

$$\begin{aligned} \dot{X} &= Xh(X,Y) \\ \dot{Y} &= -\lambda Y h(X,Y), \end{aligned} \tag{2.2}$$

with $h(X,Y) = 1+O(X,Y)$*. If* $h(X,Y)=1$*, then the system is linearizable at the origin.*

(ii) *For $\lambda = \frac{p}{q} \in \mathbb{Q}^+$ the system is orbitally normalizable at the origin if there exists an analytic change of coordinates of the form* (2.1) *transforming* (1.1) *to the semi-normal form*

$$\begin{aligned} \dot{X} &= Xk_1(U)h(X,Y) \\ \dot{Y} &= -\lambda Y k_2(U)h(X,Y), \end{aligned} \tag{2.3}$$

where k_1, k_2 and h are analytic functions such that $h(0,0) = 1$, $U = X^pY^q$ and $k_1(0) = k_2(0) = 1$. If $h(X,Y) = 1$, then the system is normalizable at the origin.

(iii) *For $\lambda = 0$ the system is orbitally normalizable at the origin if there exists an analytic change of coordinates of the form* (2.1) *transforming* (1.1) *to the semi-normal form*

$$\begin{aligned} \dot{X} &= Xk_1(Y)h(X,Y) \\ \dot{Y} &= k_2(Y)h(X,Y), \end{aligned} \tag{2.4}$$

where k_1, k_2 and h are analytic functions such that $k_1(0) = 1$ and $h(0,0) = 1$. If $h(X,Y) = 1$, then the system is normalizable at the origin.

Remarks 2.4.

(1) The case (iii) above is nothing more than case (ii) taking $p = 0$ and $q = 1$. We separate it for ease of comparison. We continue this policy throughout the paper.
(2) For $\lambda \neq 0$ the system (1.1) is integrable if and only if the holonomy of any separatrix is linearizable. This follows from the theorems of Mattei-Moussu [MM].
(3) For $\lambda \neq 0$ (resp $\lambda = 0$) the system (1.1) is orbitally normalizable if and only if the holonomy of any separatrix (resp. of the strong separatrix) is *embedable* i.e. given by the time-one map of the flow of a vector field in a neighborhood of the origin in $\mathbb{C}$ composed with a rotation (see for instance [MR, CMR]).

Definition 2.5. *We say that a meromorphic 1-form η on $\mathbb{C}^2$ is relatively exact with respect to a 1-form ω if there exists an analytic function h and a meromorphic function k such that*

$$\eta = dh + k\omega. \tag{2.5}$$

Remark 2.6. Note that in the above definition we require h to be analytic. Usually, for η meromorphic, one would require h to be only meromorphic. In fact, in our principal applications, η, ω, h and k can be chosen to be analytic.

2.2. The time-form of a vector field.

Definition 2.7. *Given a meromorphic vector field V defined on a subset of $\mathbb{C}^2$, we say that any meromorphic form ω such that*

$$i_V(\omega) = 0 \tag{2.6}$$

is dual to V.

Definition 2.8. *A time-form of a vector field V is a 1-form denoted dt, such that*

$$i_V dt = 1. \tag{2.7}$$

Remarks 2.9.

(1) The notation dt is in no way meant to imply that the form is exact, but just to indicate its role of encapsulating the time element of the vector field.

(2) All dual forms of a vector field are given as meromorphic multiples of the dual form

$$\omega = i_V(dx \wedge dy). \tag{2.8}$$

Relation (2.8) is often used for defining dual forms. We prefer definition (2.6) as we require the knowledge of ω only up to a meromorphic multiple. Geometrically, the dual form ω defines the foliations given by the vector field, but not the vector field.

(3) Similarly, time-forms of a vector field are uniquely determined up to addition of dual forms. There is however a 1-1 correspondence between, on the one hand nonzero meromorphic vector fields V and on the other, pairs of classes of 1-forms (ω, dt), where ω is dual to V and dt is a nonzero class of 1-forms. The classes of forms are taken with respect to the equivalence relation where two forms are equivalent if they differ by a meromorphic multiple of ω.

We will work mainly with analytic vector fields. Then a dual form ω can be taken analytic, but its time-form dt is in general only meromorphic.

2.3. The transformation of a vector field V under the pull-back by the time-g flow of V, Φ^g, for some non-constant time g.

The following result looks classical, but we have only found references to it in the unpublished thesis of Natali Pazii [P], and in the recent preprint of Loïc Teyssier [T1].

Let ϕ be a diffeomorphism, V a vector field. Denote $\phi^*(V)$ the pull back of the vector field V by ϕ defined by

$$\phi^*(V) = d\phi^{-1}(V \circ \phi). \tag{2.9}$$

Proposition 2.10. *Let V be an analytic vector field in the neighborhood of the origin in $\mathbb{C}^n$, and $X = X(t;x)$ represent its flow, with $X(0,x) = x$. Then the local diffeomorphism*

$$\alpha : x \mapsto X(g(x), x) \tag{2.10}$$

pulls back the vector field V to the vector field $(1 + V(g))^{-1}V$. That is

$$\alpha^*(V) = (1 + V(g))^{-1}V. \tag{2.11}$$

Proof. We write the vector field V as $\sum_{i=1}^n V^i(x)\frac{\partial}{\partial x^i}$. From the definition of the flow X, we therefore have

$$\frac{\partial}{\partial t}X^i(t,x) = V^i(X).$$

However, since X is a flow we also have

$$X(t, X(s,x)) = X(t+s, x).$$

Differentiating this expression with respect to s, we get

$$\frac{\partial}{\partial s}X^j(t+s,x) = \sum_{i=1}^n \frac{\partial X^j}{\partial x^i}(t, X(s,x))\frac{\partial}{\partial t}X^i(s,x) = \sum_{i=1}^n \frac{\partial X^j}{\partial x^i}(t, X(s,x))V^i(x)$$

from which, setting $s = 0$, we obtain

$$V^j(X) = \sum_{i=1}^n \frac{\partial X^j}{\partial x^i}V^i(x). \tag{2.12}$$

To prove the proposition, we want to find the effect of the substitution $Y = X(g(x), x)$ on the vector field $V^i(Y)\frac{\partial}{\partial Y^i}$. To do this, first note that from (2.12) we have

$$V^j(Y) = \sum_{i=1}^n \frac{\partial X^j}{\partial x^i}\Big|_{t=g(x)}V^i(x).$$

Furthermore,

$$\begin{aligned}\sum_{i=1}^n \frac{\partial Y^j}{\partial x^i}V^i(x) &= \frac{\partial X^j}{\partial t}\Big|_{t=g(x)}\sum_{i=1}^n \frac{\partial g}{\partial x^i}V^i(x) + \sum_{i=1}^n \frac{\partial X^j}{\partial x^i}\Big|_{t=g(x)}V^i(x)\\ &= V^j(Y)\sum_{i=1}^n \frac{\partial g}{\partial x^i}V^i(x) + V^j(Y)\\ &= V^j(Y)(V(g)+1),\end{aligned}$$

whence,

$$\sum_{i=1}^n V^i(x)\frac{\partial}{\partial x^i} = \sum_{i=1}^n\sum_{j=1}^n V^i(x)\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial Y^j} = (V(g)+1)\sum_{j=1}^n V^j(Y)\frac{\partial}{\partial Y^j}.$$

□

Let V be a germ of a vector field defined at the origin $(0,0)$ with an isolated singularity at the origin. Let $\mathcal{F}$ be the germ at the origin of a foliation regular in a complement of the origin whose leaves are orbits of V.

Definition 2.11. *Let V, $\mathcal{F}$ be as above and let ϕ be a germ of a diffeomorphism at $(0,0)$, fixing the origin.*

(i) *We say that ϕ is an isotropy of the foliation $\mathcal{F}$ defined by a vector field V if ϕ sends leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$.*

(ii) *We say that an isotropy ϕ fixes the leaves of the germ of the foliation $\mathcal{F}$ if for any representatives of the germ V and the germ ϕ defined on a neighborhood U of the origin, there exists a neighborhood $W \subset U$ of the origin such that $\phi(W) \subset U$ and that for any point $p \in W$, its image $\phi(p)$ belongs to the same leaf of the foliation when considered over U.*

Remark 2.12. A diffeomorphism is an isotropy of the foliation defined by a vector field V if and only if the vector fields $\phi^*V = d\phi^{-1}(V \circ \phi)$ and V are collinear. That is

$$\phi^*(V) = \tilde{u}V \tag{2.13}$$

for some analytic function germ $\tilde{u}$ at the origin.

Proposition 2.13. *Let V be a vector field of the form* (1.1) *and let ϕ be a germ of a diffeomorphism fixing the leaves of the germ of the foliation given by V. Then there exists an analytic function g such that ϕ is the time-g flow of V.*

Proof. In the saddle case we can assume without loss of generalities that the invariant manifolds are the coordinate axes. As ϕ preserves the leaves it must preserve the coordinate axes. In this form the claim is given in [BCM], Proposition 2.13 in the integrable case and Corollary 2.16 in the nonintegrable saddle case.

In the saddle-node case we assume that the vector field is in Dulac normal form, i.e. of the form

$$V = uX \quad \text{with} \quad u(0,0) = 1 \quad \text{and} \quad X = X_{k,\mu} + x^{k+1}R(x,y), \tag{2.14}$$

where

$$X_{k,\mu} = x(1 + \mu y^k)\partial/\partial x + y^{k+1}\partial/\partial x.$$

This is no loss of generality since, if the vector field is in Dulac normal form and the mapping ϕ preserves the foliation, then, by Remark 2.12, the pull-back $\phi^*(V)$ will be collinear with V, and hence again in Dulac normal form. Now we can use the results [T3] for mappings preserving the Dulac normal form: the proof relies on the stronger Corollary 4.2.24. from [T3] describing the group of isotropies of a saddle-node (not ne cessarily fixing leaves of the foliation). As ϕ fixes the strong and the formal weak invariant separatrices, it is of the form

$$\phi(x,y) = (\alpha x + \dots, \beta y + \dots). \tag{2.15}$$

Teyssier shows that any diffeomorphism ϕ of the form (2.15) can be decomposed as

$$\phi = Rot_{2\pi\theta/k} \circ N \circ \Phi_V^g, \quad n, \theta \in \mathbb{Z}, \quad c \in \mathbb{C}, \tag{2.16}$$

where g is a germ of an analytic function. Here $Rot_\gamma(x,y) = (x, e^{i\gamma}y)$, $N(x,y) = (x + o(x,y), y)$ is a fibred map and Φ_V^g is the time-g flow map of the vector field V.

We want to show that, under the additional hypothesis that ϕ fixes the leaves of the germ of the foliation, in the decomposition (2.16) we need only the last term. Note first that $\beta^k = 1$ in (2.15). Indeed, considering the first coordinate of $\phi^*(V) = \tilde{u}V$, we see that $\tilde{u}(0,0) = 1$, which gives the claim by considering the second coordinate of $\phi^*(V)$.

Let us now recall the description of the orbit space of the germ of the foliation of a saddle-node [MR]. The sectorial normalization theorem [HKM] shows the existence of a decomposition of a neighborhood of the origin $(0,0) \in \mathbb{C}^2$ as a union of $2k$ *sectors* $\{S_j^{sn}, S_j^{ns}\}$, $\{j = 0, \dots, k-1\}$. Each S_j^{sn} and S_j^{ns} is given as a product of a disc in the x-plane with a true sector of width slightly more than π/k in the y-plane and turning counterclockwise in the y-plane they appear in the cyclic order S_0^{sn}, S_0^{ns}, S_1^{sn}, S_1^{ns}, etc. The notations *sn* (and *ns*) means that turning counter clocwise in the y plane the dynamics in S_j^{sn} passes from saddle-like to node-like (and vice versa) [HKM]. Moreover, on each of these sectors there is an analytic diffeomorphism $(x, y) \mapsto (\xi, \eta)$ of the form

$$(\xi_j, \eta_j) = (x + o(x, y), y), \tag{2.17}$$

transforming the vector field V to the model vector field $X_{k,\mu}$ up to a multiplicative factor. We will not be precise about the precise choice of the sectors. For more details see [MR].

The orbit space of a germ of V restricted to any of the above sectors is analytically equivalent to a Riemann sphere. Consider two neighboring sectors intersecting first on a node-type region. In the normalizing coordinates (via the first integral) a leaf in a sector S_j^{sn} is given by $\xi_j^{sn} = c_j y^\mu e^{-1/(ky^k)}$. It extends to the neighboring sector S_j^{ns} as $\xi_j^{ns} = c_j' y^\mu e^{-1/(ky^k)}$, where $c'j = \psi^\infty(c_j) = c_j + K_j$ with ψ^∞ the translation part of the Martinet-Ramis modulus. From the behavior of the exponential function and the choice of the sectors it follows that all leaves tend to the origin on the node-type region. More precisely, for any neighborhood of the origin, it follows that any leaf of S_j^{ns} cuts $S_j^{ns} \cap S_j^{sn}$ and analogously for S_j^{sn}. That means that even at the germ level each leaf in S_j^{sn} is glued to some leaf of S_j^{ns} by the glueing map ψ^∞. That is the orbit space of a germ of a foliation above $S_j^{ss} = S_j^{ns} \cup S_j^{sn}$ is given by one copy of a Riemann sphere. An analogous claim is no longer true on the intersection of neighboring sectors intersection in a saddle-type sector. The reason is that for any given neighborhood U of the origin, and any smaller neighborhood $W \subset U$ of the origin, there are leaves belonging to $S_j^{ns} \cap W$ which do not cut $S_{j+1}^{sn} \cap W$. Hence, necessarily there are leaves in S_j^{ns} which are not glued to any leaf of S_{j+1}^{sn}. We conclude that the assumption that the germ of the diffeomorphism ϕ fixes the leaves of the germ of the foliation defined by V is only possible if $\beta = 1$ and the rotational term $Rot_{2\pi\theta/k}$ in the presentation (2.16) is trivial (i.e. equal to the identity).

We now show that the second term in (2.16) is also trivial. A diffeomorphism of the form N preserves the large sectors S_j^{ss}. In order to act nontrivially, there should be leaves in a given sector S_j^{ss} cutting twice some line $y = y_0 \neq 0$. As the set of leaves is indexed by the Riemann sphere this should be the case for all leaves except a finite number of fixed leaves. However in a sector S_j^{ss} we can use sectorial coordinates in which a leaf is given as a graph of the function

$$x = cy^\mu e^{-1/ky^k}, \tag{2.18}$$

univalued on the sector. So any graphs (2.18) cuts exactly once each line $y = y_0$. Hence the only way it can occur is that distinct graphs of the form (2.18) belong to a unique leaf, the glueing occuring outside S_j^{ss}. But only graphs corresponding to c in a neighborhood of the origin on the Riemann sphere can actually be extended outside S_j^{ss}. So any map N in (2.16) is trivial.

3. Formal normal forms for orbital equivalence and conjugacy of vector fields

In this section we give reduced formal normal forms for a critical point of saddle or saddle-node type. The following normal forms for formal (orbital) equivalence are well-known.

Proposition 3.1.

Let (1.1) *be a critical point of saddle or saddle-node type.*

(i) *If $\lambda \neq 0$ is irrational, then* (1.1) *is formally conjugate to the linear vector field*

$$\begin{aligned} \dot{X} &= X \\ \dot{Y} &= -\lambda Y. \end{aligned} \tag{3.1}$$

(ii) *If $\lambda = p/q \neq 0$ is rational, $p, q \in \mathbb{Z}$ relatively prime, then* (1.1) *is either formally orbitally equivalent to the linear form* (3.1) *(in fact, it is analytically orbitally equivalent to* (3.1) *in this case) or formally orbitally equivalent to a vector field of the form*

$$\begin{aligned} \dot{X} &= X(1 + aU^k) \\ \dot{Y} &= -\frac{p}{q} Y(1 + (a-1)U^k). \end{aligned} \tag{3.2}$$

(iii) *If $\lambda = 0$ and the origin is an isolated singular point, then* (1.1) *is formally orbitally equivalent to a vector field of the form*

$$\begin{aligned} \dot{X} &= X(1 + aY^k) \\ \dot{Y} &= Y^{k+1}. \end{aligned} \tag{3.3}$$

For λ irrational, a formally integrable system is formally linearizable. However, this is no longer true in the analytic category. In fact, in [CMR] we give a sharp condition on irrational λ to ensure that any integrable system of the form (1.1) is linearizable. The condition is that λ is not a Cremer number. That is, the denominators q_n in its continuous fraction expansion satisfy

$$\limsup_{n\to\infty} \frac{1}{q_n} \log q_{n+1} < +\infty.$$

The condition of being non Cremer is weaker than the Brjuno condition, which guarantees linearizability of (1.1) for an irrational Brjuno number λ.

We now study the normal forms under *formal conjugacy* i.e. the equivalence relation induced by formal changes of coordinates without multiplication by a function (i.e. a generalized "time scaling").

Proposition 3.2. *Let* (1.1) *be a saddle or saddle-node type vector field.*

(i) *If* $\lambda \neq 0$ *is irrational, then* (1.1) *is formally conjugate to the linear vector field*

$$\begin{aligned} \dot{X} &= X \\ \dot{Y} &= -\lambda Y. \end{aligned} \tag{3.4}$$

(ii) *If* $\lambda = p/q \neq 0$ *is rational,* $p, q \in \mathbb{Z}$ *relatively prime, and*

$$U = X^p Y^q, \tag{3.5}$$

then one of the following cases is satisfied for (1.1)*:*

(iia) If (1.1) *is integrable then it is analytically conjugate, either to the linear vector field, or to a vector field of the form*

$$\begin{aligned} \dot{X} &= X(1 + U^k) \\ \dot{Y} &= -\frac{p}{q} Y(1 + U^k), \end{aligned} \tag{3.6}$$

for some $k \in \mathbb{N}$.

(iib) If (1.1) *is not integrable then it is formally conjugate to a vector field of the form*

$$\begin{aligned} \dot{X} &= X(1 + aU^k)(1 + a_1 U + \cdots a_k U^k) \\ \dot{Y} &= -\frac{p}{q} Y(1 + (a-1)U^k)(1 + a_1 U + \cdots a_k U^k), \end{aligned} \tag{3.7}$$

for some $a_1, \ldots, a_k \in \mathbb{C}$. *If* (1.1) *is not integrable but is normalizable, then it is analytically conjugate to* (3.7)

(iii) *If* $\lambda = 0$ *and the origin is an isolated singular point of* (1.1)*, then* (1.1) *is formally conjugate to a vector field of the form*

$$\begin{aligned} \dot{X} &= X(1 + aY^k)(1 + a_1 Y + \cdots a_k Y^k), \\ \dot{Y} &= Y^{k+1}(1 + a_1 Y + \cdots a_k Y^k), \end{aligned} \tag{3.8}$$

for some $a_1, \ldots, a_k \in \mathbb{C}$. *If* $\lambda = 0$ *and* (1.1) *is normalizable, then it is analytically conjugate to* (3.8)

Proof.

(i) is well known (see for instance [B]).

(iia) It is well known (see for instance [B]) that an integrable vector field is analytically conjugate to a normal form

$$\begin{aligned} \dot{X} &= X(1 + \sum_{i=1}^{\infty} a_i U^i) \\ \dot{Y} &= -\frac{p}{q} Y(1 + \sum_{i=1}^{\infty} a_i U^i), \end{aligned} \tag{3.9}$$

If all a_i vanishes then the system is linearizable. Otherwise let a_k, $k \geq 1$, be the first non-vanishing a_i. To transform the system to the form (3.6) we take a transformation $(X,Y) = (xg(u), y)$ with g analytic and $g(0) = a_k^{1/(kp)}$. The existence of such a g is guaranteed by the implicit function theorem.

(iib) and (iii) To prove these cases we need to study the isotropies of the corresponding normal forms (i.e. the changes of coordinates which preserve the orbital normal forms). For completeness, we give the formulas for (i) and (iia) also.

Let $T_{r,s}$ denote the transformation

$$T_{r,s}(x,y) = (X,Y) = (rx, sy) \tag{3.10}$$

and let $h(x,y)$ be a non-zero analytic function.

Proposition 3.3.

(i)/(iia) *Consider the system*

$$\begin{aligned} \dot{x} &= xh(x,y), \\ \dot{y} &= -\lambda y h(x,y). \end{aligned} \tag{3.11}$$

The isotropies of this system are compositions of $T_{r(u),s(u)}$*, with* r *and* s *non-vanishing analytic functions of* u*, and transformations of the form*

$$\begin{aligned} X &= xe^{g(x,y)}, \\ Y &= ye^{-\lambda g(x,y)}, \end{aligned} \tag{3.12}$$

with $g(x,y)$ *analytic, where* $u = x^p y^q$ *when* $\lambda = p/q$ *and* 0 *otherwise.*

(iib) *Consider the system*

$$\begin{aligned} \dot{x} &= x(1 + au^k)h(x,y), \\ \dot{y} &= -\frac{p}{q} y(1 + (a-1)u^k)h(x,y), \end{aligned} \tag{3.13}$$

where $u = x^p y^q$*. Isotropies of* (3.13) *are compositions of* $T_{r,s}$*, with* $r^{pk}s^{qk} = 1$*, and transformations of the form*

$$\begin{aligned} X &= x(1 - pku^k g(x,y))^{-a/(pk)} e^g = xm(x,y), \\ Y &= y(1 - pku^k g(x,y))^{(a-1)/(qk)} e^{-gp/q} = yn(x,y), \end{aligned} \tag{3.14}$$

where $g(x,y)$ *is an analytic function.*

(iii) *Consider the system*

$$\begin{aligned} \dot{x} &= x(1 + ay^k)h(x,y), \\ \dot{y} &= y^{k+1}h(x,y). \end{aligned} \tag{3.15}$$

Isotropies of (3.15) *are compositions of* $T_{r,s}$*, with* $s^k = 1$*, and transformations of the form*

$$\begin{aligned} X &= x(1 - ky^k g(x,y))^{-a/k} e^g = xm(x,y), \\ Y &= y(1 - ky^k g(x,y))^{-1/k} = yn(x,y), \end{aligned} \tag{3.16}$$

where $g(x,y)$ is an analytic function.

(iv) *Denoting V the vector fields given by* (3.11), (3.13) *and* (3.15) *respectively and $V_0 = V/h$ the vector field of the orbital normal form, the transformations* (3.12), (3.14) *and* (3.16) *are given by the $g(x,y)$-time flow of the orbital normal form vector field V_0. They have the effect of changing the vector field hV_0 to the vector field $r(X,Y)V'$, where V' is just the vector field V_0 under the direct substitution $x = X$ and $y = Y$ and $r(X,Y)$ is obtained from $h(x,y)(1 + V_0(g))$ under the substitutions* (3.12), (3.14) *and* (3.16) *above.*

Proof. The statement (iv) is just Proposition 2.10, but with the inverse of the transformation given there.

For the rest of the proposition, we prove only (iib), as the proof of (i)/(iia) and (iii) are very similar. An isotropy of the orbital normal form must fix the invariant coordinate axes, and so must be of the form

$$\begin{aligned} X &= xm(x,y) = x(r + O(x,y)) \\ Y &= yn(x,y) = y(s + O(x,y)). \end{aligned} \tag{3.17}$$

The new system has the form

$$\begin{aligned} \dot{X} &= X(1 + aU^k)H(X,Y), \\ \dot{Y} &= -\frac{p}{q}Y(1 + (a-1)U^k)H(X,Y), \end{aligned} \tag{3.18}$$

where $U = X^pY^q$. The system (3.13) has the first integral

$$F(x,y) = x^{pk(a-1)}y^{qka}e^{-1/x^{pk}y^{qk}} = u^{ka}e^{-1/u^k}x^{-pk} = u^{k(a-1)}e^{-1/u^k}y^{kq}. \tag{3.19}$$

We claim that the first integral must be preserved, that is

$$F(X(x,y), Y(x,y)) = CF(x,y) \tag{3.20}$$

for some constant C. Indeed, $F(X(x,y), Y(x,y))$ is also a first integral of (3.13), so the quotient $F(X(x,y), Y(x,y))/F(x,y)$ is also a first integral of (3.13). It has the form

$$\frac{F(X(x,y), Y(x,y))}{F(x,y)} = K(x,y)e^{1/u^k - 1/U^k}, \tag{3.21}$$

with

$$K = m^{pk(a-1)}n^{qka} = r^{pk(a-1)}s^{qka} + O(x,y). \tag{3.22}$$

Therefore taking the logarithm of (3.21), we obtain a meromorphic first integral of (3.13). However, (3.13) being non integrable (because of the presence of resonant terms), any meromorphic first integral is trivial (i.e. constant).

Let

$$pk\, g(x,y) = \frac{1}{u^k} - \frac{1}{U^k} = \frac{1}{u^k}\frac{m^{pk}n^{qk} - 1}{m^{pk}n^{qk}}, \tag{3.23}$$

then from above, $\log(K) + pk\, g$ is a constant function, and hence $g(x,y)$ is analytic. The value of the quotient in (3.21) is then $C = r^{pk(a-1)}s^{qka}e^{pk\, g_0}$, with $g_0 = g(0,0)$. Rearranging (3.23) gives

$$m^{pk}n^{qk} = (1 - pku^k g(x,y))^{-1}. \tag{3.24}$$

Hence $n = m^{-\frac{p}{q}}(1 - pku^k g(x,y))^{-\frac{1}{qk}}$. Putting this relation in (3.21) we obtain

$$m^{-pk}(1 - pku^k g)^{-a} e^{pk\,g} = r^{pk(a-1)} s^{qka} e^{pk\,g_0}. \tag{3.25}$$

Evaluating this equation at the origin, we see that $r^{pka}s^{qka} = 1$. This gives $m/r = (1 - pku^k g)^{-\frac{a}{pk}} e^{g-g_0}$, and $n/s = (1 - pku^k g)^{\frac{a-1}{qk}} e^{-\frac{p}{q}(g-g_0)}$. After composing with $T_{e^{g_0}, e^{-pg_0/q}}$, we obtain our result. $\square$

End of proof of Proposition 3.2.

Once again, we present the proof for (iib) only, as the case(iii) follows along similar lines.

From Proposition 3.3 (iv) the system (3.13) is transformed to

$$\begin{aligned} \dot{X} &= X(1 + aU^k)r(x,y), \\ \dot{Y} &= -\frac{p}{q}Y(1 + (a-1)U^k)r(x,y), \end{aligned} \tag{3.26}$$

where $r(x,y)$ is given by

$$r = h(x,y)(1 + V_0(g)), \tag{3.27}$$

with

$$V_0 = x(1 + au^k)\frac{\partial}{\partial x} - \frac{p}{q}y(1 + (a-1)u^k)\frac{\partial}{\partial y}. \tag{3.28}$$

We therefore need to solve

$$V_0(g) = \frac{1}{h(x,y)}P(U) - 1, \tag{3.29}$$

for some choice of $P(U) = \sum_{i=0}^k a_i U^i$ with $a_0 = 1$. Formally, it is clear that if the right hand side of (3.29) contains no term in u^i for $i = 0, \dots, k$, then there is a unique solution. However, we know that $U = u(1 - pku^k g(u))^{-1/k}$, and so

$$\sum_{i=0}^{k} a_i U^k = \sum_{i=0}^{k} a_i u^k + O(u^{k+1}). \tag{3.30}$$

We write

$$1/h(x,y) = \sum_{i,j \geq 0} b_{i,j} x^i y^j. \tag{3.31}$$

Then, in order to have a formal solution the coefficients a_i of $P(U)$ must be chosen as $a_0 = 1$ and

$$a_i = -\sum_{j=0}^{i-1} a_j b_{i-j,i-j}. \tag{3.32}$$

To prove the final statement of (iib) we can assume that $h = h(u)$, so that (3.29) becomes

$$V_0(g(u)) = \frac{1}{h(u)} \sum_{i=0}^{k} a_i U^k - 1, \tag{3.33}$$

with $U = u(1-pku^k g(u))^{-1/k}$. To show that (3.33) has an analytic solution reduces to finding an analytic solution to the differential equation

$$pu^{k+1}g'(u) = \frac{1}{h(u)}P(u(1-pku^k g(u))^{-1/k}) - 1 = u^{k+1}R(u, g(u)), \tag{3.34}$$

for some analytic function R. But this follows directly from the standard existence and uniqueness results for differential equations.

In the case (iia) we have $U = u$ and $V_0 = x\frac{\partial}{\partial x} - \frac{p}{q}y\frac{\partial}{\partial y}$. The corresponding equation to (3.29), which brings (3.11) with $\lambda = p/q$ to the form (3.6) can always be solved analytically. □

Results in the same spirit can be found in [VG] and [Ya], although their reduced normal form is not exactly the same as ours. We note that the transformations (3.12), (3.14) and (3.16) fix the leaves of the foliation, whilst the transformations $T_{r,s}$ may interchange leaves.

Corollary 3.4. *There are exactly k obstructions to the existence of a formal change of variables transforming the system* (3.13) *to the form*

$$\begin{aligned} \dot{x} &= x(1 + au^k) \\ \dot{y} &= -\lambda y(1 + (a-1)u^k). \end{aligned} \tag{3.35}$$

They are given by the nonvanishing of the coefficients $a_1, \dots, a_k$ in (3.7)*. A similar statement holds for system* (3.15)*.*

Remark 3.5. The normal form (3.7) (resp. (3.8)) is not unique and the a_i are transformed under changes of coordinates of the form $T_{r,s}$ with $r^{pk}s^{qk} = 1$ and $r^p s^q \neq 1$ (resp. $s^k = 1$ and $s \neq 1$). In the case $\lambda \neq 0$ (resp. $\lambda = 0$) such transformations do not fix (resp. fix) the leaves of the foliation.

Definition 3.6. *The k normal forms* (3.7) *(resp.* (3.8)*) associated to the orbital normal forms* (3.2) *(resp.* (3.3)*) are called conjugate forms. They may not all be distinct.*

4. Linearizability, normalizability and relative exactness.

In this section we find conditions for an orbitally normalizable saddle or saddle-node to be normalizable, i.e. conjugate to one of the formal normal forms of Proposition 3.2, in terms of the relative exactness of a time-form which we will give below. However, since it is no extra work, we shall deal with the general case of deciding when two vector fields of the form (1.1) with the same orbits, V_1 and $V_2 = hV_1$ are analytically conjugate.

Lemma 4.1. *If $V = hV_0$ is a vector field of the form* (1.1)*, with $h = 1 + O(x, y)$ then, when $\lambda \neq 0$, there exists an analytic change of coordinates fixing the leaves of the foliation, which brings the vector field to the form*

$$\tilde{V} = (1 + xl(x, y))V \tag{4.1}$$

where l is analytic. In particular, an integrable system can always be brought, by a change of coordinates fixing the leaves of the foliation, to the form

$$\begin{aligned} \dot{x} &= xh(x, y) \\ \dot{y} &= -\lambda yh(x, y) \end{aligned} \tag{4.2}$$

with

$$h = 1 + xl(x, y) \tag{4.3}$$

for some analytic function l. If $\lambda = 0$ then there exists a change of coordinates which replaces h by a function with no terms in y^s, $s > k$.

Proof. If $\lambda \neq 0$, we can write $1/h = 1 + h_1(y) + xm(x, y)$, for some analytic function h_1 and m. From Proposition 4.1, we only need to show that there is some function g with

$$V_0(g) = h_1(y) + x\tilde{m} \tag{4.4}$$

and then pull back via α defined in Proposition 2.5. However, it is easy to see that we can find such an g of the form $g_1(y)$. The case when $\lambda = 0$ is similar. □

Definition 4.2.

(1) *We consider an orbitally normalizable vector field of the form* (1.1) *which has been brought by a change of coordinates to a vector field V of one of the forms* (3.11), (3.13) *and* (3.15) *in the neighborhood of a point p, and ω a reduced 1-form dual to V. Let dt be the time-form of V and dt_{norm} be the time-form of a formal normal vector field associated to V. More precisely, we take*

$$dt_{norm} = \begin{cases} \frac{dx}{x} & \textit{for} \quad (3.11) \\ \frac{dx}{x(1+au^k)P(u)} & \textit{for} \quad (3.13) \\ \frac{dx}{x(1+ay^k)P(y)} & \textit{for} \quad (3.15). \end{cases} \tag{4.5}$$

where $P(u)$ and $P(y)$ are defined as in the proof of Proposition 3.2.

(2) *In the cases* (3.11) *and* (3.15) *time-forms associated to conjugate normal forms (see Remark 3.5 and Definition 3.6) are called conjugate time-forms.*

The following Lemma is obvious:

Lemma 4.3. *In the cases* (3.13) *and* (3.15) *we define the form $\eta = dt - dt_{norm}$. After applying a change of coordinate as in Lemma 4.1 we get a form $\eta = dt - dt_{norm}$ which is holomorphic.*

Theorem 4.4.

(1) *Let V_1 and $V_2 = \tilde{h}V_1$ be two vector fields in $\mathbb{C}^2$ with $\tilde{h} \neq 0$, and let η to be the difference of their time-forms*

$$\eta = dt_2 - dt_1. \tag{4.7}$$

If η is relatively exact with respect to some 1-form ω dual to V_1 (or V_2) then V_1 and V_2 are analytically conjugate by a transformation fixing the leaves of the foliation locally.

(2) *Let V_1 and $V_2 = \tilde{h}V_1$ be two vector fields of the form* (1.1) *with $\tilde{h} \neq 0$, and take η as in* (4.7)*. Then V_1 and V_2 are analytically conjugate by a transformation fixing the leaves of the foliation locally, if and only if η is relatively exact with respect to some 1-form ω dual to V_1 (or V_2).*

(3) *We consider an orbitally normalizable vector field of the form* (1.1) *which has been brought by a change of coordinates to a vector field V of one of the forms* (3.11)*,* (3.13) *and* (3.15) *in the neighborhood of a point p, and ω a reduced 1-form dual to V. Let dt be the time-form of V and dt_{norm_i}, $i = 1, \dots, k$ be the time-forms of its conjugate formal normal forms. Then V is analytically conjugate to its normal form if and only if there exists i such that*

$$\eta = dt_{norm_i} - dt, \tag{4.8}$$

is relatively exact with respect to ω.

Proof.

(1) Let us suppose that η given in (4.7) is relatively exact. We look for a conjugacy which is a pullback of V_1 under the map $\Phi^{g(x,y)}$, where Φ^t is the time-t flow of V_1 and g is some analytic function. By Proposition 2.10 the pullback will be V_2 if $\tilde{h} = (1 + V_1(g))^{-1}$. Hence it is sufficient that the function g satisfies $V_1(g) = 1/\tilde{h} - 1$. If V_1 is of the form $Q_1\partial_x + Q_2\partial_y$, then taking $\omega = Q_1\,dy - Q_2\,dx$ we see that this is equivalent to relation

$$\eta \wedge \omega = dg \wedge \omega, \tag{4.9}$$

which follows directly from the relative exactness of η.

(2) Direct implication follows from (1). Conversely, let us suppose that V_1 and V_2 are analytically conjugate by a transformation fixing the leaves of the foliation locally. By Proposition 2.13 we have shown that the change of coordinate is necessarily of the form $\Phi^{g(x,y)}$, hence (4.9) is verified. Thus, the forms $\eta - dg$ and ω are collinear, which shows that the form η must be relatively exact.

(3) Again, one direction follows from (1). Indeed suppose (4.8) is relatively exact with respect to ω. If V_0 is the orbital normal form we have $V = hV_0$. If $\lambda > 0$ let $V_2 = P_i(u)V_0$ be the conjugate normal form corresponding to dt_{norm_i} in (4.8), so that $\tilde{h}(x,y) = P_i(u)/h(x,y)$, then (1) shows that (4.9) is equivalent to the existence of some g satisfying $V_1(g) = h/P_i - 1$, which is exactly the condition required for analytic conjugacy of V with V_2. The case $\lambda = 0$ is similar. To prove the converse, suppose V is analytically conjugate to one of its normal forms, then from Proposition 3.3 the transformations which fix the vector field orbitally are composites of $T_{r,s}$ with $r^{pk}s^{qk} = 1$ when $\lambda > 0$ (resp. $s^k = 1$ when $\lambda = 0$) and a time g map for some analytic g, and the result follows. $\square$

Remark 4.5. Note that relation (4.9) can be written also as the *homological equation*

$$V(g) = i_V(\eta). \tag{4.10}$$

in the unknown variable g. This form is used by Teyssier in [T1], [T2] and [T3]. Hence relative exactness of η with respect to the form ω dual to a vector field V is equivalent to the solvability of the homological equation (4.10).

5. Relative exactness and asymptotic cycles

In this final section we shall show that the notion of relative exactness of a 1-form η with respect to a differential form ω is equivalent to the vanishing of the integral of η along certain asymptotic cycles defined on the leaves of the foliation $\omega = 0$.

Definition 5.1. *Let $[\alpha_n, \beta_n] \subset \mathbb{R}$ be a sequence of increasing nested intervals and let $\gamma_n : [\alpha_n, \beta_n] \to \mathbb{C}^2$ be a sequence of curves, all coinciding on the intersection of their domains.*

(i) *We say that the sequence $\gamma = (\gamma_n)$, $n \in \mathbb{N}$, is an asymptotic cycle if*

$$\lim \gamma(\alpha_n) = \lim \gamma(\beta_n). \tag{5.1}$$

(ii) *We say that the form η is integrable over γ if $\lim_{n\to\infty} \int_{\gamma_n} \eta$ exists, in which case we define*

$$\int_\gamma \eta = \lim_{n\to\infty} \int_{\gamma_n} \eta. \tag{5.2}$$

Theorem 5.2. *Let ω be a 1-form dual to a vector field of the form* (1.1)*. If λ is irrational, we assume that the vector field is integrable. Then any analytic 1-form η is relatively exact with respect to ω if and only if*

$$\int_\gamma \eta = 0, \tag{5.3}$$

for any asymptotic cycle γ belonging to a leaf of the foliation defined by $\omega = 0$.

The claim of the theorem has been proven in the saddle resonant case by Berthier and Cerveau [BC] (integrable case) and Berthier and Loray [BL] (nonintegrable case) and in the saddle-node case by Teyssier [T1]. The only remaining case is the case of an integrable non resonant saddle. We shall give this proof below.

Definition 5.3. *Given two vector fields V_1 and $V_2 = hV_1$ with time-forms dt_1 and dt_2, we say that the two time-forms are synchronous if the integral of their difference is zero along every asymptotic cycle γ belonging to a leaf of the foliation defined by $\omega = 0$, where ω is some 1-form dual to V_1 (or V_2). We say that a time-form is isochronous if it is synchronous to the time-form of a linear system.*

Theorem 5.4.

(1) *Let V_1 and $V_2 = hV_1$ be two analytic vector fields in the form (1.1), and, if λ is irrational, assume that the vector fields V_i are integrable. Then the two vector fields are analytically conjugate by a change of coordinates fixing the leaves of the foliation if and only if their time-forms are synchronous.*

(2) *Let V be an orbitally normalizable vector field in the form* (3.11), (3.13) *or* (3.15). *Then V is normalizable if and only if its time-form is synchronous with one of the conjugate time-forms dt_{norm_i} given by* (4.5) *of its conjugate formal normal forms. In particular, the time-form of an integrable system of the form* (1.1) *is isochronous if and only if it is synchronous to the linear form dX/X, for some variable X which vanishes on the separatrix tangent to the x-axis of* (1.1).

Proof.

(1) The results follows directly from Theorem 4.4 and Theorem 5.2.

(2) This follows from the fact that, transforming to a normalized system (4.8) we have $X = xW(x,y)$, for some non vanishing function W, and $dX/X = dx/x + dW/W$, so the relative exactness with respect to X is equivalent to the relative exactness with respect to x (since $dW/W = d(\ln W)$). □

Remark 5.5. Unfortunately, there does not seem to be an equivalent "coordinate free" description of normalizability.

Proof of Theorem 5.2. As stated above, we only need to prove this for an integrable non-resonant saddle. The direct implication is obvious. We prove only the converse. In this case one can get a formal solution of (4.9) as in the rational case (without using the isochronicity condition). However, as shown in [BC] and [CMR], Theorem A, the formal solution does not converge in general. This is why we adopt an approach different from the approach in the rational case.

We have to define a holomorphic function g satisfying (4.9) in a neighborhood of the origin. By linear scaling we can assume that the polydisc $\Omega = \{(x,y) \in \mathbb{C}^2 : |x| \leq 1, |y| \leq 1\}$ belongs to the domain of convergence of the form η. Put $g(1,1) = 0$. Let γ_c be the curve lying in the leaf of the foliation $\omega_\lambda = 0$, passing through the point $(1,c)$, given by

$$\gamma_c(\theta) = (e^{i\theta}, ce^{-i\lambda\theta}), \quad \theta \in \mathbb{R}. \tag{5.4}$$

Put

$$g(\gamma_1(\theta)) = \int_0^\theta \gamma_1^* \eta. \tag{5.5}$$

This defines the function g on a dense set of the torus $T = \{(x,y) \in \mathbb{C}^2 : |x| = 1, |y| = 1\}$. Next, by the hypothesis (5.3), the function g can be extended without ambiguity by continuity to the torus T.

Denote by $D = \{y \in \mathbb{C} : |y| < 1\}$, the unitary disc, $\bar{D}$ its closure and S its boundary. Note that g satisfies the condition

$$g(1, ce^{2\pi i\lambda}) = g(1,c) + \int_0^{2\pi} \gamma_c^* \eta, \tag{5.6}$$

for c on the circle S. We next want to extend g to a continuous function on the disc $\{1\} \times \bar{D}$,

$$g(1,c) = u(c), \tag{5.7}$$

with u holomorphic in D. Moreover, we want condition (5.6) to hold for all points $c \in \bar{D}$.

Initially, the function u is continuous complex valued defined on the circle S. Applying separately the existence theorem for solutions of Dirichlet's problem for the real and imaginary part of u, we extend u to a continuous function on $\bar{D}$, *harmonic* in D. Let g be given by (5.7), $c \in \bar{D}$. We claim that (5.6) holds for this extended function g. Indeed, consider the function

$$\psi(c) = u(ce^{2\pi i\lambda}) - u(c) - \int_0^{2\pi} \gamma_c^* \eta. \tag{5.8}$$

The function ψ is a harmonic function in D, as the last term in (5.8) is a holomorphic function in c. Moreover, $\psi(c)$ vanishes for $c \in S$. Now by the uniqueness of solutions of Dirichlet's problem it follows that ψ is identically zero on $\bar{D}$. This proves that (5.6) holds on $\bar{D}$. We claim next that u is in fact holomorphic in D. In order to prove it, introduce the differential operators

$$\partial = \frac{1}{2}(\frac{\partial}{\partial c'} + i\frac{\partial}{\partial c''}), \quad \bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial c'} - i\frac{\partial}{\partial c''}), \tag{5.9}$$

where c' and c'' are the real and imaginary part of c. As the last term in the definition of ψ is holomorphic, it follows, from the vanishing of ψ, that

$$\bar{\partial}(u(ce^{2\pi i\lambda})) = \bar{\partial}(u(c)). \tag{5.10}$$

Now, since λ is irrational, the numbers $ce^{2\pi ik\lambda}$, $k \in \mathbb{N}$, are dense on the circle of radius $|c|$, so (5.10) shows that the function $\bar{\partial}(u(c))$ depends only on $|c|$. Say $\bar{\partial}(u(c)) = \phi(|c|)$. However, since u is harmonic, then $\partial(\phi(|c|)) = 0$. This can only happen, if ϕ is a constant. We have so far proven that $u(c) = v(c) + k\bar{c}$, where v is a holomorphic function and $k \in \mathbb{C}$. To show that $k = 0$, we integrate relation(5.8) along $|c| = 1$. As ψ vanishes in D and moreover v and the last term in (5.8) are holomorphic, we get $2\pi ik(e^{-2\pi i\lambda} - 1) = 0$, so $k = 0$ and we have shown that u is holomorphic on D.

We now extend g to a complement of the y-axis. Given any (x,y), denote $\mathcal{L}_{(x,y)}$ the leaf of the linear foliation $\omega = 0$ passing through (x,y). For (x,y), $x \neq 0$, belonging to a neighborhood of the origin, the leaf $\mathcal{L}_{(x,y)}$ cuts the disc $\{1\} \times D$ infinitely many times. Let $\gamma_{(x,y)}$ be a curve in the leaf $\mathcal{L}_{(x,y)}$ starting at a point $(1,c)$, $c = x^\lambda y \in D$, and connecting it to (x,y). The leaf also cuts the disc $\{1\} \times D$ at points $(1, ce^{2\pi i\lambda k})$ with $k \in \mathbb{Z}$. We put

$$g(x,y) = g(1,c) + \int_{\gamma_{(x,y)}} \eta. \tag{5.11}$$

We claim that g is well defined and does not depend on the choice of $\gamma_{(x,y)}$. Indeed, let $\tilde{\gamma}_{(x,y)}$ be another choice of $\gamma_{(x,y)}$ starting at $(1,\tilde{c}) \in D$ with $\tilde{c} = c\exp(2\pi ik\lambda)$. Note that the path obtained by taking $\gamma_{(x,y)}$ followed by $(\tilde{\gamma}_{(x,y)})^{-1}$ is homotopic to the path $\gamma_c : [0, 2k\pi] \to \mathcal{L}_{(x,y)}$, given by $\gamma_c(\theta) = (e^{i\theta}, ce^{-i\lambda\theta})$, for some $k \in \mathbb{Z}$. This follows from the simple connectedness of the leaf $\mathcal{L}_{(x,y)}$.

By induction (5.6) gives

$$g(1, ce^{2\pi ik\lambda}) = g(1, c) + \int_0^{2k\pi} \gamma_c^* \eta, \quad k \in \mathbb{Z}. \tag{5.12}$$

This shows that

$$g(1, c) + \int_{\gamma_{(x,y)}} \eta = g(1, \tilde{c}) + \int_{\tilde{\gamma}_{(x,y)}} \eta \tag{5.13}$$

and hence the function g is well defined on a neighborhood of the origin from which the y-axis has been deleted. Moreover, g is holomorphic, as the initial value $g(1, c)$ depends holomorphically on $c = x^\lambda y$ and g is extended by integration of the holomorphic form η.

In order to extend holomorphically g to the y-axis, note that a point (x, y) close to the y-axis can be linked to a point in the disc $\{1\} \times D$ by a path γ belonging to the leaf $\mathcal{L}_{(x,y)}$ whose length is uniformly bounded. This can be seen by taking the path obtained by following first

$$\gamma_1(\theta) = (xe^{-i\theta}, ye^{i\lambda\theta}), \tag{5.14}$$

for θ varying from 0 to $arg(x) < 2\pi$ and then following

$$\gamma_2(r) = (r, \frac{yx^\lambda}{r^\lambda}), \tag{5.15}$$

for r varying from $|x|$ to 1. The form η is bounded in the fixed neighborhood of the origin which we have chosen. As g is also holomorphic (hence bounded) on $\{1\} \times D$, it now follows from the definition (5.11) of g that it is bounded on a fixed neighborhood of the origin from which the y-axis has been deleted. By the removable singularity theorem, we can now extend g holomorphically to a full neighborhood of the origin.

Relation (4.9) follows from the definition of g. Indeed, it suffices to verify this relation locally in the complement of the origin, where ω is different from zero. By a local change of coordinates $(z, w) = (z(x, y), w(x, y))$, it can be assumed that $\omega = dw$. Taking a section $z = k$ transverse to the leaves of the foliation and noting that g is obtained by integrating η along the leaves $w = const$, it follows that the dz coordinates of dg and η coincide. Hence $\eta - dg$ is collinear to ω. Now (4.9) is proved and the proof of the Theorem is completed. □

Open Question. Can the proof of Theorem 5.2 be extended to cover 1-forms ω dual to non-integrable vector fields of the form (1.1) when λ is irrational?

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When we finished a preliminary version of this paper, we learned that Loïc Teyssier was working on a similar program relating normal forms to the solvability of the homological equation. Teyssier's work was rather oriented towards examining the saddle-node case, whereas the most original part of our work deals with the

integrable saddle case. We took Teyssier's work into account when preparing this new version of our paper.

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